Introduction to Time Series Analysis. Lecture 20.

- 1. Review: Spectral density estimation, sample autocovariance.
- 2. The periodogram and sample autocovariance.
- 3. Asymptotics of the periodogram.

Estimating the Spectrum: Outline

- We have seen that the spectral density gives an alternative view of stationary time series.
- Given a realization x_1, \ldots, x_n of a time series, how can we estimate the spectral density?
- One approach: replace $\gamma(\cdot)$ in the definition

$$f(\nu) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h},$$

with the sample autocovariance $\hat{\gamma}(\cdot)$.

• Another approach, called the *periodogram*: compute $I(\nu)$, the squared modulus of the discrete Fourier transform (at frequencies $\nu = k/n$).

Estimating the spectrum: Outline

- These two approaches are *identical* at the Fourier frequencies $\nu = k/n$.
- The asymptotic expectation of the periodogram $I(\nu)$ is $f(\nu)$. We can derive some asymptotic properties, and hence do hypothesis testing.
- Unfortunately, the asymptotic variance of I(v) is constant.
 It is not a consistent estimator of f(v).
- We can reduce the variance by smoothing the periodogram—averaging over adjacent frequencies. If we average over a narrower range as n→∞, we can obtain a consistent estimator of the spectral density.

Review: Spectral density estimation

If a time series $\{X_t\}$ has autocovariance γ satisfying $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, then we define its **spectral density** as

$$f(\nu) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h}$$

for $-\infty < \nu < \infty$.

Review: Sample autocovariance

Idea: use the sample autocovariance $\hat{\gamma}(\cdot)$, defined by

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-|h|} (x_{t+|h|} - \bar{x})(x_t - \bar{x}), \qquad \text{for } -n < h < n,$$

as an estimate of the autocovariance $\gamma(\cdot)$, and then use

$$\hat{f}(\nu) = \sum_{h=-n+1}^{n-1} \hat{\gamma}(h) e^{-2\pi i\nu h}$$

for $-1/2 \le \nu \le 1/2$.

Review: Periodogram

The periodogram is defined as

$$I(\nu) = |X(\nu)|^2$$
$$= \frac{1}{n} \left| \sum_{t=1}^n e^{-2\pi i t \nu} x_t \right|^2$$
$$= X_c^2(\nu) + X_s^2(\nu).$$

$$X_c(\nu) = \frac{1}{\sqrt{n}} \sum_{t=1}^n \cos(2\pi t\nu) x_t,$$
$$X_s(\nu) = \frac{1}{\sqrt{n}} \sum_{t=1}^n \sin(2\pi t\nu_j) x_t.$$

Periodogram

Since $I(\nu_j) = |X(\nu_j)|^2$ for one of the Fourier frequencies $\nu_j = j/n$ (for j = 0, 1, ..., n-1), the orthonormality of the e_j implies that we can write

$$x^*x = \left(\sum_{j=0}^{n-1} X(\nu_j)e_j\right)^* \left(\sum_{j=0}^{n-1} X(\nu_j)e_j\right)$$
$$= \sum_{j=0}^{n-1} |X(\nu_j)|^2 = \sum_{j=0}^{n-1} I(\nu_j).$$

We can write this as

$$\hat{\sigma}_x^2 = \frac{1}{n} \sum_{t=1}^n x_t^2 = \frac{1}{n} \sum_{j=0}^{n-1} I(\nu_j).$$

Periodogram

This is the discrete analog of the identity

$$\sigma_x^2 = \gamma_x(0) = \int_{-1/2}^{1/2} f_x(\nu) \, d\nu.$$

(Think of $I(\nu_j)$ as the discrete version of $f(\nu)$ at the frequency $\nu_j = j/n$, and think of $(1/n) \sum_{\nu_j} \cdot$ as the discrete version of $\int_{\nu} \cdot d\nu$.)

Estimating the spectrum: Periodogram

Why is the periodogram at a Fourier frequency the same as computing $f(\nu)$ from the sample autocovariance?

Almost the same—they are not the same at $\nu_0 = 0$ when $\bar{x} \neq 0$.

But if either $\bar{x} = 0$, or we consider a Fourier frequency ν_j with $j \in \{1, \ldots, n-1\}, \ldots$

Estimating the spectrum: Periodogram

$$I(\nu_j) = \frac{1}{n} \left| \sum_{t=1}^n e^{-2\pi i t\nu_j} x_t \right|^2 = \frac{1}{n} \left| \sum_{t=1}^n e^{-2\pi i t\nu_j} (x_t - \bar{x}) \right|^2$$
$$= \frac{1}{n} \left(\sum_{t=1}^n e^{-2\pi i t\nu_j} (x_t - \bar{x}) \right) \left(\sum_{t=1}^n e^{2\pi i t\nu_j} (x_t - \bar{x}) \right)$$
$$= \frac{1}{n} \sum_{s,t} e^{-2\pi i (s-t)\nu_j} (x_s - \bar{x}) (x_t - \bar{x}) = \sum_{h=-n+1}^{n-1} \hat{\gamma}(h) e^{-2\pi i h\nu_j},$$

where the fact that $\nu_j \neq 0$ implies $\sum_{t=1}^n e^{-2\pi i t \nu_j} = 0$ (we showed this when we were verifying the orthonormality of the Fourier basis) has allowed us to subtract the sample mean in that case.

We want to understand the asymptotic behavior of the periodogram $I(\nu)$ at a particular frequency ν , as n increases. We'll see that its expectation converges to $f(\nu)$.

We'll start with a simple example: Suppose that X_1, \ldots, X_n are i.i.d. $N(0, \sigma^2)$ (Gaussian white noise). From the definitions,

$$X_{c}(\nu_{j}) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \cos(2\pi t\nu_{j}) x_{t}, \qquad X_{s}(\nu_{j}) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \sin(2\pi t\nu_{j}) x_{t},$$

we have that $X_c(\nu_j)$ and $X_s(\nu_j)$ are normal, with

$$\mathbf{E}X_c(\nu_j) = \mathbf{E}X_s(\nu_j) = 0.$$

Also,

$$\operatorname{Var}(X_{c}(\nu_{j})) = \frac{\sigma^{2}}{n} \sum_{t=1}^{n} \cos^{2}(2\pi t\nu_{j})$$
$$= \frac{\sigma^{2}}{2n} \sum_{t=1}^{n} (\cos(4\pi t\nu_{j}) + 1) = \frac{\sigma^{2}}{2}$$

Similarly, $\operatorname{Var}(X_s(\nu_j)) = \sigma^2/2$.

Also,

$$\operatorname{Cov}(X_c(\nu_j), X_s(\nu_j)) = \frac{\sigma^2}{n} \sum_{t=1}^n \cos(2\pi t\nu_j) \sin(2\pi t\nu_j)$$
$$= \frac{\sigma^2}{2n} \sum_{t=1}^n \sin(4\pi t\nu_j) = 0,$$
$$\operatorname{Cov}(X_c(\nu_j), X_c(\nu_k)) = 0$$
$$\operatorname{Cov}(X_s(\nu_j), X_s(\nu_k)) = 0.$$
$$\operatorname{Cov}(X_c(\nu_j), X_s(\nu_k)) = 0.$$

for any $j \neq k$.

That is, if X_1, \ldots, X_n are i.i.d. $N(0, \sigma^2)$ (Gaussian white noise; $f(\nu) = \sigma^2$), then the $X_c(\nu_j)$ and $X_s(\nu_j)$ are all i.i.d. $N(0, \sigma^2/2)$. Thus,

$$\frac{2}{\sigma^2}I(\nu_j) = \frac{2}{\sigma^2} \left(X_c^2(\nu_j) + X_s^2(\nu_j) \right) \sim \chi_2^2.$$

So for the case of Gaussian white noise, the periodogram has a chi-squared distribution that depends on the variance σ^2 (which, in this case, is the spectral density).

Under more general conditions (e.g., normal $\{X_t\}$, or linear process $\{X_t\}$ with rapidly decaying ACF), the $X_c(\nu_j)$, $X_s(\nu_j)$ are all asymptotically independent and $N(0, f(\nu_j)/2)$.

Consider a frequency ν . For a given value of n, let $\hat{\nu}^{(n)}$ be the closest Fourier frequency (that is, $\hat{\nu}^{(n)} = j/n$ for a value of j that minimizes $|\nu - j/n|$). As n increases, $\hat{\nu}^{(n)} \to \nu$, and (under the same conditions that ensure the asymptotic normality and independence of the sine/cosine transforms), $f(\hat{\nu}^{(n)}) \to f(\nu)$.

In that case, we have

$$\frac{2}{f(\nu)}I(\hat{\nu}^{(n)}) = \frac{2}{f(\nu)} \left(X_c^2(\hat{\nu}^{(n)}) + X_s^2(\hat{\nu}^{(n)}) \right) \xrightarrow{d} \chi_2^2.$$

Thus,

$$\mathbf{E}I(\hat{\nu}^{(n)}) = \frac{f(\nu)}{2} \mathbf{E}\left(\frac{2}{f(\nu)} \left(X_c^2(\hat{\nu}^{(n)}) + X_s^2(\hat{\nu}^{(n)})\right)\right) \to \frac{f(\nu)}{2} \mathbf{E}(Z_1^2 + Z_2^2) = f(\nu),$$

where Z_1, Z_2 are independent N(0, 1). Thus, the periodogram is asymptotically unbiased.

Since we know its asymptotic distribution (chi-squared), we can compute approximate confidence intervals:

$$\Pr\left\{\frac{2}{f(\nu)}I(\hat{\nu}^{(n)}) > \chi_2^2(\alpha)\right\} \to \alpha,$$

where the cdf of a χ_2^2 at $\chi_2^2(\alpha)$ is $1 - \alpha$. Thus,

$$\Pr\left\{\frac{2I(\hat{\nu}^{(n)})}{\chi_2^2(\alpha/2)} \le f(\nu) \le \frac{2I(\hat{\nu}^{(n)})}{\chi_2^2(1-\alpha/2)}\right\} \to 1-\alpha.$$

Asymptotic properties of the periodogram: Consistency

Unfortunately, $Var(I(\hat{\nu}^{(n)})) \rightarrow f(\nu)^2 Var(Z_1^2 + Z_2^2)/4$, where Z_1, Z_2 are i.i.d. N(0, 1), that is, the variance approaches a constant.

Thus, $I(\hat{\nu}^{(n)})$ is not a consistent estimator of $f(\nu)$. In particular, if $f(\nu) > 0$, then for $\epsilon > 0$, as *n* increases,

$$\Pr\left\{\left|I(\hat{\nu}^{(n)}) - f(\nu)\right| > \epsilon\right\}$$

approaches a constant.

Asymptotic properties of the periodogram: Consistency

This means that the approximate confidence intervals we obtain are typically wide.

The source of the difficulty is that, as n increases, we have additional data (the n values of x_t), but we use it to estimate additional independent random variables, (the n independent values of $X_c(\nu_j)$, $X_s(\nu_j)$).

How can we reduce the variance? The typical approach is to average independent observations. In this case, we can take an average of "nearby" values of the periodogram, and hope that the spectral density at the frequency of interest and at those nearby frequencies will be close.