Stat153 Midterm Exam 1 Solutions (October 7, 2010)

1. (Stationarity)

   (a) The mean function of this time series is $\mathbb{E}X_t = -2t$, which varies with t. Thus, the series is not stationary.
   The autocovariance is

$$\gamma(h) = \mathsf{Cov}(-2t + W_t + 0.5W_{t-1}, -2(t+h) + W_{t+h} + 0.5W_{t+h-1})$$
$$= \begin{cases} 1.25\sigma^2 & \text{if } h = 0, \\ 0.5\sigma^2 & \text{if } |h| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (b) The differenced time series $\{Y_t\}$ is given by

$$Y_t = -2t + W_t + 0.5W_{t-1} - (-2(t-1) + W_{t-1} + 0.5W_{t-2}) = W_t - 0.5W_{t-1} - 0.5W_{t-2} - 2.$$

   This has mean function $\mathbb{E}Y_t = -2$, which is constant. Also, the autocovariance is

$$\begin{aligned} \mathsf{Cov}(Y_t, Y_{t+h}) &= \mathsf{Cov}(W_t - 0.5W_{t-1} - 0.5W_{t-2} - 2, W_{t+h} - 0.5W_{t+h-1} - 0.5W_{t+h-2} - 2) \\ &= \mathsf{Cov}(W_t, W_{t+h}) - 0.5\mathsf{Cov}(W_t, W_{t+h-1}) - 0.5\mathsf{Cov}(W_t, W_{t+h-2}) \\ &\quad - 0.5\mathsf{Cov}(W_{t-1}, W_{t+h}) + 0.25\mathsf{Cov}(W_{t-1}, W_{t+h-1}) + 0.25\mathsf{Cov}(W_{t-1}, W_{t+h-2}) \\ &\quad - 0.5\mathsf{Cov}(W_{t-2}, W_{t+h}) + 0.25\mathsf{Cov}(W_{t-2}, W_{t+h-1}) + 0.25\mathsf{Cov}(W_{t-2}, W_{t+h-2}) \\ &= \begin{cases} 1.25\sigma^2 & \text{if } h = 0, \\ -0.25\sigma^2 & \text{if } |h| = 1, \\ -0.5\sigma^2 & \text{if } |h| = 2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

   Since the mean function is constant and the autocovariance depends only on the lag h, the series is stationary.

2. (ACF, PACF)

   (a) The autocovariance function $\gamma(h)$ of an MA(q) drops to zero for $h > q$; the PACF ϕ_{hh} of an AR(p) drops to zero for $h > p$. In this case, since the PACF drops to zero for $h > 1$, we would tentatively propose an AR(1) model.

   (b) The variance $\mathsf{Var}(X_t) = \gamma(0) \approx 2.9$.

3. (Causality) Consider the following ARMA model

$$X_t = X_{t-1} - 0.25X_{t-2} + W_t - 0.25W_{t-1},$$

   where $W_t \sim N(0, 1)$.

   (a) The AR polynomial is $1 - z + 0.25z^2$, which has both roots at 2. Since the roots are outside the unit circle in the complex plane, this ARMA(2,1) model is causal.

   (b) To compute the MA(∞) representation, we need to solve

$$\left(\psi_0 + \psi_1 z + \psi_2 z^2 + \cdots\right)\left(1 - z + 0.25z^2\right) = (1 - 0.25z)$$

   for ψ_i. This is a linear difference equation. Since the AR polynomial has both roots at $z = 2$, the general form of the solution is

$$\psi_j = (c_1 + c_2 j)2^{-j}$$

for some $c_1, c_2 \in \mathbb{R}$. We use the initial conditions,

$$\psi_0 = 1, \qquad \psi_1 - \psi_0 = -0.25$$

to find c_1 and c_2. These equations imply $c_1 = 1$, $c_2 = 0.5$. Thus, the MA(∞) representation is

$$X_t = \sum_{j=0}^{\infty}(1 + 0.5j)2^{-j}W_{t-j}$$

4. (Invertibility)

   (a) The MA polynomial is $1 - 0.25z$, which has a root at $z_1 = 4$. Since $|z_1| > 1$, this ARMA(2,1) model is invertible.

   (b) To compute the AR(∞) representation, we need to solve

   $$(1 - 0.25z)\left(\pi_0 + \pi_1 z + \pi_2 z^2 + \cdots\right) = \left(1 - z + 0.25z^2\right)$$

   for π_i. The general form of the solution for the homogeneous difference equation is

   $$\pi_j = c4^{-j}.$$

   We use the initial conditions

   $$\begin{aligned} \pi_0 &= 1, \\ \pi_1 - 0.25\pi_0 &= -1, \\ \pi_2 - 0.25\pi_1 &= 0.25 \end{aligned}$$

   to find c and the initial values of the sequence. These equations imply

   $$\begin{aligned} \pi_0 &= 1, \\ \pi_1 &= -0.75, \\ \pi_j &= 4^{-j} \qquad \text{for } j \geq 2. \end{aligned}$$

   Thus, the AR(∞) representation is

   $$W_t = X_t - 0.75X_{t-1} + \sum_{j=2}^{\infty} 4^{-j}X_{t-j}.$$

5. (Forecasting)

   (a) Since we have an AR(3) model, the best linear predictor of X_{T+1} is given by

   $$\begin{aligned} P(X_{T+1}|X_T, X_{T-1}, X_{T-2}) = X_{T+1}^T = \tilde{X}_{T+1} &= \phi_1 X_T + \phi_2 X_{T-1} + \phi_3 X_{T-2} \\ &= 0.2(-0.74) - 0.2(-3.5) + 0.6(3.0) \\ &= 2.352. \end{aligned}$$

   (b) Since $X_{T+1}^T = \tilde{X}_{T+1}$ for an AR(p) model with $p \leq T$, we know that

   $$P_{T+1}^T = \sigma_w^2 \psi_0^2 = \sigma_w^2 = 1.$$

   Since W_t is Gaussian, the conditional distribution of X_{T+1} given $X_1, \ldots, X_T$ is $N(X_{T+1}^T, 1)$. So a 95% confidence interval for X_{T+1} is

   $$2.352 \pm 1.96.$$

6. (Estimation)

(a) The Yule-Walker equations are $\Gamma_2 \phi = \gamma_2$ and $\sigma^2 = \gamma(0) - \gamma_2' \phi$. In this case,

$$\begin{aligned} & \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2.5 \end{pmatrix} \\ \Leftrightarrow \quad & \hat{\phi}_1 = 0, \quad \hat{\phi}_2 = 0.5, \qquad \text{and} \\ & \hat{\sigma}_w^2 = 5 - \begin{pmatrix} 0 & 2.5 \end{pmatrix} \hat{\phi} = 3.75. \end{aligned}$$

(b) The asymptotic distribution of $\hat{\phi} = (\hat{\phi}_1, \hat{\phi}_2)'$ is

$$N\left(\phi, \frac{\sigma_w^2}{T} \Gamma_2^{-1}\right).$$

Thus, an approximate 95% confidence interval for ϕ_2 is given by

$$\hat{\phi}_2 \pm 1.96 \sqrt{\frac{\hat{\sigma}_w^2}{T} \left(\hat{\Gamma}_2^{-1}\right)_{22}} = 0.5 \pm 1.96 \sqrt{\frac{3.75}{1500 \times 5}} \approx 0.5 \pm 0.044.$$