Introduction to Time Series Analysis: Review

1. Time series modelling.
2. Time domain.
   (a) Concepts of stationarity, ACF.
   (b) Linear processes, causality, invertibility.
   (c) ARMA models, forecasting, estimation.

Objectives of Time Series Analysis

1. Compact description of data. Example:

$$X_t = T_t + S_t + f(Y_t) + W_t.$$

2. Interpretation. Example: Seasonal adjustment.
3. Forecasting. Example: Predict unemployment.
4. Control. Example: Impact of monetary policy on unemployment.
5. Hypothesis testing. Example: Global warming.
6. Simulation. Example: Estimate probability of catastrophic events.

Time Series Modelling

1. Plot the time series.

   Look for trends, seasonal components, step changes, outliers.

2. Transform data so that residuals are **stationary**.

   (a) Estimate and subtract T_t, S_t.

   (b) Differencing.

   (c) Nonlinear transformations (log, $\sqrt{\cdot}$).

3. Fit model to residuals.

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Stationarity

$\{X_t\}$ is **strictly stationary** if, for all $k, t_1, \ldots, t_k, x_1, \ldots, x_k$, and h,

$$P(X_{t_1} \le x_1, \ldots, X_{t_k} \le x_k) = P(x_{t_1+h} \le x_1, \ldots, X_{t_k+h} \le x_k).$$

i.e., shifting the time axis does not affect the distribution.

We consider **second-order properties** only:
$\{X_t\}$ is stationary if its mean function and autocovariance function satisfy

$$\mu_x(t) = \mathrm{E}[X_t] = \mu,$$
$$\gamma_x(s,t) = \mathrm{Cov}(X_s, X_t) = \gamma_x(s-t).$$

NB: Constant variance: $\gamma_x(t,t) = \mathrm{Var}(X_t) = \gamma_x(0)$.

ACF and Sample ACF

The **autocorrelation function (ACF)** is

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \mathrm{Corr}(X_{t+h}, X_t).$$

For observations $x_1, \ldots, x_n$ of a time series,

the **sample mean** is $$\bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t.$$

The **sample autocovariance function** is

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-|h|}(x_{t+|h|} - \bar{x})(x_t - \bar{x}), \qquad \text{for } -n < h < n.$$

The **sample autocorrelation function** is $\hat{\rho}(h) = \hat{\gamma}(h)/\hat{\gamma}(0)$.

Properties of the autocovariance function

For the autocovariance function γ of a stationary time series $\{X_t\}$,

1. $\gamma(0) \geq 0$,
2. $|\gamma(h)| \leq \gamma(0)$,
3. $\gamma(h) = \gamma(-h)$,
4. γ is positive semidefinite.

Linear Processes

An important class of stationary time series:

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j}$$

where $\{W_t\} \sim WN(0, \sigma_w^2)$

and μ, ψ_j are parameters satisfying

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty.$$

$\mu_X = \mu$, $\gamma_X(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{h+j}$.

e.g.: ARMA(p,q).

Causality

A linear process $\{X_t\}$ is **causal** (strictly, a **causal function of** $\{W_t\}$) if there is a

$$\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots$$

$$\text{with} \qquad \sum_{j=0}^{\infty} |\psi_j| < \infty$$

$$\text{and} \qquad X_t = \psi(B) W_t.$$

Invertibility

A linear process $\{X_t\}$ is **invertible** (strictly, an **invertible function of** $\{W_t\}$) if there is a

$$\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \cdots$$

$$\text{with} \qquad \sum_{j=0}^{\infty} |\pi_j| < \infty$$

$$\text{and} \qquad W_t = \pi(B) X_t.$$

Polynomials of a complex variable

Every degree p polynomial $a(z)$ can be factorized as

$$a(z) = a_0 + a_1 z + \cdots + a_p z^p = a_p (z - z_1)(z - z_2) \cdots (z - z_p),$$

where $z_1, \ldots, z_p \in \mathbb{C}$ are called the roots of $a(z)$. If the coefficients $a_0, a_1, \ldots, a_p$ are all real, then c is real, and the roots are all either real or come in complex conjugate pairs, $z_i = \bar{z}_j$.

Autoregressive moving average models

An **ARMA(p,q) process** $\{X_t\}$ is a stationary process that satisfies

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q},$$

where $\{W_t\} \sim WN(0, \sigma^2)$.

Also, $\phi_p, \theta_q \neq 0$ and $\phi(z), \theta(z)$ have no common factors.

Properties of ARMA(p,q) models

Theorem: If ϕ and θ have no common factors, a (unique) *stationary* solution to $\phi(B)X_t = \theta(B)W_t$ exists iff

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p = 0 \;\Rightarrow\; |z| \neq 1.$$

This ARMA(p,q) process is *causal* iff

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p = 0 \;\Rightarrow\; |z| > 1.$$

It is *invertible* iff

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q = 0. \;\Rightarrow\; |z| > 1.$$

Properties of ARMA(p,q) models

$$\phi(B)X_t = \theta(B)W_t, \qquad \Leftrightarrow \qquad X_t = \psi(B)W_t$$

$$\text{so} \qquad \theta(B) = \psi(B)\phi(B)$$

$$\Leftrightarrow \quad 1 + \theta_1 B + \cdots + \theta_q B^q = (\psi_0 + \psi_1 B + \cdots)(1 - \phi_1 B - \cdots - \phi_p B^p)$$

$$\begin{aligned} \Leftrightarrow \qquad 1 &= \psi_0, \\ \theta_1 &= \psi_1 - \phi_1\psi_0, \\ \theta_2 &= \psi_2 - \phi_1\psi_1 - \cdots - \phi_2\psi_0, \\ &\vdots \end{aligned}$$

This is equivalent to $\theta_j = \phi(B)\psi_j$, with $\theta_0 = 1$, $\theta_j = 0$ for $j < 0$, $j > q$.

Linear prediction

Given $X_1, X_2, \ldots, X_n$, the best linear predictor

$$X_{n+m}^n = \alpha_0 + \sum_{i=1}^{n} \alpha_i X_i$$

of X_{n+m} satisfies the **prediction equations**

$$\mathrm{E}\left(X_{n+m} - X_{n+m}^n\right) = 0$$

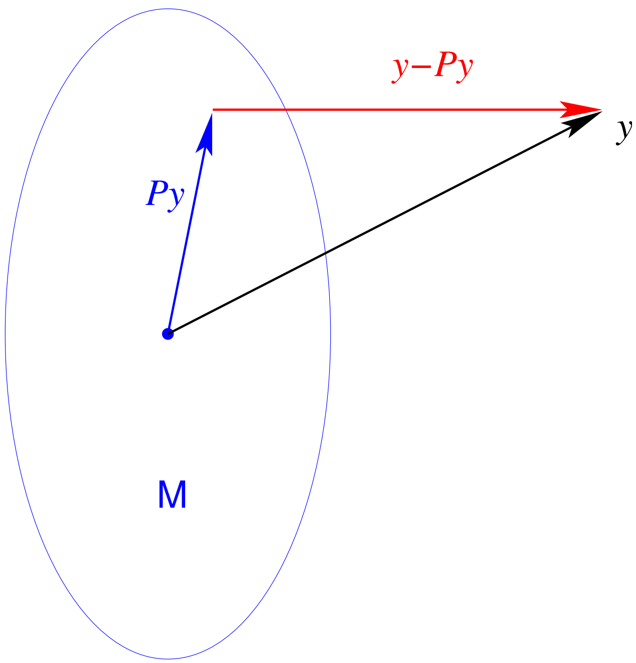
$$\mathrm{E}\left[\left(X_{n+m} - X_{n+m}^n\right) X_i\right] = 0 \qquad \text{for } i = 1, \ldots, n.$$

That is, the *prediction errors* ($X_{n+m}^n - X_{n+m}$) *are uncorrelated with the prediction variables* ($1, X_1, \ldots, X_n$).

Projection Theorem

If $\mathcal{H}$ is a Hilbert space,
$\mathcal{M}$ is a closed linear subspace of $\mathcal{H}$,
and $y \in \mathcal{H}$,
then there is a point $Py \in \mathcal{M}$
(the **projection of** y **on** $\mathcal{M}$)
satisfying

1. $\|Py - y\| \leq \|w - y\|$ for $w \in \mathcal{M}$,
2. $\|Py - y\| < \|w - y\|$ for $w \in \mathcal{M}$, $w \neq y$
3. $\langle y - Py, w\rangle = 0$ for $w \in \mathcal{M}$.



One-step-ahead linear prediction

$$X_{n+1}^n = \phi_{n1} X_n + \phi_{n2} X_{n-1} + \cdots + \phi_{nn} X_1$$

$$\Gamma_n \phi_n = \gamma_n,$$

$$P_{n+1}^n = \mathrm{E}\left(X_{n+1} - X_{n+1}^n\right)^2 = \gamma(0) - \gamma_n' \Gamma_n^{-1} \gamma_n,$$

$$\Gamma_n = \begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \\ \gamma(1) & \gamma(0) & & \gamma(n-2) \\ \vdots & & \ddots & \vdots \\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(0) \end{bmatrix},$$

$$\phi_n = (\phi_{n1}, \phi_{n2}, \ldots, \phi_{nn})', \quad \gamma_n = (\gamma(1), \gamma(2), \ldots, \gamma(n))'.$$

The innovations representation

Write the best linear predictor as

$$X_{n+1}^{n} = \theta_{n1} \underbrace{\left(X_n - X_n^{n-1}\right)}_{\text{innovation}} + \theta_{n2} \left(X_{n-1} - X_{n-1}^{n-2}\right) + \cdots + \theta_{nn} \left(X_1 - X_1^{0}\right).$$

The innovations are uncorrelated:
$\mathrm{Cov}(X_j - X_j^{j-1}, X_i - X_i^{i-1}) = 0$ for $i \neq j$.

Yule-Walker estimation

Method of moments: We choose parameters for which the moments are equal to the empirical moments.

In this case, we choose ϕ so that $\gamma = \hat{\gamma}$.

Yule-Walker equations for $\hat{\phi}$: $$\begin{cases} \hat{\Gamma}_p \hat{\phi} = \hat{\gamma}_p, \\ \hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\phi}' \hat{\gamma}_p. \end{cases}$$

These are the forecasting equations.

Recursive computation: Durbin-Levinson algorithm.

Maximum likelihood estimation

Suppose that $X_1, X_2, \ldots, X_n$ is drawn from a zero mean Gaussian ARMA(p,q) process. The likelihood of parameters $\phi \in \mathbb{R}^p$, $\theta \in \mathbb{R}^q$, $\sigma_w^2 \in \mathbb{R}_+$ is defined as the density of $X = (X_1, X_2, \ldots, X_n)'$ under the Gaussian model with those parameters:

$$L(\phi, \theta, \sigma_w^2) = \frac{1}{(2\pi)^{n/2} \left|\Gamma_n\right|^{1/2}} \exp\left(-\frac{1}{2} X' \Gamma_n^{-1} X\right),$$

where $|A|$ denotes the determinant of a matrix A, and Γ_n is the variance/covariance matrix of X with the given parameter values.

The maximum likelihood estimator (MLE) of ϕ, θ, σ_w^2 maximizes this quantity.

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