• Tuning Fixed Share for low adaptive regret (See lecture 11 notes).
• Normalised Maximum Likelihood
• Universal Portfolios
Today we look at *probability forecasting*.

*Log loss* measures mismatch of probability prediction $P$ and outcome $x$:

$$\text{loss}(P, x) = -\ln P(x)$$
Log-loss game

Protocol:

- For $t = 1, 2, \ldots$
  - Learner chooses a distribution $P_t$ on outcomes $\mathcal{X}$.
  - Adversary reveals outcome $x_t \in \mathcal{X}$.
  - Learner incurs the log loss $-\ln P_t(x_t)$. 

Joint vs predictive viewpoint

Correspondence between

- Joint distribution $P(x_1, \ldots, x_T)$ and
- Prediction strategy $P(x_t|x_1, \ldots, x_{t-1})$

Log loss telescopes:

$$\sum_{t=1}^{T} - \ln P(x_t|x_1, \ldots, x_{t-1}) = - \ln P(x_1, \ldots, x_T)$$
Reference strategies: probabilistic model

This lecture: compete with a family of distributions

\[ \{ P_\theta(x_1, \ldots, x_T) \mid \theta \in \Theta \} \]

Regret w.r.t. \( \Theta \)

\[ R_T = \sum_{t=1}^{T} - \ln P_t(x_t) - \inf_{\theta \in \Theta} - \ln P_\theta(x_1, \ldots, x_T) \]

- Loss of Learner
- Loss of best parameter \( \theta \in \Theta \)

Crucial new ingredient: Learner has white-box access to “experts” \( P_\theta \).
From $T$ rounds to one mega round

Sequential minimax reduces to single-shot

$$\min_{P_1} \max_{x_1} \ldots \min_{P_T} \max_{x_T} R_T = \min_P \max_{x_1,\ldots,x_T} R_T$$

Idea: joint distribution $P$ already encodes all sequential behaviour
Minimax game solution

**Theorem:** The $T$-round log loss prediction game w. model $\Theta$

\[
\min_P \max_{x^T} R_T \left( = \max_Q \min_P \mathbb{E}_{x^T \sim Q} R_T \right)
\]

has value given by the *parametric complexity* of model $\Theta$

\[
\mathcal{C}_T(\Theta) := \ln \left( \sum_{x_1, \ldots, x_T} \sup_{\theta \in \Theta} P_{\theta}(x_1, \ldots, x_T) \right),
\]

and both Learner’s minimax and Adversary’s maximin strategy are the *Normalised Maximum Likelihood* distribution

\[
P^{\text{NML}}_T(x_1, \ldots, x_T) = \frac{\sup_{\theta \in \Theta} P_{\theta}(x_1, \ldots, x_T)}{\sum_{z_1, \ldots, z_T} \sup_{\theta \in \Theta} P_{\theta}(z_1, \ldots, z_T)}
\]
Minimax analysis

For Learner, $P^{\text{NML}}$ equalises the regret over all sequences $x^T$:

$$- \ln P^{\text{NML}}(x^T) - \inf_{\theta} \ln P_{\theta}(x^T) = \mathcal{C}_T(\Theta)$$

For Adversary, we need to do some work:

$$\max_{Q} \min_{P} \mathbb{E}_{x^T \sim Q} \left[ - \ln P(x^T) - \inf_{\theta} \ln P_{\theta}(x^T) \right]$$

$$= \max_{Q} \mathbb{E}_{x^T \sim Q} \left[ - \ln Q(x^T) - \inf_{\theta} \ln P_{\theta}(x^T) \right]$$

To find maximin distribution $Q$ introduce Lagrange multiplier $\lambda$

$$= \min_{\lambda} \max_{Q} \mathbb{E}_{x^T \sim Q} \left[ - \ln Q(x^T) - \inf_{\theta} \ln P_{\theta}(x^T) \right] + \lambda \left( 1 - \sum_{x^T} Q(x^T) \right)$$
At the optimal $Q$ the derivative of the objective vanishes, i.e.

$$0 = -\ln Q(x^T) + 1 - \lambda - \inf_\theta - \ln P_\theta(x^T)$$

from which it follows that

$$Q(x^T) = \sup_\theta P_\theta(x^T) = \frac{\sum x^T \cdots}{P^{\text{NML}}(x^T)}$$

Plugging this in, we find that the value again equals $\mathcal{C}_T(\Theta)$. 
Discussion

• Continuous case: densities w.r.t. reference measure

• Efficient implementation tricky.

• NML strategy depends on final horizon $T$.

• Infinite parametric complexity $\Rightarrow P_{NML}$ undefined.

• Sometimes success with *Sequential NML* (SNML), also known as *Last Step Minimax*. In round $t$ play minimiser of

$$
\min_{P_t} \max_{x_t} - \ln P_t(x_t) - \inf_{\theta \in \Theta} - \ln P_\theta(x_1, \ldots, x_t)
$$
Universal portfolios
Best fixed weights

Back to the mix loss game without outcomes.

In lecture 9 we minimised regret w.r.t. the best expert.

Today we consider a more ambitious reference class:

**Definition:** Regret w.r.t. the best fixed weights

\[ R_T = \sum_{t=1}^{T} - \ln \left( \sum_{k=1}^{K} w_t^k e^{-\ell_t^k} \right) - \inf_{\mathbf{w}} \sum_{t=1}^{T} - \ln \left( \sum_{k=1}^{K} w^k e^{-\ell_t^k} \right) \]

Mix loss of Learner \hspace{5cm} Mix loss of fixed weights \( \mathbf{w} \)
**Investment interpretation**

Loss $\ell^k_t$ is negative log return of stock $k$, i.e. $\ell^k_t = -\ln \frac{\text{price}_{t+1}^k}{\text{price}_t^k}$

Weight vector $w$ is portfolio: fraction $w^k$ of capital invested in stock $k$.

Mix loss is negative log return of portfolio:

$$- \ln \sum_{k=1}^{K} w^k_t e^{-\ell^k_t} = - \ln \sum_{k=1}^{K} w^k_t \frac{\text{price}_{t+1}^k}{\text{price}_t^k}$$

Strategy playing fixed weights $w$ called *constantly rebalanced portfolio*

Strategy with low regret w.r.t. best $w$ called *universal* portfolio
Intuition

Mix loss is convex in $\mathbf{w}$, so minimiser typically in interior of simplex

<table>
<thead>
<tr>
<th>stock</th>
<th>daily return</th>
</tr>
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<tbody>
<tr>
<td>A</td>
<td>$\frac{1}{2}$ 2 $\frac{1}{2}$ 2 $\frac{1}{2}$ 2 $\frac{1}{2}$ 2</td>
</tr>
<tr>
<td>B</td>
<td>3 $\frac{1}{3}$ 3 $\frac{1}{3}$ 3 $\frac{1}{3}$ 3 $\frac{1}{3}$</td>
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Best single stock: overall return 1.

Best portfolio:

$$\max_p \left\{ 4 \ln \left( p \frac{1}{2} + (1 - p) 3 \right) + 4 \ln \left( p 2 + (1 - p) \frac{1}{3} \right) \right\}$$

results in $p = \frac{1}{2}$ and per-round log return

$$0.357 \approx \ln(1.43)$$
“Experts” are constantly rebalanced portfolios \( w \).

We again have white-box experts.

Close \( w \) have close loss.

First idea:

- Run AA on finely discretised simplex.
- Regret: overhead of AA w.r.t. best discretisation point plus overhead of best discretisation point w.r.t. best portfolio
- Balancing act.

Second idea:

- Run AA with density on simplex (discretise infinitely fine).
- Balancing act only in analysis.
Cover’s Universal “algorithm”

Put Dirichlet \((\frac{1}{2}, \ldots, \frac{1}{2})\) prior on simplex: 
\[ \pi_1(w) = \frac{\prod_{k=1}^{K} w_k^{-1/2}}{\int \cdots dw}. \]

Density in round \(t\):
\[
\pi_t(w) = \frac{\pi_1(w) e^{-\sum_{s=1}^{t-1} - \ln \sum_{k=1}^{K} w^k e^{-\ell_{st}^k}}}{\int \cdots dw} = \frac{\pi_1(w) \prod_{s=1}^{t-1} \left( \sum_{k=1}^{K} w_k e^{-\ell_{st}^k} \right)}{\int \cdots dw}
\]

Actual portfolio \(w_t\) played:
\[
w_t = \int w \pi_t(w) \, dw
\]
Cumulative mix loss of Universal again telescopes:

\[ \sum_{t=1}^{T} - \ln \left( \sum_{k=1}^{K} w_t^k e^{-\ell_t^k} \right) \]

\[ = \sum_{t=1}^{T} - \ln \left( \sum_{k=1}^{K} \left( \int w^k \pi_t(w) \, dw \right) e^{-\ell_t^k} \right) \]

\[ = \sum_{t=1}^{T} - \ln \left( \int \left( \sum_{k=1}^{K} w^k e^{-\ell_t^k} \right) \pi_t(w) \, dw \right) \]

\[ = - \ln \left( \int \pi_1(w) \prod_{t=1}^{T} \left( \sum_{k=1}^{K} w^k e^{-\ell_t^k} \right) \, dw \right) \]

However, we cannot bound using a single \( w \).
Analysis

Regret

\[ R_T = \sup_w \ln \frac{\prod_{t=1}^{T} \left( \sum_{k=1}^{K} w^k e^{-\ell_t^k} \right)}{\int \pi_1(w) \prod_{t=1}^{T} \left( \sum_{k=1}^{K} w^k e^{-\ell_t^k} \right) \, dw} \]

\[ \leq \sup_w \max_{k_1, \ldots, k_T} \ln \frac{\prod_{t=1}^{T} w_{k_t} e^{-\ell_t^{k_t}}}{\int \pi_1(w) \prod_{t=1}^{T} w_{k_t} e^{-\ell_t^{k_t}} \, dw} \]

\[ = \sup_w \max_{k_1, \ldots, k_T} \ln \frac{\prod_{t=1}^{T} w_{k_t}}{\int \pi_1(w) \prod_{t=1}^{T} w_{k_t} \, dw} \]

\[ \leq \frac{K - 1}{2} \ln T + \ln K \]

The first inequality follows from the log-sum inequality, which is tight when every round all but one expert suffer infinite loss.

The last inequality is the regret bound for the KT estimator (see Catoni 2004).

Now the worst-case is when \( k_1, \ldots, k_T \) identical.
Universal algorithm not efficient. Can integrate in $O(T^{K-1})$ time.

Algorithms based on sampling. Typically $\sqrt{T}$ regret regime.

Can use efficient Online Newton Step, which has $O(\ln T)$ regret provided the expert losses are bounded (shares do not spike/crash too much per round).

Transaction costs?

Adaptive regret? Perhaps slow drift instead of abrupt switches?