Stat153 Assignment 5 (due Tuesday, November 23)

1. In this question, we'll show that a polynomial p with real coefficients has $p(z)p(\bar{z}) = |p(z)|^2$.

   (a) Show that, for any complex z, $z\bar{z} = |z|^2$. Draw a sketch in the complex plane to illustrate.

   (b) Show that, for any complex z and any positive integer j, the complex conjugate of z^j is $\bar{z}^j$.

   (c) Hence show that, for any polynomial $p(z) = \sum_{j=1}^{k} a_j z^j$ with real coefficients a_j,

   $$p(z)p(\bar{z}) = |p(z)|^2.$$

2. **(Rational spectral densities)**
   Calculate the spectral density for the following time series models. In both cases, show in a sketch where the poles and zeros are in the complex plane, and explain how these affect the spectral density.

   (a) X_t, where

   $$\left(1 - 4\frac{\sqrt{2}}{5}B + \left(\frac{4}{5}\right)^2 B^2\right) X_t = \left(1 - \left(\frac{4}{5}\right)^2 B^2\right) W_t,$$

   and W_t is $WN(0,1)$.

   (b) Y_t, where

   $$\left(1 - \frac{5}{6}B\right) Y_t = X_t,$$

   and X_t is as defined above.

3. **(Linear filters)**
   Let $\psi(B)$ be such that $Y_t = \psi(B)W_t$ for the time series defined in Question 2b above. For this linear filter, plot (using R, for example), the function $z \mapsto |\psi(z)|^2$ defined on the complex plane.

4. **(Periodogram)**
   Consider the AR(1) time series model

   $$(1 - 0.8B)X_t = W_t,$$

   where W_t is a Gaussian white noise process. Generate four realizations of this time series, $x_1, \ldots, x_n$ for $n = 128$, 512, 1024, and 2048.

   (a) Compute and plot the periodogram in each case.
   Include the spectral density for the AR(1) model in each plot.

   (b) In each case, calculate approximate confidence intervals for $f(0.1)$, the spectral density for the time series at frequency 0.1.

   Explain your findings.

5. **(Smoothed Periodogram)**
   Consider the following smoothed spectral estimator.

   $$\hat{f}(\nu) = \frac{1}{2\lfloor\sqrt{n}\rfloor + 1} \sum_{|j|<\sqrt{n}} I(\hat{\nu}^{(n)} + j/n),$$

   where I is the periodogram and $\hat{\nu}^{(n)}$ is the value i/n closest to ν. Repeat Question 4a using this smoothed periodogram in place of the periodogram.