Introduction to Time Series Analysis. Lecture 2.

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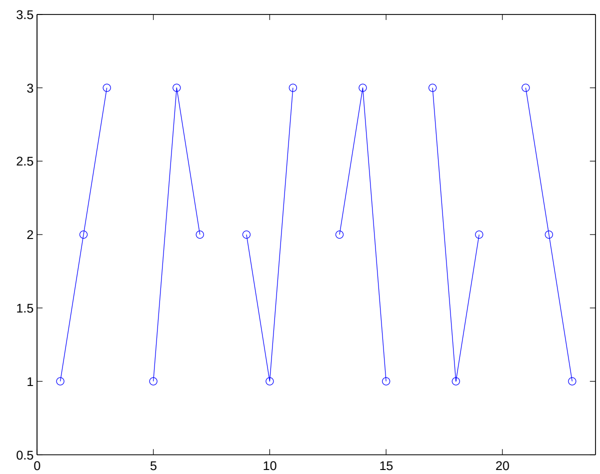
1. Tests for i.i.d.
2. Stationarity
3. Autocovariance, autocorrelation
4. MA, AR, linear processes

Testing i.i.d.: Turning point test

$\{X_t\}$ i.i.d. implies that X_t, X_{t+1} and X_{t+2} are equally likely to occur in any of six possible orders:



(provided X_t, X_{t+1}, X_{t+2} are distinct).

Four of the six are **turning points**.

Testing i.i.d.: Turning point test

Define $T = |\{t : X_t, X_{t+1}, X_{t+2} \text{ is a turning point}\}|$.

$\mathrm{E}T = (n-2)2/3$.

Can show $T \sim AN(2n/3, 8n/45)$.

Notation: $X \sim AN(\mu, \sigma^2) \Leftrightarrow \frac{X-\mu}{\sigma} \xrightarrow{d} N(0,1)$.

Reject (at 5% level) the hypothesis that the series is i.i.d. if

$$\left| T - \frac{2n}{3} \right| > 1.96 \sqrt{\frac{8n}{45}}.$$

Tests for positive/negative correlations at lag 1.

Testing i.i.d.: Difference-sign test

$$S = |\{i : X_i > X_{i-1}\}| = |\{i : (\nabla X)_i > 0\}|.$$

$$\mathrm{E}S = \frac{n-1}{2}.$$

Can show $S \sim AN(n/2, n/12)$.

Reject (at 5% level) the hypothesis that the series is i.i.d. if

$$\left| S - \frac{n}{2} \right| > 1.96 \sqrt{\frac{n}{12}}.$$

Tests for trend.

(But a periodic sequence can pass this test...)

Testing i.i.d.: Rank test

$$N = |\{(i,j) : X_i > X_j \text{ and } i > j\}|.$$

$$\mathrm{E}N = \frac{n(n-1)}{4}.$$

Can show $N \sim AN(n^2/4, n^3/36)$.

Reject (at 5% level) the hypothesis that the series is i.i.d. if

$$\left| N - \frac{n^2}{4} \right| > 1.96\sqrt{\frac{n^3}{36}}.$$

Tests for linear trend.

Testing if an i.i.d. sequence is Gaussian: qq plot

Plot the pairs $(m_1, X_{(1)}), \ldots, (m_n, X_{(n)})$,
where $m_j = \mathrm{E}X_{(j)}$,
$X_{(1)} < \cdots < X_{(n)}$ are order statistics from $N(0, 1)$ sample of size n, and
$X_{(1)} < \cdots < X_{(n)}$ are order statistics of the series $X_1, \ldots, X_n$.

Idea: If $X_i \sim N(\mu, \sigma^2)$, then

$$\mathrm{E}X_{(j)} = \mu + \sigma m_j,$$

so $(m_j, X_{(j)})$ should be *linear*.

There are tests based on how far correlation of $(m_j, X_{(j)})$ is from 1.

Stationarity

$\{X_t\}$ is **strictly stationary** if
for all $k, t_1, \ldots, t_k, x_1, \ldots, x_k$, and h,

$$P(X_{t_1} \leq x_1, \ldots, X_{t_k} \leq x_k) = P(x_{t_1+h} \leq x_1, \ldots, X_{t_k+h} \leq x_k).$$

i.e., shifting the time axis does not affect the distribution.

We shall consider **second-order properties** only.

Mean and Autocovariance

Suppose that $\{X_t\}$ is a time series with $\mathrm{E}[X_t^2] < \infty$.
Its **mean function** is

$$\mu_t = \mathrm{E}[X_t].$$

Its **autocovariance function** is

$$\begin{aligned}\gamma_X(s,t) &= \mathrm{Cov}(X_s, X_t) \\ &= \mathrm{E}[(X_s - \mu_s)(X_t - \mu_t)].\end{aligned}$$

Weak Stationarity

We say that $\{X_t\}$ is **(weakly) stationary** if

1. μ_t is independent of t, and
2. For each h, $\gamma_X(t+h, t)$ is independent of t.

In that case, we write

$$\gamma_X(h) = \gamma_X(h, 0).$$

Stationarity

The **autocorrelation function (ACF)** of $\{X_t\}$ is defined as

$$\begin{aligned}\rho_X(h) &= \frac{\gamma_X(h)}{\gamma_X(0)} \\ &= \frac{\text{Cov}(X_{t+h}, X_t)}{\text{Cov}(X_t, X_t)} \\ &= \text{Cor}(X_{t+h}, X_t).\end{aligned}$$

Stationarity

Example: i.i.d. noise, $\mathrm{E}[X_t] = 0$, $\mathrm{E}[X_t^2] = \sigma^2$. We have

$$\gamma_X(t+h, t) = \begin{cases} \sigma^2 & \text{if } h = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

1. $\mu_t = 0$ is independent of t.
2. $\gamma_X(t+h, t) = \gamma_X(h, 0)$ for all t.

So $\{X_t\}$ is stationary.

Similarly for any white noise (uncorrelated, zero mean), $X_t \sim WN(0, \sigma^2)$.

Stationarity

Example: Random walk, $S_t = \sum_{i=1}^{t} X_i$ for i.i.d., mean zero $\{X_t\}$.
We have $\mathrm{E}[S_t] = 0$, $\mathrm{E}[S_t^2] = t\sigma^2$, and

$$\begin{aligned}
\gamma_S(t+h, t) &= \mathrm{Cov}(S_{t+h}, S_t) \\
&= \mathrm{Cov}\left(S_t + \sum_{s=1}^{h} X_{t+s}, S_t\right) \\
&= \mathrm{Cov}(S_t, S_t) = t\sigma^2.
\end{aligned}$$

1. $\mu_t = 0$ is independent of t, but
2. $\gamma_S(t+h, t)$ is not.

So $\{S_t\}$ is not stationary.

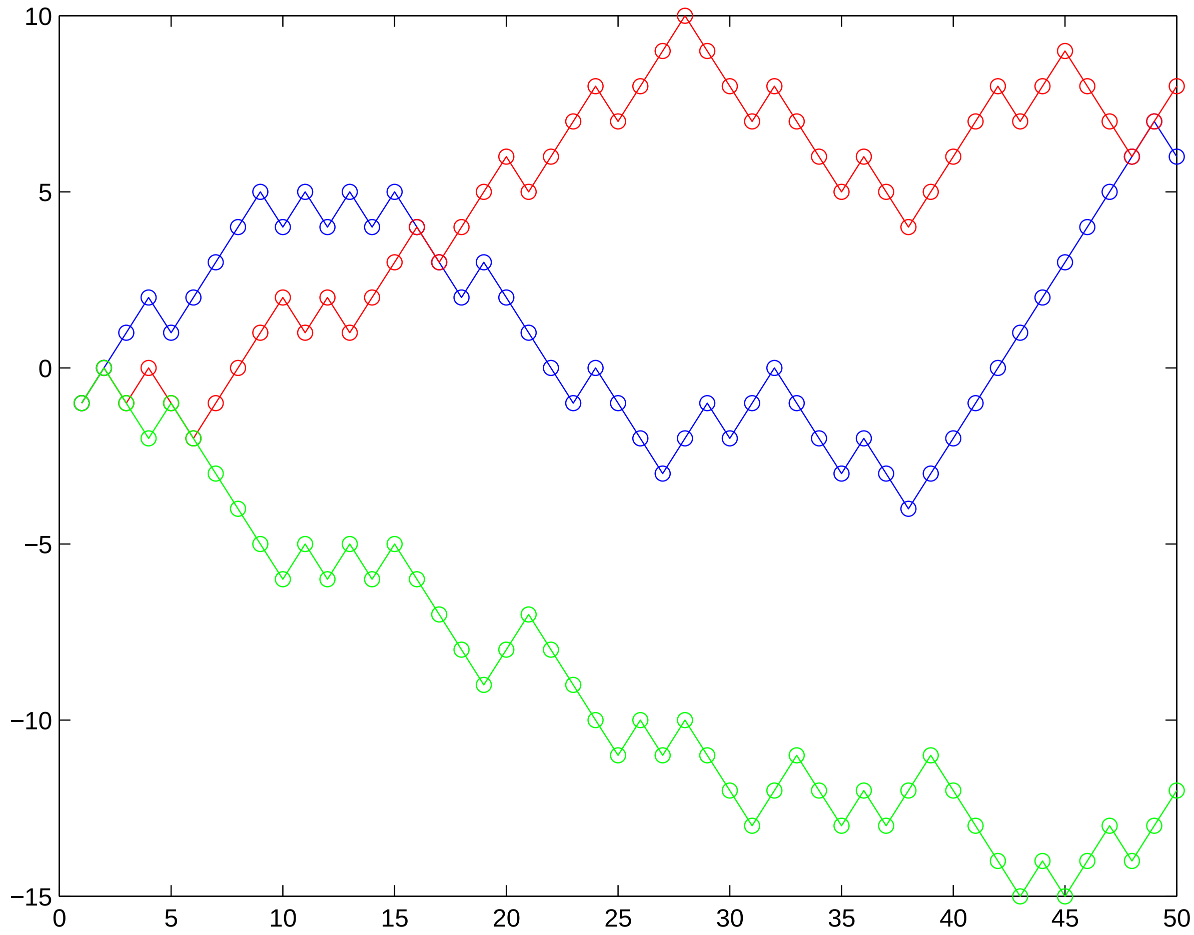
An aside: covariances

$$\text{Cov}(X+Y,Z) = \text{Cov}(X,Z) + \text{Cov}(Y,Z),$$
$$\text{Cov}(aX,Y) = a\,\text{Cov}(X,Y),$$

Also if X and Y are independent (e.g., $X = c$), then

$$\text{Cov}(X,Y) = 0.$$

Random walk



Stationarity

Example: MA(1) process (**Moving Average**):

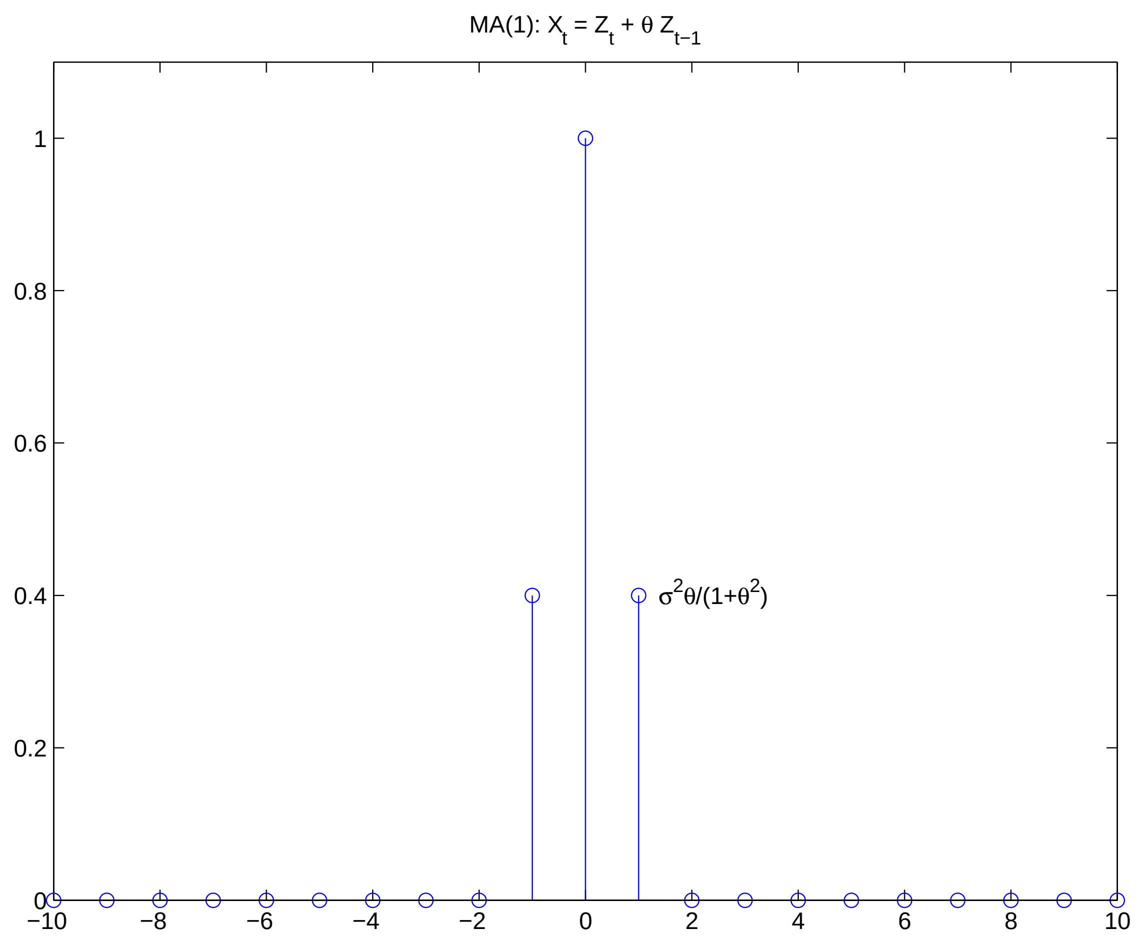
$$X_t = W_t + \theta W_{t-1}, \qquad \{W_t\} \sim WN(0, \sigma^2).$$

We have $\mathrm{E}[X_t] = 0$, and

$$\begin{aligned} \gamma_X(t+h, t) &= \mathrm{E}(X_{t+h} X_t) \\ &= \mathrm{E}[(W_{t+h} + \theta W_{t+h-1})(W_t + \theta W_{t-1})] \\ &= \begin{cases} \sigma^2(1+\theta^2) & \text{if } h = 0, \\ \sigma^2\theta & \text{if } h = \pm 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, $\{X_t\}$ is stationary.

ACF of the MA(1) process



Stationarity

Example: AR(1) process (**AutoRegressive**):

$$X_t = \phi X_{t-1} + W_t, \qquad \{W_t\} \sim WN(0, \sigma^2).$$

Assume that X_t is stationary and $|\phi| < 1$. Then we have

$$\begin{aligned} \mathrm{E}[X_t] &= \phi \mathrm{E} X_{t-1} \\ &= 0 \qquad \text{(from stationarity)} \\ \mathrm{E}[X_t^2] &= \phi^2 \mathrm{E}[X_{t-1}^2] + \sigma^2 \\ &= \frac{\sigma^2}{1-\phi^2} \qquad \text{(from stationarity)}, \end{aligned}$$

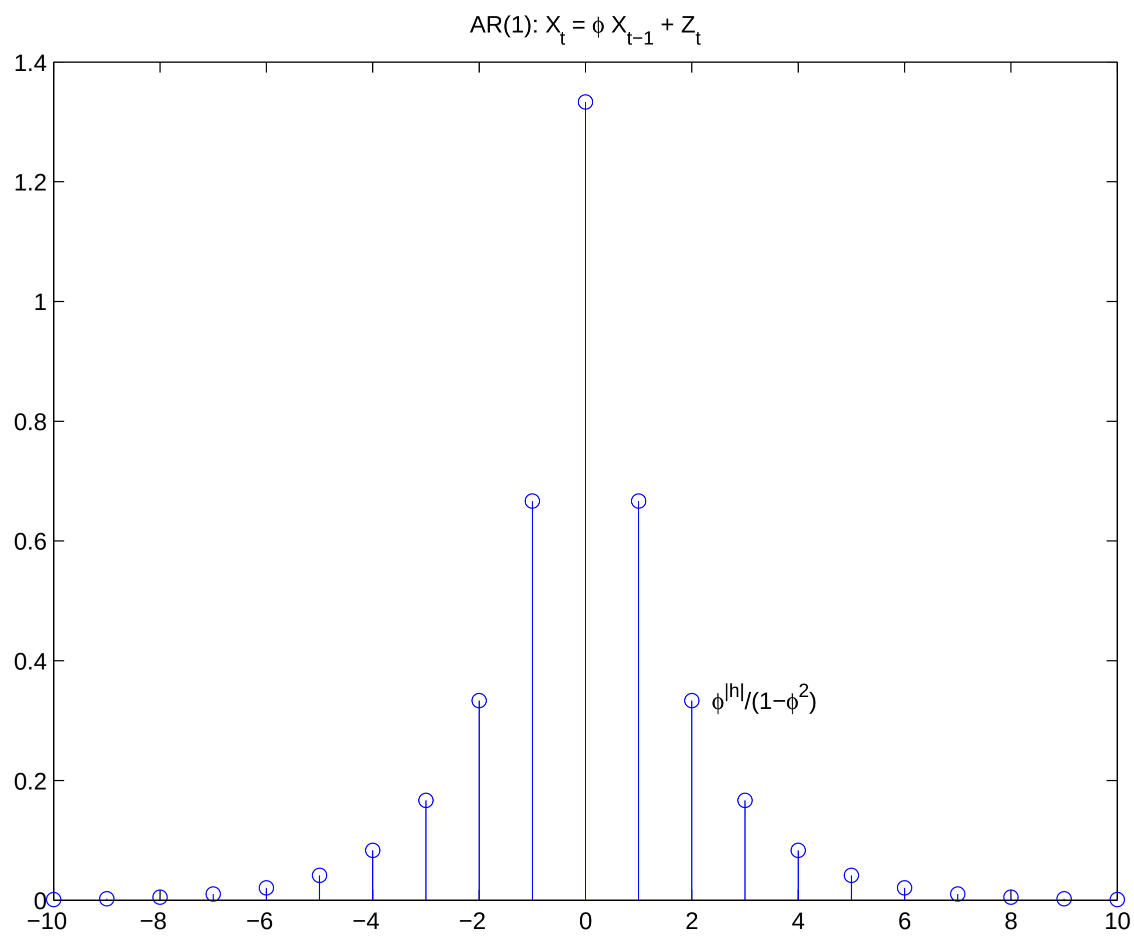
Stationarity

Example: AR(1) process, $X_t = \phi X_{t-1} + W_t, \qquad \{W_t\} \sim WN(0, \sigma^2)$.
Assume that X_t is stationary and $|\phi| < 1$. Then we have

$$E[X_t] = 0, \qquad E[X_t^2] = \frac{\sigma^2}{1-\phi^2}$$

$$\begin{aligned}
\gamma_X(h) &= \text{Cov}(\phi X_{t+h-1} + Z_{t+h}, X_t) \\
&= \phi \text{Cov}(X_{t+h-1}, X_t) \\
&= \phi \gamma_X(h-1) \\
&= \phi^{|h|} \gamma_X(0) \qquad \text{(check for } h > 0 \text{ and } h < 0\text{)} \\
&= \frac{\phi^{|h|}\sigma^2}{1-\phi^2}.
\end{aligned}$$

ACF of the AR(1) process



Linear Processes

An important class of stationary time series:

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j}$$

where $\{W_t\} \sim WN(0, \sigma_w^2)$

and μ, ψ_j are parameters satisfying

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty.$$

Linear Processes

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j}$$

We have

$$\mu_X = \mu$$

$$\gamma_X(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{h+j}. \qquad \text{(why?)}$$

Examples of Linear Processes: White noise

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j}$$

Choose μ,

$$\psi_j = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{X_t\} \sim WN(\mu, \sigma_W^2)$. (why?)

Examples of Linear Processes: MA(1)

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j}$$

Choose $\mu = 0$

$$\psi_j = \begin{cases} 1 & \text{if } j = 0, \\ \theta & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $X_t = W_t + \theta W_{t-1}$. (why?)

Examples of Linear Processes: AR(1)

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j}$$

Choose $\mu = 0$

$$\psi_j = \begin{cases} \phi^j & \text{if } j \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then for $|\phi| < 1$, we have $X_t = \phi X_{t-1} + W_t$. (why?)