Introduction to Time Series Analysis. Lecture 6.

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1. Review: Causality, invertibility, AR(p) models
2. ARMA(p,q) models
3. Stationarity, causality and invertibility
4. The linear process representation of ARMA processes: ψ.
5. Autocovariance of an ARMA process.

Review: Causality

A linear process $\{X_t\}$ is **causal** (strictly, a **causal function of** $\{W_t\}$) if there is a

$$\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots$$

$$\text{with} \qquad \sum_{j=0}^{\infty} |\psi_j| < \infty$$

$$\text{and} \qquad X_t = \psi(B) W_t.$$

Review: Invertibility

A linear process $\{X_t\}$ is **invertible** (strictly, an **invertible function of** $\{W_t\}$) if there is a

$$\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \cdots$$

$$\text{with} \qquad \sum_{j=0}^{\infty} |\pi_j| < \infty$$

$$\text{and} \qquad W_t = \pi(B) X_t.$$

Review: AR(p), Autoregressive models of order p

An **AR(p) process** $\{X_t\}$ is a stationary process that satisfies

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t,$$

where $\{W_t\} \sim WN(0, \sigma^2)$.

Equivalently, $\phi(B)X_t = W_t,$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p.$

Review: AR(p), Autoregressive models of order p

Theorem: A (unique) *stationary* solution to $\phi(B)X_t = W_t$ exists iff

$$|z| = 1 \Rightarrow \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0.$$

This AR(p) process is *causal* iff

$$|z| \leq 1 \Rightarrow \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0.$$

Polynomials of a complex variable

Every degree p polynomial $a(z)$ can be factorized as

$$a(z) = a_0 + a_1 z + \cdots + a_p z^p = a_p(z - z_1)(z - z_2) \cdots (z - z_p),$$

where $z_1, \ldots, z_p \in \mathbb{C}$ are called the roots of $a(z)$. If the coefficients $a_0, a_1, \ldots, a_p$ are all real, then c is real, and the roots are all either real or come in complex conjugate pairs, $z_i = \bar{z}_j$.

Example: $z + z^3 = z(1 + z^2) = (z - 0)(z - i)(z + i)$,
that is, $c = 1$, $z_1 = 0$, $z_2 = i$, $z_3 = -i$. So $z_1 \in \mathbb{R}$; $z_2, z_3 \notin \mathbb{R}$; $z_2 = \bar{z}_3$.

Recall notation: A complex number $z = a + ib$ has $\mathrm{Re}(z) = a$, $\mathrm{Im}(z) = b$, $\bar{z} = a - ib$, $|z| = a^2 + b^2$, $\arg(z) = \tan^{-1}(b/a) \in (-\pi, \pi]$.

Calculating ψ for an AR(p): matching coefficients

Example: $X_t = \psi(B)W_t \quad \Leftrightarrow \quad (1 - 0.5B + 0.6B^2)X_t = W_t,$

$$
\begin{aligned}
\text{so} \quad & 1 = \psi(B)(1 - 0.5B + 0.6B^2) \\
\Leftrightarrow \quad & 1 = (\psi_0 + \psi_1 B + \psi_2 B^2 + \cdots)(1 - 0.5B + 0.6B^2) \\
\Leftrightarrow \quad & 1 = \psi_0, \\
& 0 = \psi_1 - 0.5\psi_0, \\
& 0 = \psi_2 - 0.5\psi_1 + 0.6\psi_0, \\
& 0 = \psi_3 - 0.5\psi_2 + 0.6\psi_1, \\
& \vdots
\end{aligned}
$$

Calculating ψ for an AR(p): example

$$
\begin{aligned}
\Leftrightarrow \qquad & 1 = \psi_0, \qquad 0 = \psi_j \qquad (j \le 0), \\
& 0 = \psi_j - 0.5\psi_{j-1} + 0.6\psi_{j-2} \\
\Leftrightarrow \qquad & 1 = \psi_0, \qquad 0 = \psi_j \qquad (j \le 0), \\
& 0 = \phi(B)\psi_j.
\end{aligned}
$$

We can solve these *linear difference equations* in several ways:

- numerically, or
- by guessing the form of a solution and using an inductive proof, or
- by using the theory of linear difference equations.

Calculating ψ for an AR(p): general case

$$\phi(B)X_t = W_t, \qquad \Leftrightarrow \qquad X_t = \psi(B)W_t$$

so $\qquad 1 = \psi(B)\phi(B)$

$$\Leftrightarrow \qquad 1 = (\psi_0 + \psi_1 B + \cdots)(1 - \phi_1 B - \cdots - \phi_p B^p)$$

$$\Leftrightarrow \qquad 1 = \psi_0,$$

$$0 = \psi_1 - \phi_1\psi_0,$$

$$0 = \psi_2 - \phi_1\psi_1 - \phi_2\psi_0,$$

$$\vdots$$

$$\Leftrightarrow \qquad 1 = \psi_0, \qquad 0 = \psi_j \qquad (j < 0),$$

$$0 = \phi(B)\psi_j.$$

ARMA(p,q): Autoregressive moving average models

An **ARMA(p,q) process** $\{X_t\}$ is a stationary process that satisfies

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q},$$

where $\{W_t\} \sim WN(0, \sigma^2)$.

- AR(p) $=$ ARMA(p,0): $\theta(B) = 1$.
- MA(q) $=$ ARMA(0,q): $\phi(B) = 1$.

ARMA processes

Can accurately approximate many stationary processes:

For any stationary process with autocovariance γ, and any $k > 0$, there is an ARMA process $\{X_t\}$ for which

$$\gamma_X(h) = \gamma(h), \qquad h = 0, 1, \ldots, k.$$

ARMA(p,q): Autoregressive moving average models

An **ARMA(p,q) process** $\{X_t\}$ is a stationary process that satisfies

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q},$$

where $\{W_t\} \sim WN(0, \sigma^2)$.

Usually, we insist that $\phi_p, \theta_q \neq 0$ and that the polynomials

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p, \qquad \theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$$

have no common factors. This implies it is not a lower order ARMA model.

ARMA(p,q): An example of parameter redundancy

Consider a white noise process W_t. We can write

$$\begin{aligned} X_t &= W_t \\ \Rightarrow \qquad X_t - X_{t-1} + 0.25X_{t-2} &= W_t - W_{t-1} + 0.25W_{t-2} \\ (1 - B + 0.25B^2)X_t &= (1 - B + 0.25B^2)W_t \end{aligned}$$

This is in the form of an ARMA(2,2) process, with

$$\phi(B) = 1 - B + 0.25B^2, \qquad \theta(B) = 1 - B + 0.25B^2.$$

But it is white noise.

ARMA(p,q): An example of parameter redundancy

$$
\begin{aligned}
\text{ARMA model:} \qquad & \phi(B)X_t = \theta(B)W_t, \\
\text{with} \qquad & \phi(B) = 1 - B + 0.25B^2, \\
& \theta(B) = 1 - B + 0.25B^2 \\
& X_t = \psi(B)W_t \\
\Leftrightarrow \qquad & \psi(B) = \frac{\theta(B)}{\phi(B)} = 1.
\end{aligned}
$$

i.e., $X_t = W_t$.

ARMA(p,q): Stationarity, causality, and invertibility

Theorem: If ϕ and θ have no common factors, a (unique) *stationary* solution to $\phi(B)X_t = \theta(B)W_t$
exists iff

$$|z| = 1 \Rightarrow \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0.$$

This ARMA(p,q) process is *causal* iff

$$|z| \leq 1 \Rightarrow \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0.$$

It is *invertible* iff

$$|z| \leq 1 \Rightarrow \theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0.$$

Recall: Causality and Invertibility

A linear process $\{X_t\}$ is **causal** if there is a

$$\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots$$

$$\text{with} \quad \sum_{j=0}^{\infty} |\psi_j| < \infty \quad \text{and} \quad X_t = \psi(B) W_t.$$

It is **invertible** if there is a

$$\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \cdots$$

$$\text{with} \quad \sum_{j=0}^{\infty} |\pi_j| < \infty \quad \text{and} \quad W_t = \pi(B) X_t.$$

ARMA(p,q): Stationarity, causality, and invertibility

Example: $(1 - 1.5B)X_t = (1 + 0.2B)W_t.$

$$\phi(z) = 1 - 1.5z = -\frac{3}{2}\left(z - \frac{2}{3}\right),$$
$$\theta(z) = 1 + 0.2z = \frac{1}{5}(z + 5).$$

1. ϕ and θ have no common factors, and ϕ's root is at $2/3$, which is not on the unit circle, so $\{X_t\}$ is an ARMA(1,1) process.
2. ϕ's root (at $2/3$) is inside the unit circle, so $\{X_t\}$ is *not causal*.
3. θ's root is at -5, which is outside the unit circle, so $\{X_t\}$ is *invertible*.

ARMA(p,q): Stationarity, causality, and invertibility

Example: $(1 + 0.25B^2)X_t = (1 + 2B)W_t.$

$$\phi(z) = 1 + 0.25z^2 = \frac{1}{4}\left(z^2 + 4\right) = \frac{1}{4}(z + 2i)(z - 2i),$$
$$\theta(z) = 1 + 2z = 2\left(z + \frac{1}{2}\right).$$

1. ϕ and θ have no common factors, and ϕ's roots are at $\pm 2i$, which is not on the unit circle, so $\{X_t\}$ is an ARMA(2,1) process.
2. ϕ's roots (at $\pm 2i$) are outside the unit circle, so $\{X_t\}$ is *causal*.
3. θ's root (at $-1/2$) is inside the unit circle, so $\{X_t\}$ is *not invertible*.

Causality and Invertibility

Theorem: Let $\{X_t\}$ be an ARMA process defined by $\phi(B)X_t = \theta(B)W_t$. If $\theta(z) \neq 0$ for all $|z| = 1$, then there are polynomials $\tilde{\phi}$ and $\tilde{\theta}$ and a white noise sequence $\tilde{W}_t$ such that $\{X_t\}$ satisfies $\tilde{\phi}(B)X_t = \tilde{\theta}(B)\tilde{W}_t$, and this is a causal, invertible ARMA process.

So we'll stick to causal, invertible ARMA processes.

Calculating ψ for an ARMA(p,q): matching coefficients

Example: $X_t = \psi(B)W_t \quad \Leftrightarrow \quad (1 + 0.25B^2)X_t = (1 + 0.2B)W_t$

so $1 + 0.2B = (1 + 0.25B^2)\psi(B)$

$$\Leftrightarrow \quad 1 + 0.2B = (1 + 0.25B^2)(\psi_0 + \psi_1 B + \psi_2 B^2 + \cdots)$$

$$\begin{aligned} \Leftrightarrow \quad 1 &= \psi_0, \\ 0.2 &= \psi_1, \\ 0 &= \psi_2 + 0.25\psi_0, \\ 0 &= \psi_3 + 0.25\psi_1, \\ &\vdots \end{aligned}$$

Calculating ψ for an ARMA(p,q): example

$$\Leftrightarrow \qquad 1 = \psi_0, \qquad 0.2 = \psi_1,$$
$$0 = \psi_j + 0.25\psi_{j-2}.$$

We can think of this as $\theta_j = \phi(B)\psi_j$, with $\theta_0 = 1$, $\theta_j = 0$ for $j < 0$, $j > q$.

This is a *first order difference equation* in the ψ_js.
We can use the θ_js to give the initial conditions and solve it using the theory of homogeneous difference equations.

$\psi_j = \left(1, \frac{1}{5}, -\frac{1}{4}, -\frac{1}{20}, \frac{1}{16}, \frac{1}{80}, -\frac{1}{64}, -\frac{1}{320}, \ldots\right).$

Calculating ψ for an ARMA(p,q): general case

$$\phi(B)X_t = \theta(B)W_t, \qquad \Leftrightarrow \qquad X_t = \psi(B)W_t$$

$$\text{so} \qquad \theta(B) = \psi(B)\phi(B)$$

$$\Leftrightarrow \; 1 + \theta_1 B + \cdots + \theta_q B^q = (\psi_0 + \psi_1 B + \cdots)(1 - \phi_1 B - \cdots - \phi_p B^p)$$

$$\begin{aligned} \Leftrightarrow \qquad 1 &= \psi_0, \\ \theta_1 &= \psi_1 - \phi_1\psi_0, \\ \theta_2 &= \psi_2 - \phi_1\psi_1 - \cdots - \phi_2\psi_0, \\ &\vdots \end{aligned}$$

This is equivalent to $\theta_j = \phi(B)\psi_j$, with $\theta_0 = 1$, $\theta_j = 0$ for $j < 0$, $j > q$.