Introduction to Time Series Analysis. Lecture 18.

1. Review: Spectral density. Spectral distribution function.
2. Rational spectra. Poles and zeros.
3. Examples.
4. Linear filters.
5. Frequency response.

Review: Spectral density and spectral distribution function

If a time series $\{X_t\}$ has autocovariance γ satisfying $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, then we define its **spectral density** as

$$f(\nu) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h}$$

for $-\infty < \nu < \infty$. We have

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i \nu h} f(\nu)\, d\nu = \int_{-1/2}^{1/2} e^{2\pi i \nu h}\, dF(\nu),$$

where $dF(\nu) = f(\nu) d\nu$.

f measures how the variance of X_t is distributed across the spectrum.

Review: Spectral density and spectral distribution function

For any stationary $\{X_t\}$ with autocovariance γ, we can write

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i\nu h} dF(\nu),$$

where F is the *spectral distribution function* of $\{X_t\}$.

If F has no singular part, we can write $F = F^{(c)} + F^{(d)}$, where $F^{(c)}$ is absolutely continuous with respect to Lebesgue measure, that is, $dF^{(c)}(\nu) = f(\nu)d\nu$, and $F^{(d)}$ is discrete.

Review: Spectral density of a linear process

If X_t is a linear process, it can be written $X_t = \sum_{i=0}^{\infty} \psi_i W_{t-i} = \psi(B) W_t$. Then

$$f(\nu) = \sigma_w^2 \left| \psi\left(e^{-2\pi i \nu}\right) \right|^2 .$$

That is, the spectral density $f(\nu)$ of a linear process measures the modulus of the ψ (MA(∞)) polynomial at the point $e^{2\pi i \nu}$ on the unit circle.

Spectral density of a linear process

For an ARMA(p,q), $\psi(B) = \theta(B)/\phi(B)$, so

$$f(\nu) = \sigma_w^2 \frac{\theta(e^{-2\pi i\nu})\theta(e^{2\pi i\nu})}{\phi\left(e^{-2\pi i\nu}\right)\phi\left(e^{2\pi i\nu}\right)}$$

$$= \sigma_w^2 \left| \frac{\theta(e^{-2\pi i\nu})}{\phi\left(e^{-2\pi i\nu}\right)} \right|^2 .$$

This is known as a *rational spectrum*.

Rational spectra

Consider the factorization of θ and ϕ as

$$\theta(z) = \theta_q (z - z_1)(z - z_2) \cdots (z - z_q)$$
$$\phi(z) = \phi_p (z - p_1)(z - p_2) \cdots (z - p_p),$$

where $z_1, \ldots, z_q$ and $p_1, \ldots, p_p$ are called the *zeros* and *poles*.

$$f(\nu) = \sigma_w^2 \left| \frac{\theta_q \prod_{j=1}^{q} (e^{-2\pi i\nu} - z_j)}{\phi_p \prod_{j=1}^{p} (e^{-2\pi i\nu} - p_j)} \right|^2$$
$$= \sigma_w^2 \frac{\theta_q^2 \prod_{j=1}^{q} \left| e^{-2\pi i\nu} - z_j \right|^2}{\phi_p^2 \prod_{j=1}^{p} \left| e^{-2\pi i\nu} - p_j \right|^2}.$$

Rational spectra

$$f(\nu) = \sigma_w^2 \frac{\theta_q^2 \prod_{j=1}^{q} \left| e^{-2\pi i \nu} - z_j \right|^2}{\phi_p^2 \prod_{j=1}^{p} \left| e^{-2\pi i \nu} - p_j \right|^2}.$$

As ν varies from 0 to $1/2$, $e^{-2\pi i \nu}$ moves clockwise around the unit circle from 1 to $e^{-\pi i} = -1$.
And the value of $f(\nu)$ goes up as this point moves closer to (further from) the poles p_j (zeros z_j).

Example: ARMA

Recall AR(1): $\phi(z) = 1 - \phi_1 z$. The pole is at $1/\phi_1$. If $\phi_1 > 0$, the pole is to the right of 1, so the spectral density decreases as ν moves away from 0. If $\phi_1 < 0$, the pole is to the left of -1, so the spectral density is at its maximum when $\nu = 0.5$.

Recall MA(1): $\theta(z) = 1 + \theta_1 z$. The zero is at $-1/\theta_1$. If $\theta_1 > 0$, the zero is to the left of -1, so the spectral density decreases as ν moves towards -1. If $\theta_1 < 0$, the zero is to the right of 1, so the spectral density is at its minimum when $\nu = 0$.

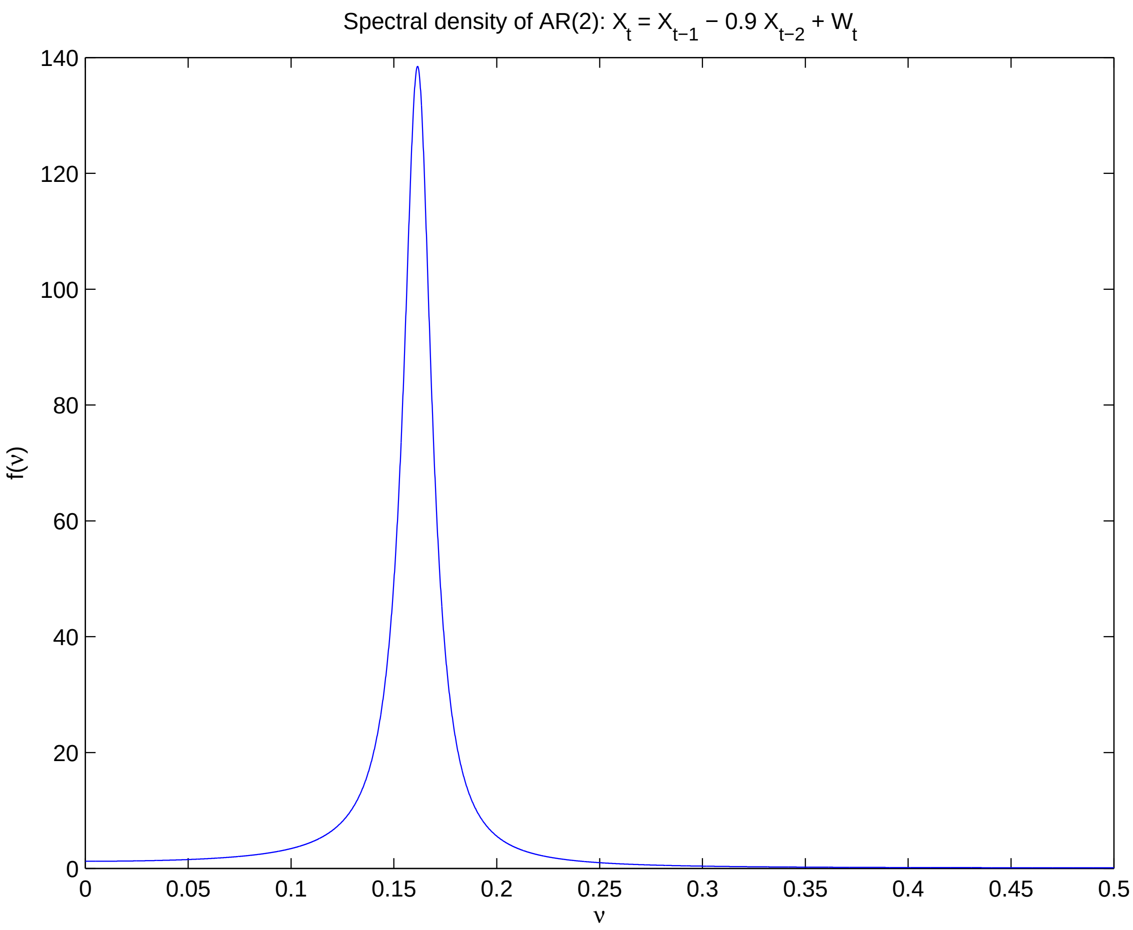
Example: AR(2)

Consider $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t$. Example 3.5 in the text considers this model with $\phi_1 = 1$, $\phi_2 = -0.9$, and $\sigma_w^2 = 1$. In this case, the poles are at $p_1, p_2 \approx 0.5555 \pm i0.8958 \approx 1.054e^{\pm i1.01567} \approx 1.054e^{\pm 2\pi i0.16165}$.
Thus, we have

$$f(\nu) = \frac{\sigma_2^2}{|e^{-2\pi i\nu} - p_1|^2 |e^{-2\pi i\nu} - p_2|^2},$$

and this gets very peaked when $e^{-2\pi i\nu}$ passes near $1.054e^{-2\pi i0.16165}$.

Example: AR(2)



Spectral density of AR(2): $X_t = X_{t-1} - 0.9\,X_{t-2} + W_t$
140
120
100
80
60
40
20
0
f(ν)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
ν

Example: Seasonal ARMA

Consider $X_t = \Phi_1 X_{t-12} + W_t$.

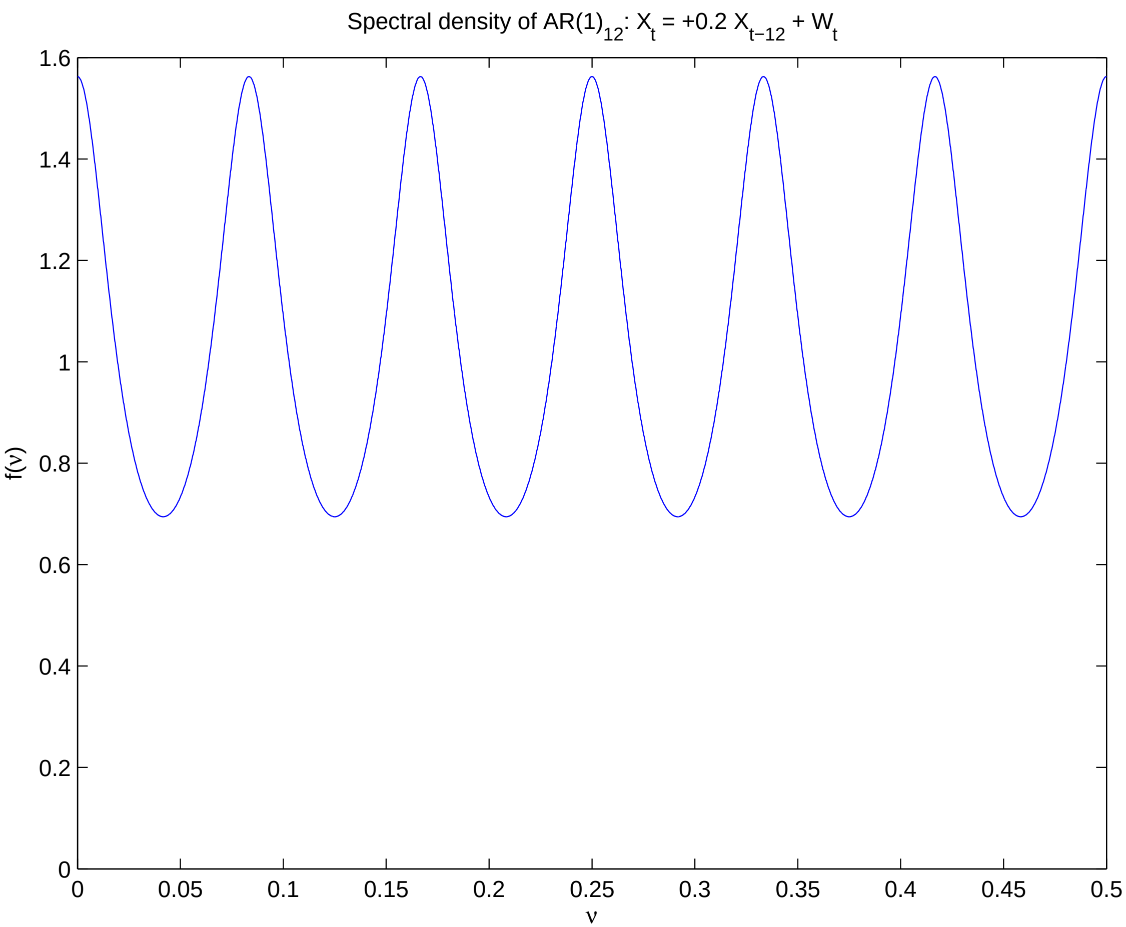
$$\psi(B) = \frac{1}{1 - \Phi_1 B^{12}},$$

$$f(\nu) = \sigma_w^2 \frac{1}{(1 - \Phi_1 e^{-2\pi i 12\nu})(1 - \Phi_1 e^{2\pi i 12\nu})}$$

$$= \sigma_w^2 \frac{1}{1 - 2\Phi_1 \cos(24\pi\nu) + \Phi_1^2}.$$

Notice that $f(\nu)$ is periodic with period $1/12$.

Example: Seasonal ARMA



Example: Seasonal ARMA

Another view:

$$1 - \Phi_1 z^{12} = 0 \quad \Leftrightarrow \quad z = re^{i\theta},$$

$$\text{with} \qquad r = |\Phi_1|^{-1/12}, \qquad e^{i12\theta} = e^{-i\arg(\Phi_1)}.$$

For $\Phi_1 > 0$, the twelve poles are at $|\Phi_1|^{-1/12} e^{ik\pi/6}$ for $k = 0, \pm 1, \ldots, \pm 5, 6$.

So the spectral density gets peaked as $e^{-2\pi i\nu}$ passes near $|\Phi_1|^{-1/12} \times \left\{1, e^{-i\pi/6}, e^{-i\pi/3}, e^{-i\pi/2}, e^{-i2\pi/3}, e^{-i5\pi/6}, -1\right\}$.

Example: Multiplicative seasonal ARMA

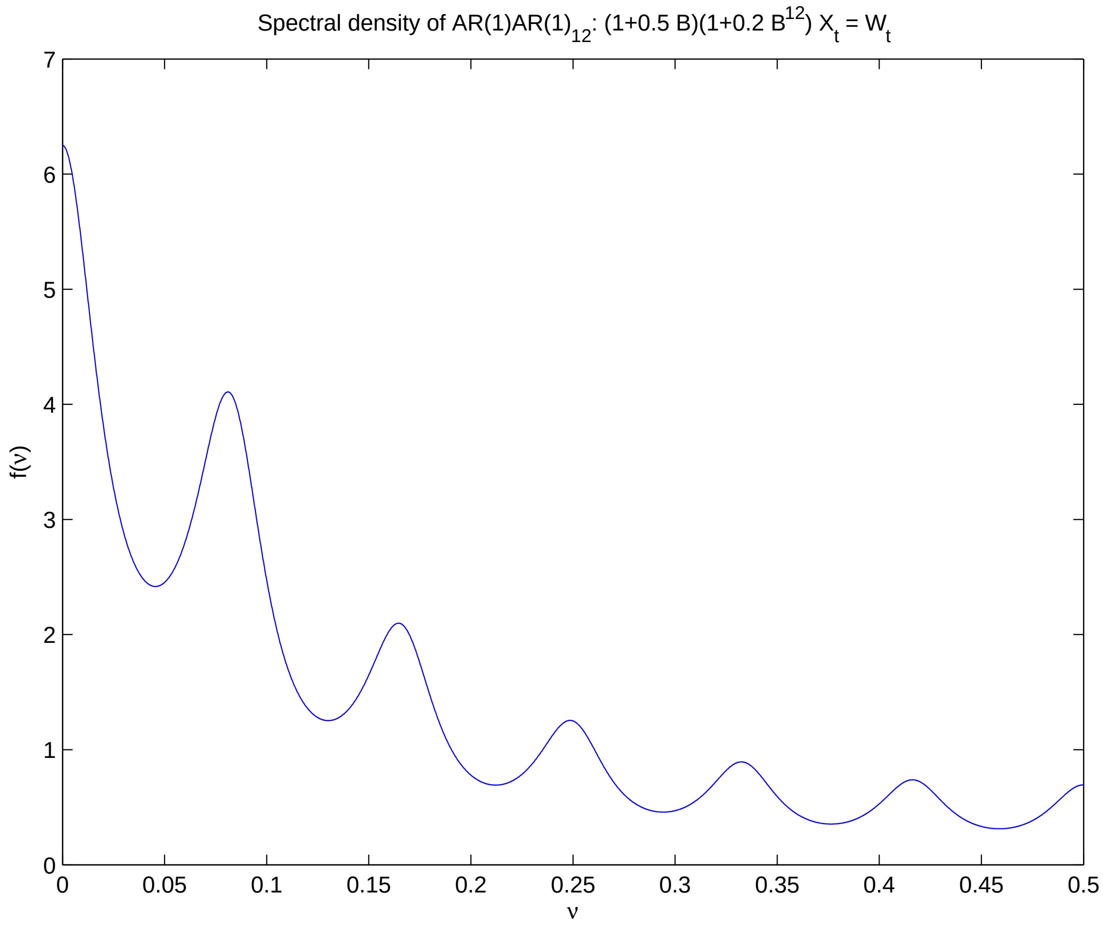
Consider $(1 - \Phi_1 B^{12})(1 - \phi_1 B)X_t = W_t$.

$$f(\nu) = \sigma_w^2 \frac{1}{(1 - 2\Phi_1 \cos(24\pi\nu) + \Phi_1^2)(1 - 2\phi_1 \cos(2\pi\nu) + \phi_1^2)}.$$

This is a scaled product of the AR(1) spectrum and the (periodic) AR(1)$_{12}$ spectrum.

The AR(1)$_{12}$ poles give peaks when $e^{-2\pi i\nu}$ is at one of the 12th roots of 1; the AR(1) poles give a peak near $e^{-2\pi i\nu} = 1$.

Example: Multiplicative seasonal ARMA



Time-invariant linear filters

A filter is an operator; given a time series $\{X_t\}$, it maps to a time series $\{Y_t\}$. We can think of a linear process $X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j}$ as the output of a *causal linear filter* with a white noise input.

A time series $\{Y_t\}$ is the output of a linear filter $A = \{a_{t,j} : t, j \in \mathbb{Z}\}$ with input $\{X_t\}$ if

$$Y_t = \sum_{j=-\infty}^{\infty} a_{t,j} X_j.$$

If $a_{t,t-j}$ is independent of t ($a_{t,t-j} = \psi_j$), then we say that the filter is *time-invariant*.
If $\psi_j = 0$ for $j < 0$, we say the filter ψ is *causal*.

We'll see that the name 'filter' arises from the frequency domain viewpoint.

Time-invariant linear filters: Examples

1. $Y_t = X_{-t}$ is linear, but not time-invariant.
2. $Y_t = \frac{1}{3}(X_{t-1} + X_t + X_{t+1})$ is linear, time-invariant, but not causal:

$$\psi_j = \begin{cases} \frac{1}{3} & \text{if } |j| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

3. For polynomials $\phi(B), \theta(B)$ with roots outside the unit circle, $\psi(B) = \theta(B)/\phi(B)$ is a linear, time-invariant, causal filter.

Time-invariant linear filters

The operation

$$\sum_{j=-\infty}^{\infty} \psi_j X_{t-j}$$

is called the *convolution* of X with ψ.

Time-invariant linear filters

The sequence ψ is also called the *impulse response*, since the output $\{Y_t\}$ of the linear filter in response to a *unit impulse*,

$$X_t = \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{otherwise,} \end{cases}$$

is

$$Y_t = \psi(B)X_t = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j} = \psi_t.$$

Frequency response of a time-invariant linear filter

Suppose that $\{X_t\}$ has spectral density $f_x(\nu)$ and ψ is *stable*, that is, $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$. Then $Y_t = \psi(B)X_t$ has spectral density

$$f_y(\nu) = \left|\psi\left(e^{2\pi i\nu}\right)\right|^2 f_x(\nu).$$

The function $\nu \mapsto \psi(e^{2\pi i\nu})$ (the polynomial $\psi(z)$ evaluated on the unit circle) is known as the *frequency response* or *transfer function* of the linear filter.

The squared modulus, $\nu \mapsto |\psi(e^{2\pi i\nu})|^2$ is known as the *power transfer function* of the filter.

Frequency response of a time-invariant linear filter

For stable ψ, $Y_t = \psi(B)X_t$ has spectral density

$$f_y(\nu) = \left|\psi\left(e^{2\pi i\nu}\right)\right|^2 f_x(\nu).$$

We have seen that a linear process, $Y_t = \psi(B)W_t$, is a special case, since $f_y(\nu) = |\psi(e^{2\pi i\nu})|^2\sigma_w^2 = |\psi(e^{2\pi i\nu})|^2 f_w(\nu)$.

When we pass a time series $\{X_t\}$ through a linear filter, the spectral density is multiplied, frequency-by-frequency, by the squared modulus of the frequency response $\nu \mapsto |\psi(e^{2\pi i\nu})|^2$.

This is a version of the equality $\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$, but the equality is true for the component of the variance at every frequency.

This is also the origin of the name ‘filter.’

Frequency response of a time-invariant linear filter

Why is $f_y(\nu) = \left|\psi\left(e^{2\pi i\nu}\right)\right|^2 f_x(\nu)$? First,

$$\gamma_y(h) = \mathrm{E}\left[\sum_{j=-\infty}^{\infty} \psi_j X_{t-j} \sum_{k=-\infty}^{\infty} \psi_k X_{t+h-k}\right]$$

$$= \sum_{j=-\infty}^{\infty} \psi_j \sum_{k=-\infty}^{\infty} \psi_k \mathrm{E}\left[X_{t+h-k} X_{t-j}\right]$$

$$= \sum_{j=-\infty}^{\infty} \psi_j \sum_{k=-\infty}^{\infty} \psi_k \gamma_x(h+j-k) = \sum_{j=-\infty}^{\infty} \psi_j \sum_{l=-\infty}^{\infty} \psi_{h+j-l} \gamma_x(l).$$

It is easy to check that $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ and $\sum_{h=-\infty}^{\infty} |\gamma_x(h)| < \infty$ imply that $\sum_{h=-\infty}^{\infty} |\gamma_y(h)| < \infty$. Thus, the spectral density of y is defined.

Frequency response of a time-invariant linear filter

$$
\begin{aligned}
f_y(\nu) &= \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h} \\
&= \sum_{h=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \psi_j \sum_{l=-\infty}^{\infty} \psi_{h+j-l} \gamma_x(l) e^{-2\pi i \nu h} \\
&= \sum_{j=-\infty}^{\infty} \psi_j e^{2\pi i \nu j} \sum_{l=-\infty}^{\infty} \gamma_x(l) e^{-2\pi i \nu l} \sum_{h=-\infty}^{\infty} \psi_{h+j-l} e^{-2\pi i \nu (h+j-l)} \\
&= \psi(e^{2\pi i \nu j}) f_x(\nu) \sum_{h=-\infty}^{\infty} \psi_h e^{-2\pi i \nu h} \\
&= \left| \psi(e^{2\pi i \nu j}) \right|^2 f_x(\nu).
\end{aligned}
$$