# Introduction to Time Series Analysis. Lecture 5.

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1. Causality

2. Invertibility

3. AR(p) models

# AR(1) and Causality

Let $X_t$ be the stationary solution to

$$X_t - \phi X_{t-1} = W_t,$$

where $W_t \sim WN(0, \sigma^2)$.

If $|\phi| < 1$,

$$X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}.$$

$\phi = 1$?

$\phi = -1$?

$|\phi| > 1$?

# AR(1) and Causality

If $|\phi| > 1$, $\pi(B)W_t$ does not converge.
But we can rearrange

$$X_t = \phi X_{t-1} + W_t$$

as $$X_{t-1} = \frac{1}{\phi}X_t - \frac{1}{\phi}W_t,$$

and we can check that the unique stationary solution is

$$X_t = -\sum_{j=1}^{\infty} \phi^{-j} W_{t+j}.$$

But... $X_t$ depends on **future** values of $W_t$.

# Causality

A linear process $\{X_t\}$ is **causal** (strictly, a **causal function of** $\{W_t\}$) if there is a

$$\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots$$

$$\text{with} \qquad \sum_{j=0}^{\infty} |\psi_j| < \infty$$

$$\text{and} \qquad X_t = \psi(B) W_t.$$

# AR(1) and Causality

- Causality is a property of $\{X_t\}$ **and** $\{W_t\}$.
- The AR(1) process defined by $\phi(B)X_t = W_t$ (with $\phi(B) = 1 - \phi B$) is causal iff $|\phi| < 1$, iff the root $z_1$ of the polynomial $\phi(z) = 1 - \phi z$ satisfies $|z_1| > 1$.
- If $|\phi| > 1$, we can define an equivalent causal model, $X_t - \phi^{-1}X_{t-1} = \tilde{W}_t$, where $\tilde{W}_t$ is a new white noise sequence.
- Is an MA(1) process causal?

# MA(1) and Invertibility

Define

$$\begin{aligned} X_t &= W_t + \theta W_{t-1} \\ &= (1 + \theta B) W_t. \end{aligned}$$

If $|\theta| < 1$, we can write

$$\begin{aligned} & (1 + \theta B)^{-1} X_t = W_t \\ \Leftrightarrow \quad & (1 - \theta B + \theta^2 B^2 - \theta^3 B^3 + \cdots) X_t = W_t \\ \Leftrightarrow \quad & \sum_{j=0}^{\infty} (-\theta)^j X_{t-j} = W_t. \end{aligned}$$

That is, we can write $W_t$ as a *causal* function of $X_t$.
We say that this MA(1) is **invertible**.

# MA(1) and Invertibility

$$X_t = W_t + \theta W_{t-1}$$

If $|\theta| > 1$, the sum $\sum_{j=0}^{\infty}(-\theta)^j X_{t-j}$ diverges, but we can write

$$W_{t-1} = -\theta^{-1} W_t + \theta^{-1} X_t.$$

Just like the noncausal AR(1), we can show that

$$W_t = -\sum_{j=1}^{\infty}(-\theta)^{-j} X_{t+j}.$$

That is, we can write $W_t$ as a linear function of $X_t$, but it is not causal. We say that this MA(1) is not **invertible**.

# Invertibility

A linear process $\{X_t\}$ is **invertible** (strictly, an **invertible function of** $\{W_t\}$) if there is a

$$\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \cdots$$

with $$\sum_{j=0}^{\infty} |\pi_j| < \infty$$

and $$W_t = \pi(B) X_t.$$

# MA(1) and Invertibility

- Invertibility is a property of $\{X_t\}$ and $\{W_t\}$.
- The MA(1) process defined by $X_t = \theta(B)W_t$ (with $\theta(B) = 1 + \theta B$) is invertible iff $|\theta| < 1$ iff the root $z_1$ of the polynomial $\theta(z) = 1 + \theta z$ satisfies $|z_1| > 1$.
- If $|\theta| > 1$, we can define an equivalent invertible model in terms of a new white noise sequence.
- Is an AR(1) process invertible?

# AR(p): Autoregressive models of order $p$

An **AR(p) process** $\{X_t\}$ is a stationary process that satisfies

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t,$$

where $\{W_t\} \sim WN(0, \sigma^2)$.

Equivalently, $\phi(B) X_t = W_t,$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p.$

# AR(p): Constraints on $\phi$

Recall: For $p = 1$ (AR(1)), $\phi(B) = 1 - \phi_1 B$.

This is an AR(1) model only if there is a *stationary* solution to $\phi(B)X_t = W_t$, which is equivalent to $|\phi_1| \neq 1$.

This is equivalent to the following condition on $\phi(z) = 1 - \phi_1 z$:

$$\forall z \in \mathbb{R},\ \phi(z) = 0 \Rightarrow z \neq \pm 1$$

$$\text{equivalently, } \forall z \in \mathbb{C},\ \phi(z) = 0 \Rightarrow |z| \neq 1,$$

where $\mathbb{C}$ is the set of complex numbers.

# AR(p): Constraints on $\phi$

$$\text{Stationarity:} \qquad \forall z \in \mathbb{C},\ \phi(z) = 0 \Rightarrow |z| \neq 1,$$

where $\mathbb{C}$ is the set of complex numbers.

$\phi(z) = 1 - \phi_1 z$ has one root at $z_1 = 1/\phi_1 \in \mathbb{R}$.
But the roots of a degree $p > 1$ polynomial might be complex.
For stationarity, we want the roots of $\phi(z)$ to avoid the **unit circle**, $\{z \in \mathbb{C} : |z| = 1\}$.

# AR(p): Stationarity and causality

**Theorem:** A (unique) *stationary* solution to $\phi(B)X_t = W_t$ exists iff

$$|z| = 1 \Rightarrow \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0.$$

This AR(p) process is *causal* iff

$$|z| \leq 1 \Rightarrow \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0.$$

# Recall: Causality

A linear process $\{X_t\}$ is **causal** (strictly, a **causal function of** $\{W_t\}$) if there is a

$$\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots$$

$$\text{with} \qquad \sum_{j=0}^{\infty} |\psi_j| < \infty$$

$$\text{and} \qquad X_t = \psi(B) W_t.$$

# AR(p): All roots outside the unit circle implies causal

$$\forall z \in \mathbb{C},\ |z| \le 1 \ \Rightarrow\ \phi(z) \ne 0$$

$$\Leftrightarrow \qquad \exists \{\psi_j\}, \delta > 0,\ \forall |z| \le 1+\delta,\ \frac{1}{\phi(z)} = \sum_{j=0}^{\infty} \psi_j z^j.$$

$$\Rightarrow \qquad \forall |z| \le 1+\delta,\ |\psi_j z^j| \to 0,\ \left(|\psi_j|^{1/j}|z|\right)^j \to 0$$

$$\Rightarrow \qquad \exists j_0,\ \forall j \ge j_0,\ |\psi_j|^{1/j} \le \frac{1}{1+\delta/2} \qquad \Rightarrow \qquad \sum_{j=0}^{\infty} |\psi_j| < \infty.$$

So if $|z| \le 1 \Rightarrow \phi(z) = 0$, then $S_m = \sum_{j=0}^{m} \psi_j B^j W_t$ converges in mean square, so we have a stationary, causal time series $X_t = \phi^{-1}(B) W_t$.

# Calculating $\psi$ for an AR(p): matching coefficients

Example: $X_t = \psi(B)W_t \quad \Leftrightarrow \quad (1 - 0.5B + 0.6B^2)X_t = W_t,$

$$\begin{aligned}
\text{so} \quad & 1 = \psi(B)(1 - 0.5B + 0.6B^2) \\
\Leftrightarrow \quad & 1 = (\psi_0 + \psi_1 B + \psi_2 B^2 + \cdots)(1 - 0.5B + 0.6B^2) \\
\Leftrightarrow \quad & 1 = \psi_0, \\
& 0 = \psi_1 - 0.5\psi_0, \\
& 0 = \psi_2 - 0.5\psi_1 + 0.6\psi_0, \\
& 0 = \psi_3 - 0.5\psi_2 + 0.6\psi_1, \\
& \vdots
\end{aligned}$$

# Calculating $\psi$ for an AR(p): example

$$\begin{aligned}
\Leftrightarrow \qquad & 1 = \psi_0, \qquad 0 = \psi_j \qquad (j \leq 0), \\
& 0 = \psi_j - 0.5\psi_{j-1} + 0.6\psi_{j-2} \\
\Leftrightarrow \qquad & 1 = \psi_0, \qquad 0 = \psi_j \qquad (j \leq 0), \\
& 0 = \phi(B)\psi_j.
\end{aligned}$$

We can solve these *linear difference equations* in several ways:

- numerically, or
- by guessing the form of a solution and using an inductive proof, or
- by using the theory of linear difference equations.

## Calculating $\psi$ for an AR(p): general case

$$\phi(B)X_t = W_t, \qquad \Leftrightarrow \qquad X_t = \psi(B)W_t$$

so

$$\begin{aligned}
&1 = \psi(B)\phi(B) \\
\Leftrightarrow\quad &1 = (\psi_0 + \psi_1 B + \cdots)(1 - \phi_1 B - \cdots - \phi_p B^p) \\
\Leftrightarrow\quad &1 = \psi_0, \\
&0 = \psi_1 - \phi_1\psi_0, \\
&0 = \psi_2 - \phi_1\psi_1 - \phi_2\psi_0, \\
&\vdots \\
\Leftrightarrow\quad &1 = \psi_0, \qquad 0 = \psi_j \qquad (j < 0), \\
&0 = \phi(B)\psi_j.
\end{aligned}$$