Introduction to Time Series Analysis. Lecture 18.

1. Review: Spectral density. Spectral distribution function.
2. Autocovariance generating function and spectral density.
3. Rational spectra. Poles and zeros.
4. Examples.

Review: Spectral density and spectral distribution function

If a time series $\{X_t\}$ has autocovariance γ satisfying $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, then we define its **spectral density** as

$$f(\nu) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h}$$

for $-\infty < \nu < \infty$. We have

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i \nu h} f(\nu)\, d\nu = \int_{-1/2}^{1/2} e^{2\pi i \nu h}\, dF(\nu),$$

where $dF(\nu) = f(\nu) d\nu$.

f measures how the variance of X_t is distributed across the spectrum.

Review: Spectral density and spectral distribution function

For any stationary $\{X_t\}$ with autocovariance γ, we can write

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i \nu h} dF(\nu),$$

where F is the *spectral distribution function* of $\{X_t\}$.

If F has no singular part, we can write $F = F^{(c)} + F^{(d)}$, where $F^{(c)}$ is continuous, that is, $dF^{(c)}(\nu) = f(\nu)d\nu$, and $F^{(d)}$ is discrete.

Autocovariance generating function and spectral density

Suppose X_t is a linear process, so it can be written
$X_t = \sum_{i=0}^{\infty} \psi_i W_{t-i} = \psi(B) W_t$.
Consider the autocovariance sequence,

$$\begin{aligned}
\gamma_h &= \text{Cov}(X_t, X_{t+h}) \\
&= \text{E}\left[\sum_{i=0}^{\infty} \psi_i W_{t-i} \sum_{j=0}^{\infty} \psi_j W_{t+h-j} \right] \\
&= \sigma_w^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+h}.
\end{aligned}$$

Autocovariance generating function and spectral density

Define the autocovariance generating function as

$$\gamma(B) = \sum_{h=-\infty}^{\infty} \gamma_h B^h.$$

Then,

$$\begin{aligned}\gamma(B) &= \sigma_w^2 \sum_{h=-\infty}^{\infty} \sum_{i=0}^{\infty} \psi_i \psi_{i+h} B^h \\ &= \sigma_w^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \psi_i \psi_j B^{j-i} \\ &= \sigma_w^2 \sum_{i=0}^{\infty} \psi_i B^{-i} \sum_{j=0}^{\infty} \psi_j B^j = \sigma_w^2 \psi(B^{-1}) \psi(B).\end{aligned}$$

Autocovariance generating function and spectral density

Notice that

$$\gamma(B) = \sum_{h=-\infty}^{\infty} \gamma_h B^h.$$

$$\begin{aligned} f(\nu) &= \sum_{h=-\infty}^{\infty} \gamma_h e^{-2\pi i \nu h} \\ &= \gamma\left(e^{-2\pi i \nu}\right) \\ &= \sigma_w^2 \psi\left(e^{-2\pi i \nu}\right) \psi\left(e^{2\pi i \nu}\right) \\ &= \sigma_w^2 \left|\psi\left(e^{2\pi i \nu}\right)\right|^2 . \end{aligned}$$

Autocovariance generating function and spectral density

For example, for an MA(q), we have $\psi(B) = \theta(B)$, so

$$\begin{aligned} f(\nu) &= \sigma_w^2 \theta\left(e^{-2\pi i\nu}\right)\theta\left(e^{2\pi i\nu}\right) \\ &= \sigma_w^2 \left|\theta\left(e^{-2\pi i\nu}\right)\right|^2 . \end{aligned}$$

For MA(1),

$$\begin{aligned} f(\nu) &= \sigma_w^2 \left|1 + \theta_1 e^{-2\pi i\nu}\right|^2 \\ &= \sigma_w^2 \left|1 + \theta_1 \cos(-2\pi\nu) + i\theta_1 \sin(-2\pi\nu)\right|^2 \\ &= \sigma_w^2 \left(1 + 2\theta_1 \cos(2\pi\nu) + \theta_1^2\right) . \end{aligned}$$

Autocovariance generating function and spectral density

For an AR(p), we have $\psi(B) = 1/\phi(B)$, so

$$f(\nu) = \frac{\sigma_w^2}{\phi\left(e^{-2\pi i\nu}\right)\phi\left(e^{2\pi i\nu}\right)} = \frac{\sigma_w^2}{\left|\phi\left(e^{-2\pi i\nu}\right)\right|^2}.$$

For AR(1),

$$f(\nu) = \frac{\sigma_w^2}{\left|1-\phi_1 e^{-2\pi i\nu}\right|^2} = \frac{\sigma_w^2}{1-2\phi_1\cos(2\pi\nu)+\phi_1^2}.$$

Spectral density of a linear process

If X_t is a linear process, it can be written $X_t = \sum_{i=0}^{\infty} \psi_i W_{t-i} = \psi(B) W_t$. Then

$$f(\nu) = \sigma_w^2 \left| \psi \left(e^{-2\pi i \nu} \right) \right|^2 .$$

That is, the spectral density $f(\nu)$ of a linear process measures the modulus of the ψ (MA(∞)) polynomial at the point $e^{2\pi i \nu}$ on the unit circle.

Spectral density of a linear process

For an ARMA(p,q), $\psi(B) = \theta(B)/\phi(B)$, so

$$f(\nu) = \sigma_w^2 \frac{\theta(e^{-2\pi i\nu})\theta(e^{2\pi i\nu})}{\phi\left(e^{-2\pi i\nu}\right)\phi\left(e^{2\pi i\nu}\right)}$$

$$= \sigma_w^2 \left| \frac{\theta(e^{-2\pi i\nu})}{\phi\left(e^{-2\pi i\nu}\right)} \right|^2 .$$

This is known as a *rational spectrum*.

Rational spectra

Consider the factorization of θ and ϕ as

$$\theta(z) = \theta_q(z - z_1)(z - z_2)\cdots(z - z_q)$$
$$\phi(z) = \phi_p(z - p_1)(z - p_2)\cdots(z - p_p),$$

where $z_1, \ldots, z_q$ and $p_1, \ldots, p_p$ are called the *zeros* and *poles*.

$$f(\nu) = \sigma_w^2 \left| \frac{\theta_q \prod_{j=1}^{q}(e^{-2\pi i\nu} - z_j)}{\phi_p \prod_{j=1}^{p}(e^{-2\pi i\nu} - p_j)} \right|^2$$
$$= \sigma_w^2 \frac{\theta_q^2 \prod_{j=1}^{q} \left|e^{-2\pi i\nu} - z_j\right|^2}{\phi_p^2 \prod_{j=1}^{p} \left|e^{-2\pi i\nu} - p_j\right|^2}.$$

Rational spectra

$$f(\nu) = \sigma_w^2 \frac{\theta_q^2 \prod_{j=1}^{q} \left|e^{-2\pi i\nu} - z_j\right|^2}{\phi_p^2 \prod_{j=1}^{p} \left|e^{-2\pi i\nu} - p_j\right|^2}.$$

As ν varies from 0 to $1/2$, $e^{-2\pi i\nu}$ moves clockwise around the unit circle from 1 to $e^{-\pi i} = -1$.
And the value of $f(\nu)$ goes up as this point moves closer to (further from) the poles p_j (zeros z_j).

Example: ARMA

Recall AR(1): $\phi(z) = 1 - \phi_1 z$. The pole is at $1/\phi_1$. If $\phi_1 > 0$, the pole is to the right of 1, so the spectral density decreases as ν moves away from 0. If $\phi_1 < 0$, the pole is to the left of -1, so the spectral density is at its maximum when $\nu = 0.5$.

Recall MA(1): $\theta(z) = 1 + \theta_1 z$. The zero is at $-1/\theta_1$. If $\theta_1 > 0$, the zero is to the left of -1, so the spectral density decreases as ν moves towards -1. If $\theta_1 < 0$, the zero is to the right of 1, so the spectral density is at its minimum when $\nu = 0$.

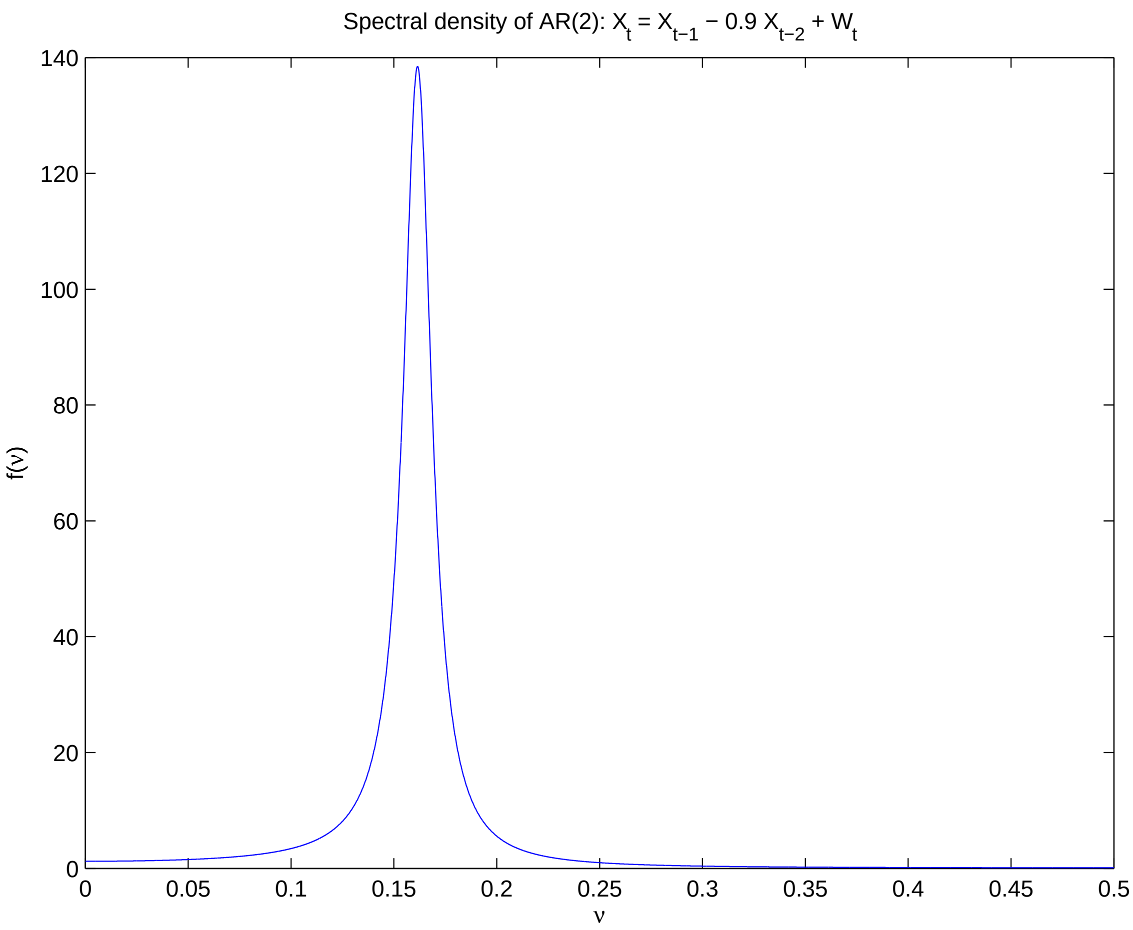
Example: AR(2)

Consider $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t$. Example 4.6 in the text considers this model with $\phi_1 = 1$, $\phi_2 = -0.9$, and $\sigma_w^2 = 1$. In this case, the poles are at $p_1, p_2 \approx 0.5555 \pm i0.8958 \approx 1.054e^{\pm i1.01567} \approx 1.054e^{\pm 2\pi i0.16165}$. Thus, we have

$$f(\nu) = \frac{\sigma_w^2}{\phi_2^2 |e^{-2\pi i\nu} - p_1|^2 |e^{-2\pi i\nu} - p_2|^2},$$

and this gets very peaked when $e^{-2\pi i\nu}$ passes near $1.054e^{-2\pi i0.16165}$.

Example: AR(2)



Spectral density of AR(2): X_t = X_{t−1} − 0.9 X_{t−2} + W_t
140
120
100
80
60
40
20
0
f(ν)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
ν

Example: Seasonal ARMA

Consider $X_t = \Phi_1 X_{t-12} + W_t$.

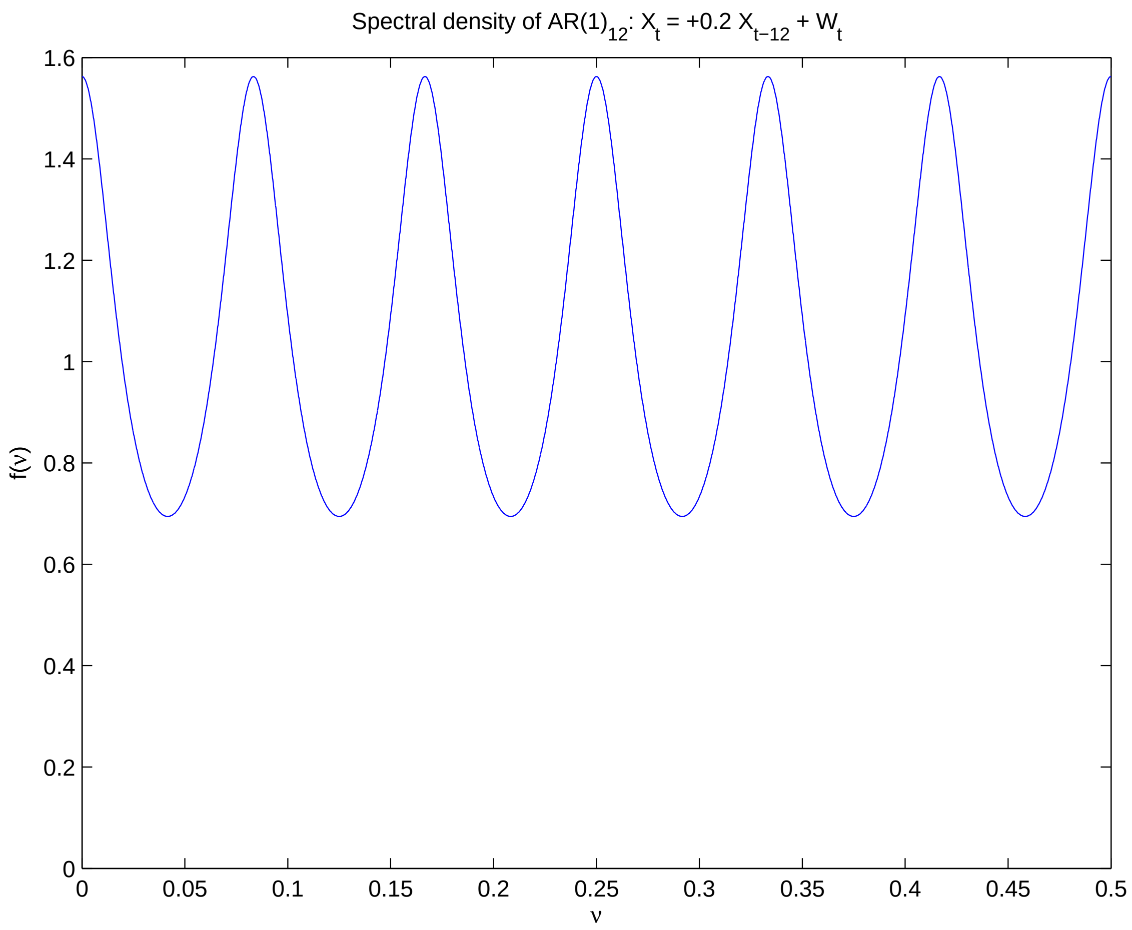
$$\psi(B) = \frac{1}{1 - \Phi_1 B^{12}},$$

$$f(\nu) = \sigma_w^2 \frac{1}{(1 - \Phi_1 e^{-2\pi i 12\nu})(1 - \Phi_1 e^{2\pi i 12\nu})}$$

$$= \sigma_w^2 \frac{1}{1 - 2\Phi_1 \cos(24\pi\nu) + \Phi_1^2}.$$

Notice that $f(\nu)$ is periodic with period $1/12$.

Example: Seasonal ARMA



Example: Seasonal ARMA

Another view:

$$1 - \Phi_1 z^{12} = 0 \quad \Leftrightarrow \quad z = re^{i\theta},$$

$$\text{with} \qquad r = |\Phi_1|^{-1/12}, \qquad e^{i12\theta} = e^{-i\arg(\Phi_1)}.$$

For $\Phi_1 > 0$, the twelve poles are at $|\Phi_1|^{-1/12} e^{ik\pi/6}$ for $k = 0, \pm 1, \ldots, \pm 5, 6$.
So the spectral density gets peaked as $e^{-2\pi i\nu}$ passes near $|\Phi_1|^{-1/12} \times \left\{1, e^{-i\pi/6}, e^{-i\pi/3}, e^{-i\pi/2}, e^{-i2\pi/3}, e^{-i5\pi/6}, -1\right\}$.

Example: Multiplicative seasonal ARMA

Consider $(1 - \Phi_1 B^{12})(1 - \phi_1 B)X_t = W_t$.

$$f(\nu) = \sigma_w^2 \frac{1}{(1 - 2\Phi_1 \cos(24\pi\nu) + \Phi_1^2)(1 - 2\phi_1 \cos(2\pi\nu) + \phi_1^2)}.$$

This is a scaled product of the AR(1) spectrum and the (periodic) $\text{AR}(1)_{12}$ spectrum.

The $\text{AR}(1)_{12}$ poles give peaks when $e^{-2\pi i\nu}$ is at one of the 12th roots of 1; the AR(1) poles give a peak near $e^{-2\pi i\nu} = 1$.

Example: Multiplicative seasonal ARMA