Introduction to Time Series Analysis. Lecture 6.

Peter Bartlett

www.stat.berkeley.edu/∼bartlett/courses/153-fall2010

Last lecture:

1. Causality
2. Invertibility
3. AR(p) models
4. ARMA(p,q) models

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1. ARMA(p,q) models
2. Stationarity, causality and invertibility
3. The linear process representation of ARMA processes: ψ.
4. Autocovariance of an ARMA process.
5. Homogeneous linear difference equations.

Review: Causality

A linear process $\{X_t\}$ is **causal** (strictly, a **causal function of** $\{W_t\}$) if there is a

$$\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots$$

$$\text{with} \qquad \sum_{j=0}^{\infty} |\psi_j| < \infty$$

$$\text{and} \qquad X_t = \psi(B) W_t.$$

Review: Invertibility

A linear process $\{X_t\}$ is **invertible** (strictly, an **invertible function of** $\{W_t\}$) if there is a

$$\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \cdots$$

$$\text{with} \qquad \sum_{j=0}^{\infty} |\pi_j| < \infty$$

$$\text{and} \qquad W_t = \pi(B) X_t.$$

Review: AR(p), Autoregressive models of order p

An **AR(p) process** $\{X_t\}$ is a stationary process that satisfies

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t,$$

where $\{W_t\} \sim WN(0, \sigma^2)$.

Equivalently, $\phi(B) X_t = W_t,$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p.$

Review: AR(p), Autoregressive models of order p

Theorem: A (unique) *stationary* solution to $\phi(B)X_t = W_t$ exists iff the roots of $\phi(z)$ *avoid* the unit circle:

$$|z| = 1 \Rightarrow \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0.$$

This AR(p) process is *causal* iff the roots of $\phi(z)$ are *outside* the unit circle:

$$|z| \leq 1 \Rightarrow \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0.$$

Reminder: Polynomials of a complex variable

Every degree p polynomial $a(z)$ can be factorized as

$$a(z) = a_0 + a_1 z + \cdots + a_p z^p = a_p(z - z_1)(z - z_2)\cdots(z - z_p),$$

where $z_1, \ldots, z_p \in \mathbb{C}$ are the *roots* of $a(z)$. If the coefficients $a_0, a_1, \ldots, a_p$ are all real, then the roots are all either real or come in complex conjugate pairs, $z_i = \bar{z}_j$.

Example: $z + z^3 = z(1 + z^2) = (z - 0)(z - i)(z + i)$,
that is, $z_1 = 0$, $z_2 = i$, $z_3 = -i$. So $z_1 \in \mathbb{R}$; $z_2, z_3 \notin \mathbb{R}$; $z_2 = \bar{z}_3$.

Recall notation: A complex number $z = a + ib$ has $\mathrm{Re}(z) = a$, $\mathrm{Im}(z) = b$, $\bar{z} = a - ib$, $|z| = \sqrt{a^2 + b^2}$, $\arg(z) = \tan^{-1}(b/a) \in (-\pi, \pi]$.

Review: Calculating ψ for an AR(p): general case

$$\phi(B)X_t = W_t, \qquad \Leftrightarrow \qquad X_t = \psi(B)W_t$$

$$\text{so} \qquad 1 = \psi(B)\phi(B)$$

$$\Leftrightarrow \qquad 1 = (\psi_0 + \psi_1 B + \cdots)(1 - \phi_1 B - \cdots - \phi_p B^p)$$

$$\Leftrightarrow \qquad 1 = \psi_0, \qquad 0 = \psi_j \qquad (j < 0),$$

$$0 = \phi(B)\psi_j \qquad (j > 0).$$

We can solve these *linear difference equations* in several ways:

- numerically, or
- by guessing the form of a solution and using an inductive proof, or
- by using the theory of linear difference equations.

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ARMA(p,q): Autoregressive moving average models

An **ARMA(p,q) process** $\{X_t\}$ is a stationary process that satisfies

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q},$$

where $\{W_t\} \sim WN(0, \sigma^2)$.

- AR(p) $=$ ARMA(p,0): $\theta(B) = 1$.
- MA(q) $=$ ARMA(0,q): $\phi(B) = 1$.

ARMA(p,q): Autoregressive moving average models

An **ARMA(p,q) process** $\{X_t\}$ is a stationary process that satisfies

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q},$$

where $\{W_t\} \sim WN(0, \sigma^2)$.

Usually, we insist that $\phi_p, \theta_q \neq 0$ and that the polynomials

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p, \qquad \theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$$

have no common factors. This implies it is not a lower order ARMA model.

ARMA(p,q): An example of parameter redundancy

Consider a white noise process W_t. We can write

$$
\begin{aligned}
X_t &= W_t \\
\Rightarrow \qquad X_t - X_{t-1} + 0.25X_{t-2} &= W_t - W_{t-1} + 0.25W_{t-2} \\
(1 - B + 0.25B^2)X_t &= (1 - B + 0.25B^2)W_t
\end{aligned}
$$

This is in the form of an ARMA(2,2) process, with

$$\phi(B) = 1 - B + 0.25B^2, \qquad \theta(B) = 1 - B + 0.25B^2.$$

But it is white noise.

ARMA(p,q): An example of parameter redundancy

ARMA model: $\phi(B)X_t = \theta(B)W_t,$

with $\phi(B) = 1 - B + 0.25B^2,$

$\theta(B) = 1 - B + 0.25B^2$

$$X_t = \psi(B)W_t$$

$$\Leftrightarrow \qquad \psi(B) = \frac{\theta(B)}{\phi(B)} = 1.$$

i.e., $X_t = W_t$.

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Recall: Causality and Invertibility

A linear process $\{X_t\}$ is **causal** if there is a

$$\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots$$

$$\text{with} \qquad \sum_{j=0}^{\infty} |\psi_j| < \infty \qquad \text{and} \qquad X_t = \psi(B) W_t.$$

It is **invertible** if there is a

$$\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \cdots$$

$$\text{with} \qquad \sum_{j=0}^{\infty} |\pi_j| < \infty \qquad \text{and} \qquad W_t = \pi(B) X_t.$$

ARMA(p,q): Stationarity, causality, and invertibility

Theorem: If ϕ and θ have no common factors, a (unique) *stationary* solution to $\phi(B)X_t = \theta(B)W_t$ exists iff the roots of $\phi(z)$ *avoid* the unit circle:

$$|z| = 1 \Rightarrow \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0.$$

This ARMA(p,q) process is *causal* iff the roots of $\phi(z)$ are *outside* the unit circle:

$$|z| \leq 1 \Rightarrow \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0.$$

It is *invertible* iff the roots of $\theta(z)$ are *outside* the unit circle:

$$|z| \leq 1 \Rightarrow \theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0.$$

ARMA(p,q): Stationarity, causality, and invertibility

Example: $(1 - 1.5B)X_t = (1 + 0.2B)W_t.$

$$\phi(z) = 1 - 1.5z = -\frac{3}{2}\left(z - \frac{2}{3}\right),$$

$$\theta(z) = 1 + 0.2z = \frac{1}{5}\left(z + 5\right).$$

1. ϕ and θ have no common factors, and ϕ's root is at $2/3$, which is not on the unit circle, so $\{X_t\}$ is an ARMA(1,1) process.
2. ϕ's root (at $2/3$) is inside the unit circle, so $\{X_t\}$ is *not causal*.
3. θ's root is at -5, which is outside the unit circle, so $\{X_t\}$ is *invertible*.

ARMA(p,q): Stationarity, causality, and invertibility

Example: $(1 + 0.25B^2)X_t = (1 + 2B)W_t.$

$$\phi(z) = 1 + 0.25z^2 = \frac{1}{4}\left(z^2 + 4\right) = \frac{1}{4}(z + 2i)(z - 2i),$$

$$\theta(z) = 1 + 2z = 2\left(z + \frac{1}{2}\right).$$

1. ϕ and θ have no common factors, and ϕ's roots are at $\pm 2i$, which is not on the unit circle, so $\{X_t\}$ is an ARMA(2,1) process.

2. ϕ's roots (at $\pm 2i$) are outside the unit circle, so $\{X_t\}$ is *causal*.

3. θ's root (at $-1/2$) is inside the unit circle, so $\{X_t\}$ is *not invertible*.

Causality and Invertibility

Theorem: Let $\{X_t\}$ be an ARMA process defined by $\phi(B)X_t = \theta(B)W_t$. If all $|z| = 1$ have $\theta(z) \neq 0$, then there are polynomials $\tilde{\phi}$ and $\tilde{\theta}$ and a white noise sequence $\tilde{W}_t$ such that $\{X_t\}$ satisfies $\tilde{\phi}(B)X_t = \tilde{\theta}(B)\tilde{W}_t$, and this is a causal, invertible ARMA process.

So we'll stick to causal, invertible ARMA processes.

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Calculating ψ for an ARMA(p,q): matching coefficients

Example: $X_t = \psi(B)W_t \quad \Leftrightarrow \quad (1 + 0.25B^2)X_t = (1 + 0.2B)W_t$

so $\quad 1 + 0.2B = (1 + 0.25B^2)\psi(B)$

$$\Leftrightarrow \quad 1 + 0.2B = (1 + 0.25B^2)(\psi_0 + \psi_1 B + \psi_2 B^2 + \cdots)$$

$$\begin{aligned} \Leftrightarrow \quad 1 &= \psi_0, \\ 0.2 &= \psi_1, \\ 0 &= \psi_2 + 0.25\psi_0, \\ 0 &= \psi_3 + 0.25\psi_1, \\ &\vdots \end{aligned}$$

Calculating ψ for an ARMA(p,q): example

$$\Leftrightarrow \qquad 1 = \psi_0, \qquad 0.2 = \psi_1,$$
$$0 = \psi_j + 0.25\psi_{j-2} \qquad (j \geq 2).$$

We can think of this as $\theta_j = \phi(B)\psi_j$, with $\theta_0 = 1$, $\theta_j = 0$ for $j < 0$, $j > q$.

This is a *first order difference equation* in the ψ_js.
We can use the θ_js to give the initial conditions and solve it using the theory of homogeneous difference equations.

$$\psi_j = \left(1, \frac{1}{5}, -\frac{1}{4}, -\frac{1}{20}, \frac{1}{16}, \frac{1}{80}, -\frac{1}{64}, -\frac{1}{320}, \ldots\right).$$

Calculating ψ for an ARMA(p,q): general case

$$\phi(B)X_t = \theta(B)W_t, \qquad \Leftrightarrow \qquad X_t = \psi(B)W_t$$

$$\text{so} \qquad \theta(B) = \psi(B)\phi(B)$$

$$\Leftrightarrow \; 1 + \theta_1 B + \cdots + \theta_q B^q = (\psi_0 + \psi_1 B + \cdots)(1 - \phi_1 B - \cdots - \phi_p B^p)$$

$$\begin{aligned} \Leftrightarrow \qquad 1 &= \psi_0, \\ \theta_1 &= \psi_1 - \phi_1\psi_0, \\ \theta_2 &= \psi_2 - \phi_1\psi_1 - \cdots - \phi_2\psi_0, \\ &\vdots \end{aligned}$$

This is equivalent to $\theta_j = \phi(B)\psi_j$, with $\theta_0 = 1$, $\theta_j = 0$ for $j < 0$, $j > q$.

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Autocovariance functions of linear processes

Consider a (mean 0) linear process $\{X_t\}$ defined by $X_t = \psi(B)W_t$.

$$\begin{aligned}
\gamma(h) &= \mathrm{E}\left(X_t X_{t+h}\right) \\
&= \mathrm{E}\left(\psi_0 W_t + \psi_1 W_{t-1} + \psi_2 W_{t-2} + \cdots\right) \\
&\qquad \times \left(\psi_0 W_{t+h} + \psi_1 W_{t+h-1} + \psi_2 W_{t+h-2} + \cdots\right) \\
&= \sigma_w^2 \left(\psi_0 \psi_h + \psi_1 \psi_{h+1} + \psi_2 \psi_{h+2} + \cdots\right).
\end{aligned}$$

Autocovariance functions of MA processes

Consider an MA(q) process $\{X_t\}$ defined by $X_t = \theta(B)W_t$.

$$\gamma(h) = \begin{cases} \sigma_w^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h} & \text{if } h \leq q, \\ 0 & \text{if } h > q. \end{cases}$$

Autocovariance functions of ARMA processes

ARMA process: $\phi(B)X_t = \theta(B)W_t$.

To compute γ, we can compute ψ, and then use

$$\gamma(h) = \sigma_w^2 \left(\psi_0\psi_h + \psi_1\psi_{h+1} + \psi_2\psi_{h+2} + \cdots\right).$$

Autocovariance functions of ARMA processes

An alternative approach:

$$\begin{aligned} X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} \\ = W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q}, \end{aligned}$$

$$\begin{aligned} \text{so } \mathrm{E}\left(\left(X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p}\right) X_{t-h}\right) \\ = \mathrm{E}\left(\left(W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q}\right) X_{t-h}\right), \end{aligned}$$

$$\begin{aligned} \text{that is, } \gamma(h) - \phi_1 \gamma(h-1) - \cdots - \phi_p \gamma(h-p) \\ = \mathrm{E}\left(\theta_h W_{t-h} X_{t-h} + \cdots + \theta_q W_{t-q} X_{t-h}\right) \\ = \sigma_w^2 \sum_{j=0}^{q-h} \theta_{h+j} \psi_j. \qquad \text{(Write } \theta_0 = 1\text{).} \end{aligned}$$

This is a linear difference equation.

Autocovariance functions of ARMA processes: Example

$$(1+0.25B^2)X_t = (1+0.2B)W_t, \qquad \Leftrightarrow \qquad X_t = \psi(B)W_t,$$

$$\psi_j = \left(1, \frac{1}{5}, -\frac{1}{4}, -\frac{1}{20}, \frac{1}{16}, \frac{1}{80}, -\frac{1}{64}, -\frac{1}{320}, \ldots\right).$$

$$\gamma(h) - \phi_1\gamma(h-1) - \phi_2\gamma(h-2) = \sigma_w^2 \sum_{j=0}^{q-h} \theta_{h+j}\psi_j$$

$$\Leftrightarrow \gamma(h) + 0.25\gamma(h-2) = \begin{cases} \sigma_w^2\left(\psi_0 + 0.2\psi_1\right) & \text{if } h=0, \\ 0.2\sigma_w^2\psi_0 & \text{if } h=1, \\ 0 & \text{otherwise.} \end{cases}$$

Autocovariance functions of ARMA processes: Example

We have the homogeneous linear difference equation

$$\gamma(h) + 0.25\gamma(h-2) = 0$$

for $h \geq 2$, with initial conditions

$$\gamma(0) + 0.25\gamma(-2) = \sigma_w^2 \left(1 + 1/25\right)$$
$$\gamma(1) + 0.25\gamma(-1) = \sigma_w^2/5.$$

We can solve these linear equations to determine γ.

Or we can use the theory of linear difference equations...

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Difference equations

Examples:

$$x_t - 3x_{t-1} = 0 \qquad \text{(first order, linear)}$$

$$x_t - x_{t-1}x_{t-2} = 0 \qquad \text{(2nd order, nonlinear)}$$

$$x_t + 2x_{t-1} - x_{t-3}^2 = 0 \qquad \text{(3rd order, nonlinear)}$$

Homogeneous linear diff eqns with constant coefficients

$$
\begin{aligned}
a_0 x_t + a_1 x_{t-1} + \cdots + a_k x_{t-k} &= 0 \\
\Leftrightarrow \qquad \left(a_0 + a_1 B + \cdots + a_k B^k\right) x_t &= 0 \\
\Leftrightarrow \qquad a(B) x_t &= 0 \\
\text{auxiliary equation:} \qquad a_0 + a_1 z + \cdots + a_k z^k &= 0 \\
\Leftrightarrow \qquad (z - z_1)(z - z_2) \cdots (z - z_k) &= 0
\end{aligned}
$$

where $z_1, z_2, \ldots, z_k \in \mathbb{C}$ are the roots of this *characteristic polynomial*.

Thus,

$$
a(B) x_t = 0 \qquad \Leftrightarrow \qquad (B - z_1)(B - z_2) \cdots (B - z_k) x_t = 0.
$$

Homogeneous linear diff eqns with constant coefficients

$$a(B)x_t = 0 \qquad \Leftrightarrow \qquad (B - z_1)(B - z_2)\cdots(B - z_k)x_t = 0.$$

So any $\{x_t\}$ satisfying $(B - z_i)x_t = 0$ for some i also satisfies $a(B)x_t = 0$.

Three cases:

1. The z_i are real and distinct.
2. The z_i are complex and distinct.
3. Some z_i are repeated.

Homogeneous linear diff eqns with constant coefficients

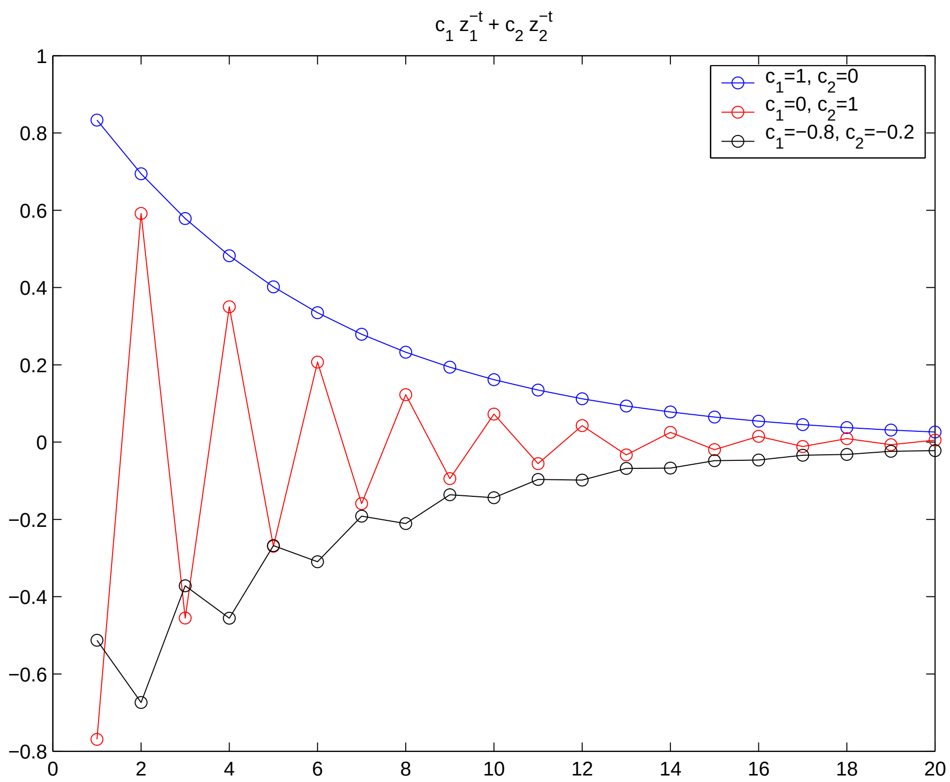
1. The z_i are real and distinct.

$$
\begin{aligned}
& & a(B)x_t &= 0 \\
&\Leftrightarrow & (B - z_1)(B - z_2)\cdots(B - z_k)x_t &= 0 \\
&\Leftrightarrow & x_t \text{ is a linear combination of solutions to} & \\
& & (B - z_1)x_t = 0,\ (B - z_2)x_t = 0, \ldots,\ (B - z_k)x_t &= 0 \\
&\Leftrightarrow & x_t = c_1 z_1^{-t} + c_2 z_2^{-t} + \cdots + c_k z_k^{-t}&,
\end{aligned}
$$

for some constants $c_1, \ldots, c_k$.

Homogeneous linear diff eqns with constant coefficients

1. The z_i are real and distinct. e.g., $z_1 = 1.2$, $z_2 = -1.3$



Reminder: Complex exponentials

$$a + ib = re^{i\theta} = r(\cos\theta + i\sin\theta),$$

$$\text{where} \quad r = |a + ib| = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \in (-\pi, \pi].$$

Thus,

$$r_1 e^{i\theta_1} r_2 e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1 + \theta_2)},$$

$$z\bar{z} = |z|^2.$$

Homogeneous linear diff eqns with constant coefficients

2. The z_i are complex and distinct.

As before, $$a(B)x_t = 0$$
$$\Leftrightarrow \qquad x_t = c_1 z_1^{-t} + c_2 z_2^{-t} + \cdots + c_k z_k^{-t}.$$

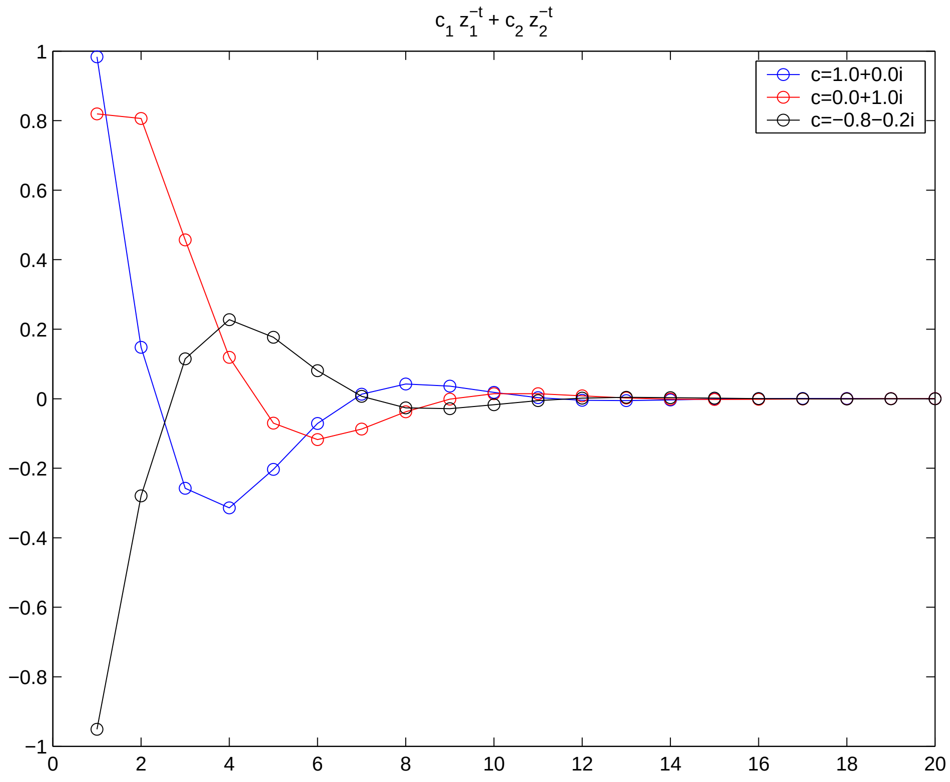
If $z_1 \notin \mathbb{R}$, since $a_1, \ldots, a_k$ are real, we must have the complex conjugate root, $z_j = \bar{z_1}$. And for x_t to be real, we must have $c_j = \bar{c_1}$. For example:

$$\begin{aligned} x_t &= c\, z_1^{-t} + \bar{c}\, \bar{z_1}^{-t} \\ &= r\, e^{i\theta}|z_1|^{-t}e^{-i\omega t} + r\, e^{-i\theta}|z_1|^{-t}e^{i\omega t} \\ &= r|z_1|^{-t}\left(e^{i(\theta-\omega t)} + e^{-i(\theta-\omega t)}\right) \\ &= 2r|z_1|^{-t}\cos(\omega t - \theta) \end{aligned}$$

where $z_1 = |z_1|e^{i\omega}$ and $c = re^{i\theta}$.

Homogeneous linear diff eqns with constant coefficients

2. The z_i are complex and distinct. e.g., $z_1 = 1.2 + i$, $z_2 = 1.2 - i$



Homogeneous linear diff eqns with constant coefficients

2. The z_i are complex and distinct. e.g., $z_1 = 1 + 0.1i$, $z_2 = 1 - 0.1i$