Homework 2 solutions

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1. (a) We compute three cases: since the W_t are uncorrelated, we can ignore any cross-terms of the form $\mathbb{E}W_sW_t$ when $s \neq t$. Then

$$\gamma(0) = \mathbb{E}W_t^2 + \frac{25}{4}\mathbb{E}W_{t-1}^2 + \frac{9}{4}\mathbb{E}W_{t-2}^2 = \frac{19}{2}$$
$$\gamma(1) = \frac{5}{2}\mathbb{E}W_t^2 + -\frac{15}{4}\mathbb{E}W_{t-1}^2 = -\frac{5}{4}$$
$$\gamma(2) = -\frac{3}{2}\mathbb{E}W_t^2 = -\frac{3}{2}.$$

For $h \geq 3$, $\gamma(h) = 0$.

(b) Similarly to the previous part,

$$\gamma(0) = \mathbb{E}\widetilde{W}_t^2 + \frac{1}{36}\mathbb{E}\widetilde{W}_{t-1}^2 + \frac{1}{36}\mathbb{E}\widetilde{W}_{t-2}^2 = \frac{19}{2}.$$
$$\gamma(1) = -\frac{1}{6}\mathbb{E}\widetilde{W}_t^2 + \frac{1}{36}\mathbb{E}\widetilde{W}_{t-1}^2 = -\frac{5}{4}$$
$$\gamma(2) = -\frac{1}{6}\mathbb{E}\widetilde{W}_t^2 = -\frac{3}{2}.$$

For $h \geq 3$, $\gamma(h) = 0$. This is exactly the same covariance function as in part 1(a).

(c) Let $\theta_a(z) = 1 + \frac{5}{2}z - \frac{3}{2}z^2$ and $\theta_b(z) = 1 - \frac{1}{6}z - \frac{1}{6}z^2$ be the MA polynomials of parts (a) and (b) respectively. By the quadratic formula, the roots of θ_a are 2 and $-1/3$; similarly, the roots of θ_b are -3 and 2. By Proposition P3.2 in the text, the MA model of part (a) is not invertible, but the MA model of part (b) is invertible.

2. (a) The AR polynomial is $\phi(z) = 1+0.81z^2$, which has roots $z = \pm 10i/9$. The MA polynomial is $\theta(z) = 1 + z/3$, which has root $z = -3$. Thus, this is an ARMA$(2, 1)$ process which is causal and invertible.

(b) The AR polynomial is $\phi(z) = 1 - z$, which has root 1. The MA polynomial is $\theta(z) = 1 - z/2 - z^2/2$, which has roots -2 and 1. Since these polynomials share a common root, they have the common factor $1 - z$. Factoring these out, the irredundant representation has AR polynomial $\phi(z) = 1$ (which has no roots) and MA polynomial

$\theta(z) = 1 + z/2$ (which has root -2). Thus, this is an ARMA$(0, 1)$ process (or, in other words, an MA(1) process) which is causal and invertible.

(c) The AR polynomial is $\phi(z) = 1 - 3z$, which has root $1/3$. The MA polynomial is $\theta(z) = 1 + 2z - 8z^2$, which has roots $-1/4$ and $1/2$. Thus, this is an ARMA$(1, 2)$ process which is neither causal nor invertible.

(d) The AR polynomial is $\phi(z) = 1 - 2z + 2z^2$, which has roots $1/2 \pm i/2$. The MA polynomial is $\theta(z) = 1 - 8z/9$, which has root $9/8$. Thus, this is an ARMA$(2, 1)$ process which is invertible but not causal.

(e) The AR polynomial is $\phi(z) = 1 - 4z^2$, which has roots $\pm 1/2$. The MA polynomial is $\theta(z) = 1 - z + z^2/2$, which has roots $1 \pm i$. Thus, this is an ARMA$(2, 2)$ process which is invertible, but not causal.

(f) The AR polynomial is $\phi(z) = 1 - 9z/4 - 9z^2/4$, which has roots $1/3$ and $-4/3$. The MA polynomial is $\theta(z) = 1$, which has no roots. Thus, this is an ARMA$(2, 0)$ process (an AR(2) process) which is invertible, but not causal.

(g) The AR polynomial is $\phi(z) = 1 - 9z/2 - 9z^2/4$, which has roots $1/3$ and $-4/3$. The MA polynomial is $\theta(z) = 1 - 3z + z^2/9 - z^3/3$, which has roots $1/3$ and $\pm 3i$. As in part (b), we can factorize out the common factor of $1 - 3z$, to obtain the irredundant form $\phi(z) = 1 + 3z/4$ and $\theta(z) = 1 + z^2/9$. Thus, this is an ARMA$(1, 2)$ process which is causal and invertible.

3. Parts (a), (b) and (g) from question 2 are causal:

(a) The power series ψ is given by the expansion of

$$\begin{aligned}\frac{\theta(z)}{\phi(z)} &= \frac{1 + z/3}{1 + 81z^2/100} \\ &= (1 + z/3)(1 - \frac{81}{100}z^2 + \frac{81^2}{100^2}z^4 - \dots) \\ &= 1 + z/3 - \frac{81}{100}z^2 - \frac{27}{100}z^3 + \frac{6561}{10000}z^4 + \dots\end{aligned}$$

(b) The power series ψ is given by the expansion of

$$\begin{aligned}\frac{\theta(z)}{\phi(z)} &= \frac{1 - z/2 - z^2/2}{1} \\ &= 1 - \frac{1}{2}z - \frac{1}{2}z^2 + 0z^3 + 0z^4.\end{aligned}$$

(g) The power series ψ is given by the expansion of

$$\begin{aligned}\frac{\theta(z)}{\phi(z)} &= \frac{1+z^2/9}{1+3z/4} \\ &= (1+z^2/9)(1-3z/4+9z^2/16-27z^3/64+81z^4/256+\dots) \\ &= 1-\frac{3}{4}z+\left(\frac{1}{9}+\frac{9}{16}\right)z^2-\left(\frac{1}{12}+\frac{27}{64}\right)z^3+\frac{97}{256}z^4+\dots\end{aligned}$$

4. The simulation code for all three parts is as follows:

```
simAndPlot <- function(ar, ma, file) {
  p <- list()
  p[["ar"]] <- ar
  p[["ma"]] <- ma
  x <- arima.sim(p, 100)
  true_acf <- ARMAacf(ar=ar, ma=ma, 20)

  postscript(file=file)
  par(mfcol=c(3,1))
  plot(x)
  a <- acf(x, ylab="Sample ACF")
  a$acf <- array(true_acf, dim=c(21, 1, 1))
  plot(a)
  dev.off()
}

simAndPlot(c(0, -0.81), 1/3, "stat_153_solutions2_4a.eps")
simAndPlot(0, c(-1/2, -1/2), "stat_153_solutions2_4b.eps")
simAndPlot(-3/4, c(0, 1/9), "stat_153_solutions2_4g.eps")
```

(a) The recurrence relation for the autocorrelation function is

$$\gamma(h)+\frac{81}{100}\gamma(h-2)=0 \tag{1}$$

for $h \geq 2$. There are two ways to solve this. The easier way is to notice that this decomposes into two first-order recurrence relations: one for $\gamma(0), \gamma(2), \gamma(4), \dots$ and one for $\gamma(1), \gamma(3), \gamma(5), \dots$. However, let's follow the general procedure for solving recurrence relations: the characteristic polynomial is $r^2+\frac{81}{100}=0$ and its roots are $9i/10$ and $-9i/10$ (in the notation of the lecture slides, $z_1^{-1}=9i/10$ and $z_2^{-1}=-9i/10$). Therefore, the general solution has the form

$$\begin{aligned}\gamma(h) &= C((9i/10)^t+(-9i/10)^t) \\ &= r\left(\frac{9}{10}\right)^t(e^{-i\omega t}+e^{i\omega t}) \\ &= 2r\left(\frac{9}{10}\right)^t\cos(\omega t-\theta)\end{aligned}$$

where ω is the argument of $9i/10$ (which is $\pi/2$), $r = |C|$ and θ is the argument of C.

The initial conditions for the recurrence relation are

$$\gamma(0) + \frac{81}{100}\gamma(2) = \frac{10}{9}$$
$$\gamma(1) + \frac{81}{100}\gamma(1) = \frac{1}{3}.$$

We can solve the first of these simultaneously with (1) (for $h = 2$) to obtain $\gamma(0) = \frac{100000}{30951}$; we can solve the second directly to obtain $\gamma(1) = \frac{100}{543}$.

Now we need to use these to find r and θ in the general solution. We have

$$\frac{100}{543} = \gamma(1) = \frac{9r}{5}\cos(\pi t/2 - \theta) = \frac{9r}{5}\sin\theta$$
$$\frac{100000}{30951} = \gamma(0) = 2r\cos(-\theta) = 2r\cos\theta.$$

Thus, $r = 500/(4887\sin\theta)$, which we plug into the second equation to obtain

$$\frac{1000\cos\theta}{4887\sin\theta} = \frac{100000}{30951}$$

and so $\theta = \tan^{-1}\frac{19}{300} \approx 0.632$ and $\sin\theta = 19/\sqrt{90361}$. Solving for r, we get $r = 500\sqrt{90361}/92853$. This gives us the general solution

$$\gamma(h) = \frac{100\sqrt{90361}}{92853} \cdot \left(\frac{9}{10}\right)^t \cos(\pi t/2 - \tan^{-1}(19/300)).$$

Fortunately, this can be simplified: we use the formula $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$ to see that

$$\cos(\pi t/2 - \theta) = \begin{cases} (-1)^{t/2}\cos\theta & \text{if } t \text{ is even} \\ (-1)^{(t-1)/2}\sin\theta & \text{if } t \text{ is odd.} \end{cases}$$

We can substitute this back in to obtain the general solution

$$\gamma(2h) = \frac{100000}{30951}\left(\frac{9}{10}\right)^{2h}$$
$$\gamma(2h+1) = \frac{100}{543}\left(\frac{9}{10}\right)^{2h}.$$

Dividing everything by $\gamma(0)$ gives us

$$\rho(2h) = \left(\frac{9}{10}\right)^{2h}$$
$$\rho(2h+1) = \frac{57}{1000}\left(\frac{9}{10}\right)^{2h}.$$

The plot is in Figure 1.

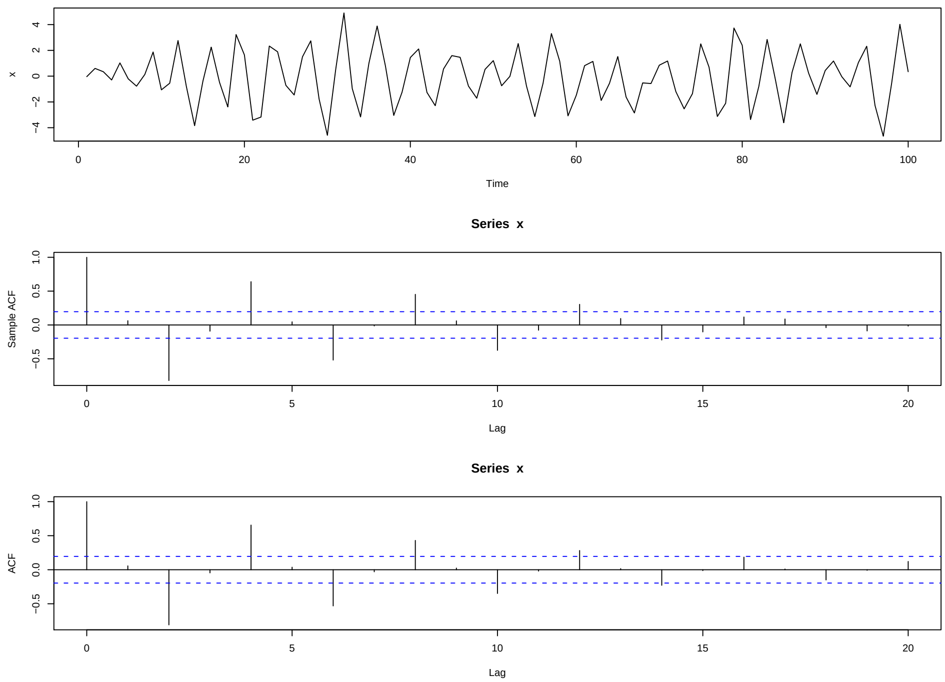

Figure 1: The simulated series, empirical autocorrelation function and true autocorrelation function for the model of question 2(a).

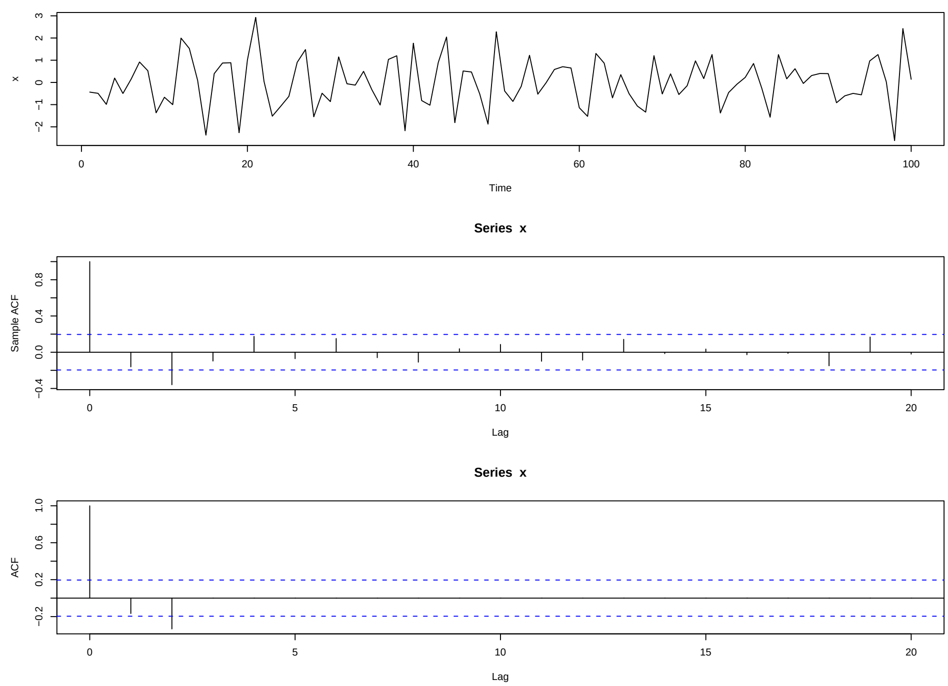

Figure 2: The simulated series, empirical autocorrelation function and true autocorrelation function for the model of question 2(b).

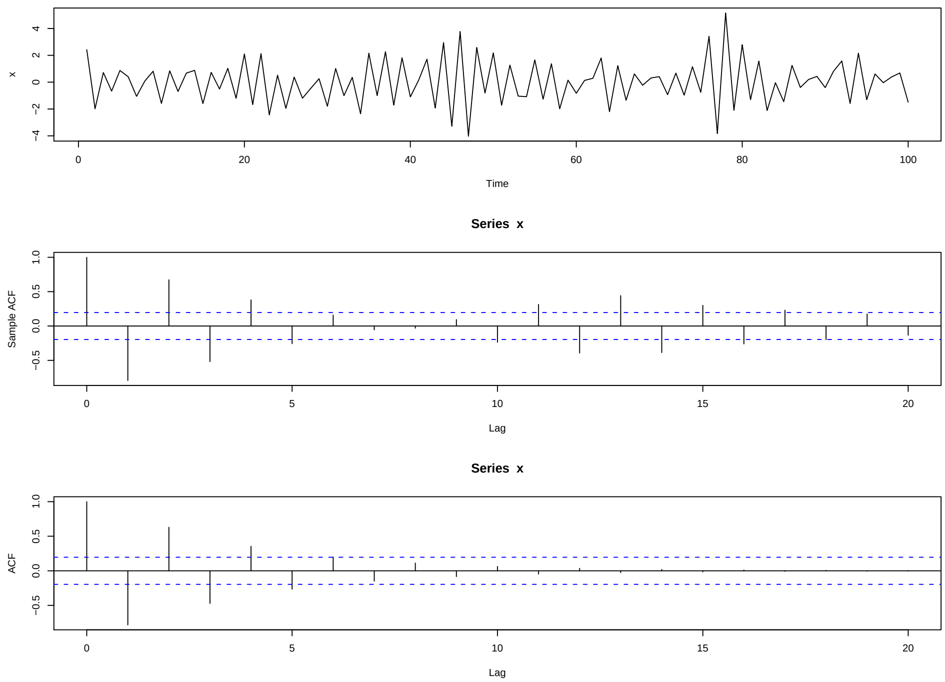

Figure 3: The simulated series, empirical autocorrelation function and true autocorrelation function for the model of question 2(g).

(b) Part (b), when written with irredundant parameters, is just an MA model, so we can compute the autocovariance function without solving any recurrence relations. The autocovariance function is $\gamma(0) = 5/4$, $\gamma(1) = 1/2$ and $\gamma = 0$ otherwise. Thus, the autocorrelation function is $\rho(0) = 1$, $\rho(1) = 2/5$ and $\rho = 0$ otherwise.

The plot is in Figure 2.

(g) The recurrence relation for the autocorrelation function is

$$\gamma(h) + \frac{3/4}{\gamma}(h-1) = 0.$$

The characteristic polynomial for this relation is $r + 3/4 = 0$, which has a single root at $-3/4$. Therefore, the general solution to the recurrence relation is $\gamma(h) = A(-3/4)^h$. The initial conditions are

$$\gamma(0) + \frac{3}{4}\gamma(1) = \frac{1393}{1296}$$
$$\gamma(1) + \frac{3}{4}\gamma(0) = -\frac{1}{12}$$
$$\gamma(2) + \frac{3}{4}\gamma(1) = \frac{1}{9},$$

which we can solve to obtain $\gamma(0) = \frac{1474}{567}$, $\gamma(1) = -\frac{1537}{756}$ and $\gamma(2) = \frac{1649}{1008}$. Thus,

$$\rho(0) = 1$$
$$\rho(1) = -\frac{4611}{5896}$$
$$\rho(2) = \frac{14841}{23584}$$
$$\rho(h) = \frac{14841}{23584} \cdot \left(-\frac{3}{4}\right)^{h-2},$$

where the last equation holds for $h > 2$.

The plot is in Figure 3.