1.12 By definition,

$$\rho_{xy}(h) = \frac{\gamma_{xy}(h)}{\sqrt{\gamma_x(0)\gamma_y(0)}}$$

and

$$\rho_{yx}(-h) = \frac{\gamma_{yx}(-h)}{\sqrt{\gamma_y(0)\gamma_x(0)}}.$$

To show that these two expressions are the same, it is enough to show that $\gamma_{xy}(h) = \gamma_{yx}(-h)$. But for any t and h,

$$\gamma_{xy}(h) = \mathbb{E}(X_{t+h} - \mu_x)(Y_t - \mu_y) = \mathbb{E}(X_s - \mu_x)(Y_{s-h} - \mu_y) = \gamma_{yx}(-h)$$

with the change of variables $s = t + h$.

1.13 (a) X_t is just a white noise, so $\rho_X(0) = 1$ and $\rho(h) = 0$ for $h \neq 0$. For Y_t, we note that the mean is zero and so

$$\begin{aligned}
\gamma_Y(h) &= \mathbb{E}(Y_{t+h}Y_t) \\
&= \mathbb{E}(W_{t+h} - \theta W_{t+h-1} + U_{t+h})(W_t - \theta W_{t-1} + U_t) \\
&= \mathbb{E}W_{t+h}W_t - \theta\mathbb{E}W_{t+h-1}W_t - \theta\mathbb{E}W_{t+h}W_{t-1} + \theta^2\mathbb{E}W_{t+h-1}W_{t-1} + \mathbb{E}U_{t+h}U_t \\
&= \begin{cases} \sigma_W^2(1+\theta^2) + \sigma_U^2 & \text{if } h = 0 \\ -\theta\sigma_W^2 & \text{if } |h| = 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Therefore,

$$\rho_Y(h) = \begin{cases} 1 & \text{if } h = 0 \\ -\frac{\theta\sigma_W^2}{(1+\theta^2)\sigma_W^2 + \sigma_U^2} & \text{if } |h| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) We begin by computing the cross-covariance function:

$$\begin{aligned}
\gamma_{XY}(h) &= \mathbb{E}X_{t+h}Y_t \\
&= \mathbb{E}W_{t+h}(W_t - \theta W_{t-1} + U_t) \\
&= \begin{cases} \sigma_W^2 & \text{if } h = 0 \\ -\theta\sigma_W^2 & \text{if } h = -1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Then the CCF is

$$\rho_{XY}(h) = \begin{cases} \frac{\sigma_W}{\sqrt{\sigma_W^2(1+\theta^2)+\sigma_U^2}} & \text{if } h = 0 \\ -\frac{\theta\sigma_W}{\sqrt{\sigma_W^2(1+\theta^2)+\sigma_U^2}} & \text{if } h = -1 \\ 0 & \text{otherwise.} \end{cases}$$

(c) In part (b), we saw that $\mathbb{E}X_{t+h}Y_t$ does not depend on t. Since X and Y are both stationary, it follows that they are jointly stationary.

4.16 (a) Consider the cross-covariance:

$$\begin{aligned}
\mathbb{E}X_{t+h}Y_t &= \frac{1}{2}\mathbb{E}(W_{t+h} - W_{t+h-1})(W_t + W_{t-1}) \\
&= \frac{1}{2}\left(\mathbb{E}W_{t+h}W_t + \mathbb{E}W_{t+h}W_{t-1} - \mathbb{E}W_tW_{t+h-1} - \mathbb{E}W_{t+h}W_t\right) \\
&= \begin{cases} 0 & \text{if } h = 0 \\ -\frac{1}{2} & \text{if } h = 1 \\ \frac{1}{2} & \text{if } h = -1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Since this doesn't depend on t, X and Y are jointly stationary.

(b) Using the formula for the spectrum of a moving average process,

$$f_X(\omega) = 2 - 2\cos(2\pi\omega)$$

and

$$f_Y(\omega) = \frac{1}{2} + \frac{1}{2}\cos(2\pi\omega).$$

These two spectral densities differ by a factor of 4 and the fact that one is a translation of the other. In particular, f_X is at its largest at $\omega = \pm\frac{1}{2}$ and so X has oscillatory behavior at a period of 2, whereas f_Y is at its largest at $\omega = 0$ and so it does not exhibit oscillatory behavior.

(c) The true value of the spectral density f_Y at 0.1 is $\frac{1}{2} + \frac{1}{2}\cos(\pi/5) \approx 0.90$. By (4.48) in the text, the estimate $\bar{f}_Y$ is distributed as $\frac{0.90}{6}\chi^2_6$. The 95% and 5% quantiles for a χ^2_6 variable are about 12.59 and 1.64; hence the 95% and 5% quantiles for $\bar{f}_Y(0.1)$ are about 0.25 and 1.90:

$$P(0.25 \leq \bar{f}_Y(0.1) \leq 1.90) \approx 0.9$$

and 5% of the area is in each tail.

4.18a Since X is a white noise, its spectral density is the constant function $f_X(\nu) = \sigma^2$. Since X is independent of V, Y is also a white noise and its variance is $\sigma^2(1 + \phi^2)$, so its spectral density is $f_Y(\nu) = \sigma^2(1 + \phi^2)$. The cross-covariance is

$$\begin{aligned}
\gamma_{XY}(h) &= \mathbb{E}X_{t+h}Y_t \\
&= \mathbb{E}W_{t+h}(W_{t-D} + V_t) \\
&= \begin{cases} \sigma^2 & \text{if } h = -D \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

and so the cross-spectrum is $f_{XY}(\nu) = \sigma^2 e^{2\pi i D\nu}$. The coherence, therefore, is

$$\rho_{X \cdot Y}(\nu) = \frac{|\sigma^2 e^{2\pi i D\nu}|^2}{\sigma^4(1+\phi^2)} = \frac{1}{1+\phi^2}.$$