Introduction to Time Series Analysis. Lecture 19.

1. Review: Spectral density estimation, sample autocovariance.
2. The periodogram and sample autocovariance.
3. Asymptotics of the periodogram.

Estimating the Spectrum: Outline

- We have seen that the spectral density gives an alternative view of stationary time series.
- Given a realization $x_1, \ldots, x_n$ of a time series, how can we estimate the spectral density?
- One approach: replace $\gamma(\cdot)$ in the definition

$$f(\nu) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h},$$

  with the sample autocovariance $\hat{\gamma}(\cdot)$.
- Another approach, called the *periodogram*: compute $I(\nu)$, the squared modulus of the discrete Fourier transform (at frequencies $\nu = k/n$).

Estimating the spectrum: Outline

- These two approaches are *identical* at the Fourier frequencies $\nu = k/n$.
- The asymptotic expectation of the periodogram $I(\nu)$ is $f(\nu)$. We can derive some asymptotic properties, and hence do hypothesis testing.
- Unfortunately, the asymptotic variance of $I(\nu)$ is constant.
It is not a consistent estimator of $f(\nu)$.

Review: Spectral density estimation

If a time series $\{X_t\}$ has autocovariance γ satisfying $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, then we define its **spectral density** as

$$f(\nu) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h}$$

for $-\infty < \nu < \infty$.

Review: Sample autocovariance

Idea: use the sample autocovariance $\hat{\gamma}(\cdot)$, defined by

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-|h|} (x_{t+|h|} - \bar{x})(x_t - \bar{x}), \qquad \text{for } -n < h < n,$$

as an estimate of the autocovariance $\gamma(\cdot)$, and then use

$$\hat{f}(\nu) = \sum_{h=-n+1}^{n-1} \hat{\gamma}(h) e^{-2\pi i \nu h}$$

for $-1/2 \leq \nu \leq 1/2$.

Discrete Fourier transform

For a sequence $(x_1, \ldots, x_n)$, define the *discrete Fourier transform (DFT)* as $(X(\nu_0), X(\nu_1), \ldots, X(\nu_{n-1}))$, where

$$X(\nu_k) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} x_t e^{-2\pi i \nu_k t},$$

and $\nu_k = k/n$ (for $k = 0, 1, \ldots, n-1$) are called the *Fourier frequencies*. (Think of $\{\nu_k : k = 0, \ldots, n-1\}$ as the discrete version of the frequency range $\nu \in [0, 1]$.)

First, let's show that we can view the DFT as a representation of x in a different basis, the *Fourier basis*.

Discrete Fourier transform

Consider the space $\mathbb{C}^n$ of vectors of n complex numbers, with inner product $\langle a, b\rangle = a^* b$, where a^* is the complex conjugate transpose of the vector $a \in \mathbb{C}^n$.

Suppose that a set $\{\phi_j : j = 0, 1, \ldots, n-1\}$ of n vectors in $\mathbb{C}^n$ are orthonormal:

$$\langle \phi_j, \phi_k \rangle = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Then these $\{\phi_j\}$ span the vector space $\mathbb{C}^n$, and so for any vector x, we can write x in terms of this new orthonormal basis,

$$x = \sum_{j=0}^{n-1} \langle \phi_j, x \rangle \phi_j .$$

(picture)

Discrete Fourier transform

Consider the following set of n vectors in $\mathbb{C}^n$:

$$\left\{ e_j = \frac{1}{\sqrt{n}} \left(e^{2\pi i \nu_j}, e^{2\pi i 2\nu_j}, \ldots, e^{2\pi i n \nu_j} \right)' : j = 0, \ldots, n-1 \right\}.$$

It is easy to check that these vectors are orthonormal:

$$\begin{aligned}
\langle e_j, e_k \rangle &= \frac{1}{n} \sum_{t=1}^{n} e^{2\pi i t(\nu_k - \nu_j)} = \frac{1}{n} \sum_{t=1}^{n} \left(e^{2\pi i (k-j)/n} \right)^t \\
&= \begin{cases} 1 & \text{if } j = k, \\ \frac{1}{n} e^{2\pi i (k-j)/n} \frac{1 - (e^{2\pi i (k-j)/n})^n}{1 - e^{2\pi i (k-j)/n}} & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

Discrete Fourier transform

where we have used the fact that $S_n = \sum_{t=1}^{n} \alpha^t$ satisfies $\alpha S_n = S_n + \alpha^{n+1} - \alpha$ and so $S_n = \alpha(1 - \alpha^n)/(1 - \alpha)$ for $\alpha \neq 1$.

So we can represent the real vector $x = (x_1, \ldots, x_n)' \in \mathbb{C}^n$ in terms of this orthonormal basis,

$$x = \sum_{j=0}^{n-1} \langle e_j, x \rangle e_j = \sum_{j=0}^{n-1} X(\nu_j) e_j.$$

That is, the vector of discrete Fourier transform coefficients $(X(\nu_0), \ldots, X(\nu_{n-1}))$ is the representation of x in the Fourier basis.

Discrete Fourier transform

An alternative way to represent the DFT is by separately considering the real and imaginary parts,

$$
\begin{aligned}
X(\nu_j) = \langle e_j, x\rangle &= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e^{-2\pi i t \nu_j} x_t \\
&= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \cos(2\pi t \nu_j) x_t - i \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \sin(2\pi t \nu_j) x_t \\
&= X_c(\nu_j) - i X_s(\nu_j),
\end{aligned}
$$

where this defines the sine and cosine transforms, X_s and X_c, of x.

Periodogram

The periodogram is defined as

$$
\begin{aligned}
I(\nu_j) &= |X(\nu_j)|^2 \\
&= \frac{1}{n}\left|\sum_{t=1}^{n} e^{-2\pi i t\nu_j} x_t\right|^2 \\
&= X_c^2(\nu_j) + X_s^2(\nu_j).
\end{aligned}
$$

$$
X_c(\nu_j) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\cos(2\pi t\nu_j)x_t,
$$

$$
X_s(\nu_j) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sin(2\pi t\nu_j)x_t.
$$

Periodogram

Since $I(\nu_j) = |X(\nu_j)|^2$ for one of the Fourier frequencies $\nu_j = j/n$ (for $j = 0, 1, \ldots, n-1$), the orthonormality of the e_j implies that we can write

$$x^* x = \left(\sum_{j=0}^{n-1} X(\nu_j) e_j \right)^* \left(\sum_{j=0}^{n-1} X(\nu_j) e_j \right)$$

$$= \sum_{j=0}^{n-1} |X(\nu_j)|^2 = \sum_{j=0}^{n-1} I(\nu_j).$$

For $\bar{x} = 0$, we can write this as

$$\hat{\sigma}_x^2 = \frac{1}{n} \sum_{t=1}^{n} x_t^2 = \frac{1}{n} \sum_{j=0}^{n-1} I(\nu_j).$$

Periodogram

This is the discrete analog of the identity

$$\sigma_x^2 = \gamma_x(0) = \int_{-1/2}^{1/2} f_x(\nu)\, d\nu.$$

(Think of $I(\nu_j)$ as the discrete version of $f(\nu)$ at the frequency $\nu_j = j/n$, and think of $(1/n)\sum_{\nu_j} \cdot$ as the discrete version of $\int_{\nu} \cdot d\nu$.)

Estimating the spectrum: Periodogram

Why is the periodogram at a Fourier frequency (that is, $\nu = \nu_j$) the same as computing $f(\nu)$ from the sample autocovariance?

Almost the same—they are not the same at $\nu_0 = 0$ when $\bar{x} \neq 0$.

But if either $\bar{x} = 0$, or we consider a Fourier frequency ν_j with $j \in \{1, \ldots, n-1\}, \ldots$

Estimating the spectrum: Periodogram

$$
\begin{aligned}
I(\nu_j) &= \frac{1}{n}\left|\sum_{t=1}^{n} e^{-2\pi i t\nu_j} x_t\right|^2 = \frac{1}{n}\left|\sum_{t=1}^{n} e^{-2\pi i t\nu_j}(x_t-\bar{x})\right|^2 \\
&= \frac{1}{n}\left(\sum_{t=1}^{n} e^{-2\pi i t\nu_j}(x_t-\bar{x})\right)\left(\sum_{t=1}^{n} e^{2\pi i t\nu_j}(x_t-\bar{x})\right) \\
&= \frac{1}{n}\sum_{s,t} e^{-2\pi i(s-t)\nu_j}(x_s-\bar{x})(x_t-\bar{x}) = \sum_{h=-n+1}^{n-1} \hat{\gamma}(h)e^{-2\pi i h\nu_j},
\end{aligned}
$$

where the fact that $\nu_j \neq 0$ implies $\sum_{t=1}^{n} e^{-2\pi i t\nu_j} = 0$ (we showed this when we were verifying the orthonormality of the Fourier basis) has allowed us to subtract the sample mean in that case.

Asymptotic properties of the periodogram

We want to understand the asymptotic behavior of the periodogram $I(\nu)$ at a particular frequency ν, as n increases. We'll see that its expectation converges to $f(\nu)$.

We'll start with a simple example: Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(0, \sigma^2)$ (Gaussian white noise). From the definitions,

$$X_c(\nu_j) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \cos(2\pi t \nu_j) x_t, \qquad X_s(\nu_j) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \sin(2\pi t \nu_j) x_t,$$

we have that $X_c(\nu_j)$ and $X_s(\nu_j)$ are normal, with

$$\mathbf{E} X_c(\nu_j) = \mathbf{E} X_s(\nu_j) = 0.$$

Asymptotic properties of the periodogram

Also,

$$\begin{aligned}\mathrm{Var}(X_c(\nu_j)) &= \frac{\sigma^2}{n}\sum_{t=1}^{n}\cos^2(2\pi t\nu_j) \\ &= \frac{\sigma^2}{2n}\sum_{t=1}^{n}(\cos(4\pi t\nu_j)+1) = \frac{\sigma^2}{2}.\end{aligned}$$

Similarly, $\mathrm{Var}(X_s(\nu_j)) = \sigma^2/2$.

Asymptotic properties of the periodogram

Also,

$$\begin{aligned}\mathrm{Cov}(X_c(\nu_j), X_s(\nu_j)) &= \frac{\sigma^2}{n}\sum_{t=1}^{n}\cos(2\pi t\nu_j)\sin(2\pi t\nu_j)\\ &= \frac{\sigma^2}{2n}\sum_{t=1}^{n}\sin(4\pi t\nu_j) = 0,\end{aligned}$$

$$\mathrm{Cov}(X_c(\nu_j), X_c(\nu_k)) = 0$$

$$\mathrm{Cov}(X_s(\nu_j), X_s(\nu_k)) = 0$$

$$\mathrm{Cov}(X_c(\nu_j), X_s(\nu_k)) = 0.$$

for any $j \neq k$.

Asymptotic properties of the periodogram

That is, if $X_1, \ldots, X_n$ are i.i.d. $N(0, \sigma^2)$ (Gaussian white noise; $f(\nu) = \sigma^2$), then the $X_c(\nu_j)$ and $X_s(\nu_j)$ are all i.i.d. $N(0, \sigma^2/2)$. Thus,

$$\frac{2}{\sigma^2} I(\nu_j) = \frac{2}{\sigma^2} \left(X_c^2(\nu_j) + X_s^2(\nu_j) \right) \sim \chi_2^2.$$

So for the case of Gaussian white noise, the periodogram has a chi-squared distribution that depends on the variance σ^2 (which, in this case, is the spectral density).

Asymptotic properties of the periodogram

Under more general conditions (e.g., normal $\{X_t\}$, or linear process $\{X_t\}$ with rapidly decaying ACF), the $X_c(\nu_j)$, $X_s(\nu_j)$ are all asymptotically independent and $N(0, f(\nu_j)/2)$.

Consider a frequency ν. For a given value of n, let $\hat{\nu}^{(n)}$ be the closest Fourier frequency (that is, $\hat{\nu}^{(n)} = j/n$ for a value of j that minimizes $|\nu - j/n|$). As n increases, $\hat{\nu}^{(n)} \to \nu$, and (under the same conditions that ensure the asymptotic normality and independence of the sine/cosine transforms), $f(\hat{\nu}^{(n)}) \to f(\nu)$. (picture)

In that case, we have

$$\frac{2}{f(\nu)} I(\hat{\nu}^{(n)}) = \frac{2}{f(\nu)} \left(X_c^2(\hat{\nu}^{(n)}) + X_s^2(\hat{\nu}^{(n)}) \right) \xrightarrow{\text{d}} \chi_2^2.$$

Asymptotic properties of the periodogram

Thus,

$$\mathbf{E}I(\hat{\nu}^{(n)}) = \frac{f(\nu)}{2}\mathbf{E}\left(\frac{2}{f(\nu)}\left(X_c^2(\hat{\nu}^{(n)}) + X_s^2(\hat{\nu}^{(n)})\right)\right)$$
$$\to \frac{f(\nu)}{2}\mathbf{E}(Z_1^2 + Z_2^2) = f(\nu),$$

where Z_1, Z_2 are independent $N(0,1)$. Thus, the periodogram is asymptotically unbiased.

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