Theoretical Statistics. Lecture 24.

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1. Relative efficiency of tests. [vdv14]
   (a) Asymptotic power functions.
   (b) Asymptotic relative efficiency of tests.

Recall: Relative efficiency of tests

Example: Suppose $X_1, \ldots, X_n \sim P_\theta$, where

1. P_θ has density $f(x-\theta)$ on $\mathbb{R}$,
2. f is symmetric about zero (so the mean=median of P_θ is θ),
3. f has a unique median ($f(0) \neq 0$),
4. f has a finite variance.

We wish to test $H_0 : \theta = 0$ versus $H_1 : \theta > 0$.

Recall: Relative efficiency of tests

Example: Candidate tests:

1. Sign test: $S_n = \dfrac{1}{n}\sum_{i=1}^{n} 1[X_i > 0]$.
2. t-test: $T_n = \dfrac{1}{n}\sum_{i=1}^{n} \dfrac{X_i}{S_n}$.

Which is better?

Recall: Relative efficiency of tests. Sign test

Definition: The power function of a test that rejects the null hypothesis when the statistic T_n falls in the critical region K_n is

$$\pi_n(\theta) = P_\theta(T_n \in K_n).$$

For the sign test,

$$\pi_n(\theta) = 1 - \Phi\left(\frac{\sigma(0)z_\alpha + \sqrt{n}\,(\mu(0) - \mu(\theta))}{\sigma(\theta)}\right) + o(1)$$
$$\to \begin{cases} \alpha & \text{if } \theta = 0, \\ 1 & \text{if } \theta > 0. \end{cases}$$

So the limiting power function is perfect. (Typical for a reasonable test.)

Recall: Relative efficiency of tests

How do we compare tests? We need to make the problem of discriminating between the null and the alternative more difficult as n increases. It is natural to consider a **shrinking alternative**, that converges to the null.

We wish to test $H_0 : \theta = 0$ versus $H_1 : \theta_n > 0$, with $\theta_n \to 0$.

Recall: Relative efficiency of tests

For the sign test,

$$\pi_n(\theta_n) = 1 - \Phi\left(\frac{\sigma(0)z_\alpha + \sqrt{n}\,(\mu(0) - \mu(\theta_n))}{\sigma(\theta_n)}\right) + o(1).$$

The power depends on the asymptotics of $\sqrt{n}\,(\mu(0) - \mu(\theta_n))$. Since F is differentiable at 0,

$$\sqrt{n}\,(\mu(0) - \mu(\theta_n)) = \sqrt{n}\,(F(-\theta_n) - F(0)) = -\sqrt{n}\theta_n f(0) + o(\sqrt{n}\theta_n).$$

Recall: Relative efficiency of tests

If $\theta_n \to \theta$ faster than $1/\sqrt{n}$, $\sqrt{n}\,(\mu(0) - \mu(\theta_n)) \to 0$, so $\pi_n(\theta_n) \to \alpha$. The test fails: these alternatives are too hard.

For $\theta_n \to \theta$ slower than $1/\sqrt{n}$, $\sqrt{n}\,(\mu(0) - \mu(\theta_n)) \to -\infty$, so $\pi_n(\theta_n) \to 1$. These slowly shrinking alternatives are too easy.

Consider an intermediate rate:
$\sqrt{n}\theta_n \to h$.

Relative efficiency of tests

If $\sqrt{n}\theta_n \to h$, then $\sqrt{n}\left(\mu(0) - \mu(\theta_n)\right) \to -hf(0)$, so

$$\begin{aligned}\pi_n(\theta_n) &\to 1 - \Phi\left(\frac{\sigma(0)z_\alpha - hf(0)}{\sigma(0)}\right)\\ &= 1 - \Phi\left(z_\alpha - 2hf(0)\right)\\ &= \Phi\left(2hf(0) - z_\alpha\right).\end{aligned}$$

Relative efficiency of tests

This leads to a natural asymptotic comparison of two tests for $H_0 : \theta = 0$ versus $H_1 : \theta > 0$:
Compare the **local limiting power functions,**

$$\pi(h) = \lim_{n\to\infty} \pi_n \left(\frac{h}{\sqrt{n}} \right)$$

for $h \geq 0$.

Relative efficiency of tests

Theorem: Suppose that (1) T_n, μ, and σ are such that, for all h and $\theta_n = h/\sqrt{n}$,

$$\frac{\sqrt{n}\,(T_n - \mu(\theta_n))}{\sigma(\theta_n)} \overset{\theta_n}{\rightsquigarrow} N(0,1),$$

(2) μ is differentiable at 0, (3) σ is continuous at 0.

Then a test that rejects $H_0 : \theta = 0$ for large values of T_n and is asymptotically of level α satisfies, for all h,

$$\pi_n\left(\frac{h}{\sqrt{n}}\right) \to 1 - \Phi\left(z_\alpha - h\frac{\mu'(0)}{\sigma(0)}\right).$$

Relative efficiency of tests

Proof:

Substituting $h = 0$ shows that the asymptotic level of the test is α iff we reject $H_0 : \theta = 0$ for

$$\frac{\sqrt{n}\,(T_n - \mu(0))}{\sigma(0)} > z_\alpha.$$

Thus,

$$\begin{aligned}\pi_n(\theta_n) &= P_{\theta_n}\left(\sqrt{n}\,(T_n - \mu(0)) > \sigma(0) z_\alpha\right) \\ &= P_{\theta_n}\left(\sqrt{n}\frac{(T_n - \mu(\theta_n))}{\sigma(\theta_n)} > \frac{\sigma(0) z_\alpha - \sqrt{n}\,(\mu(\theta_n) - \mu(0))}{\sigma(\theta_n)}\right) \\ &\to 1 - \Phi\left(z_\alpha - h\frac{\mu'(0)}{\sigma(0)}\right).\end{aligned}$$

Relative efficiency of tests

So we have an easy comparison between tests based on locally asymptotically normal statistics: compare the **slope** of the tests, $\mu'(0)/\sigma(0)$. The bigger the slope, the faster $\pi_n(h/\sqrt{n})$ increases from α as h increases from 0.

Relative efficiency of tests

Example: sign test

$$
\begin{aligned}
\mu(\theta) &= 1 - F(-\theta), \\
\mu'(\theta) &= f(-\theta), \\
\sigma^2(\theta) &= (1 - F(-\theta))F(-\theta), \\
\frac{\mu'(0)}{\sigma(0)} &= 2f(0).
\end{aligned}
$$

Relative efficiency of tests: t-test

$$T_n = \frac{\bar{X}_n}{S_n}.$$

$$\sqrt{n}\frac{\bar{X}_n - \theta}{S_n} \overset{\theta}{\rightsquigarrow} N(0,1).$$

Reject H_0 if $\sqrt{n}T_n > z_\alpha$.

Relative efficiency of tests: t-test

So

$$\begin{aligned}
\pi_n(\theta) &= P_\theta\left(\sqrt{n}T_n > z_\alpha\right) \\
&= P_\theta\left(\sqrt{n}\frac{\bar{X}_n - \theta}{S_n} > z_\alpha - \sqrt{n}\frac{\theta}{S_n}\right) \\
&= 1 - \Phi\left(z_\alpha - \sqrt{n}\frac{\theta}{\sigma}\right) + o(1).
\end{aligned}$$

As before,

$$\pi_n(\theta) \to \begin{cases} \alpha & \text{if } \theta = 0, \\ 1 & \text{if } \theta > 0. \end{cases}$$

The limiting power function is perfect.

Relative efficiency of tests: t-test

$$T_n = \frac{\bar{X}_n}{S_n}.$$

$$\sqrt{n}\frac{\bar{X}_n - \theta}{S_n} \overset{\theta}{\rightsquigarrow} N(0,1).$$

$$\sqrt{n}\left(\frac{\bar{X}_n}{S_n} - \frac{h/\sqrt{n}}{\sigma}\right) = \sqrt{n}\left(\frac{\bar{X}_n - h/\sqrt{n}}{S_n}\right) + h\left(\frac{1}{S} - \frac{1}{\sigma}\right) \overset{h/\sqrt{n}}{\rightsquigarrow} N(0,1).$$

$$\mu(\theta) = \frac{\theta}{\sigma},$$

$$\sigma(\theta) = 1.$$

$$\frac{\mu'(0)}{\sigma(0)} = \frac{1}{\sigma}.$$

Relative efficiency of tests

sign test: $$\frac{\mu'(0)}{\sigma(0)} = 2f(0).$$

t-test: $$\frac{\mu'(0)}{\sigma(0)} = \frac{1}{\sigma}.$$

Laplace: $$2f(0)\sigma = 2.$$

Logistic: $$2f(0)\sigma = \frac{\pi^2}{12} = 0.82246703.$$

Normal: $$2f(0)\sigma = \frac{2}{\pi} = 0.63661977.$$

Uniform: $$2f(0)\sigma = \frac{1}{3}.$$

Relative efficiency of tests

But the fact that the local limiting power function for the sign test depends on the density at a single point (0) should raise a red flag!
Consider a uniform distribution with its density slightly modified to give a huge, narrow peak at 0. The sign test will have better asymptotics, but unless we have a huge sample, this distribution would be hard to distinguish from a uniform. That is, the asymptotics would need a very large n to be relevant.

Asymptotic relative efficiency of tests

Definition: For level α and power $\gamma \in (\alpha, 1)$, the **asymptotic relative efficiency** or **Pitman efficiency** of test 1 with respect to test 2 is

$$\lim_{\nu \to \infty} \frac{n_{\nu,1}}{n_{\nu,2}},$$

where $n_{\nu,1}$ is the minimal number of observations such that

$$\pi_{n_{\nu,1}}(0) \leq \alpha, \qquad \text{and} \qquad \pi_{n_{\nu,1}}(\theta_\nu) \geq \gamma.$$

Asymptotic relative efficiency of tests

Theorem: For a model P_θ, suppose $\|P_\theta - P_0\| \to 0$ as $\theta \to 0$. Suppose tests $i = 1, 2$ satisfy: (1) Test i rejects the null hypothesis $H_0 : \theta = 0$ for large values of a statistic $T_{n,i}$, and $T_{n,i}$ satisfies

$$\frac{\sqrt{n}(T_{n,i} - \mu_i(\theta_n))}{\sigma_i(\theta_n)} \overset{\theta_n}{\rightsquigarrow} N(0,1) \qquad \text{for } \sqrt{n}\theta_n \to h.$$

(2) μ_i is differentiable at 0, σ_i is continuous at 0, $\mu_i'(0) > 0$, $\sigma_i(0) > 0$. (3) The power function of test i is nondecreasing for each n. Then the relative efficiency of these tests is

$$\left(\frac{\mu_1'(0)\sigma_2(0)}{\mu_2'(0)\sigma_1(0)}\right)^2.$$

Asymptotic relative efficiency of tests: Proof

The condition that P_θ approaches P_0 in total variation distance as $\theta \to 0$ implies that the minimal numbers $n_{\nu,i}$ must go to infinity as $\nu \to \infty$.

Then the limiting normal distribution reveals the appropriate threshold to ensure that $\pi_{n_{\nu,1}}(0) = \alpha$:

$$\sqrt{n_{\nu,i}}(T_{n_{\nu,i},i} - \mu_i(0)) > \sigma_i(0) z_\alpha + o(1).$$

Then

$$\pi_{n_{\nu,i}}(\theta_\nu) = 1 - \Phi\left(z_\alpha - \sqrt{n_{\nu,i}}\theta_\nu \frac{\mu_i'(0)}{\sigma_i(0)} \right) + o(1).$$

Asymptotic relative efficiency of tests: Proof

For the power to approach γ as $\nu \to \infty$, the argument of Φ must approach z_γ, which means

$$\sqrt{n_{\nu,i}}\theta_\nu \frac{\mu_i'(0)}{\sigma_i(0)} \to z_\alpha - z_\gamma.$$

Hence,

$$\lim_{\nu\to\infty} \frac{n_{\nu,2}}{n_{\nu,1}} = \lim_{\nu\to\infty} \left(\frac{\sqrt{n_{\nu,2}}\theta_\nu}{\sqrt{n_{\nu,1}}\theta_\nu} \right)^2 = \left(\frac{\mu_1'(0)\sigma_2(0)}{\mu_2'(0)\sigma_1(0)} \right)^2.$$