Introduction to Time Series Analysis. Lecture 17.

1. Review: Spectral density. Examples.
2. Spectral distribution function.
3. Wold's decomposition.
4. Autocovariance generating function and spectral density.

Review: Spectral density

If a time series $\{X_t\}$ has autocovariance γ satisfying $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, then we define its **spectral density** as

$$f(\nu) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h}$$

for $-\infty < \nu < \infty$. We have

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i \nu h} f(\nu)\, d\nu.$$

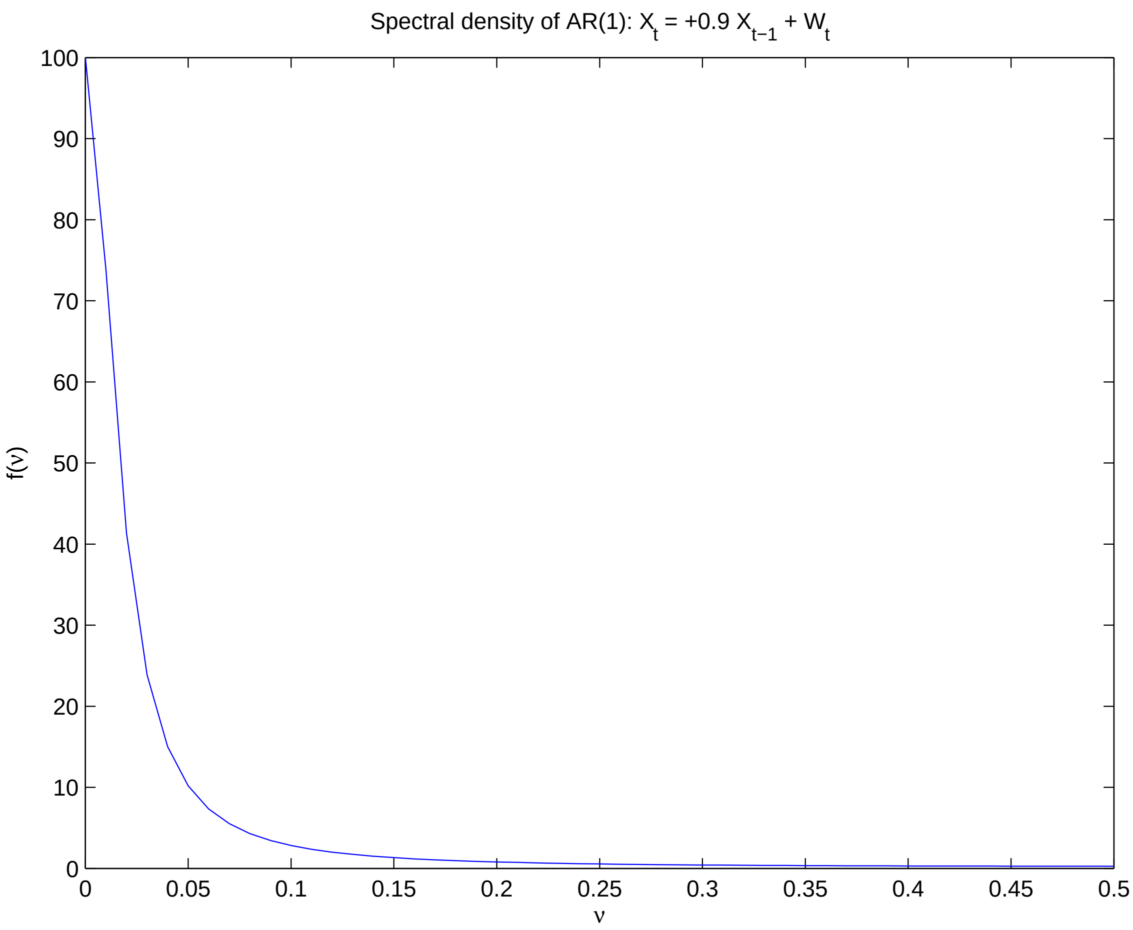
Review: Examples

White noise: $\{W_t\}$, $\gamma(0) = \sigma_w^2$ and $\gamma(h) = 0$ for $h \neq 0$.
$f(\nu) = \gamma(0) = \sigma_w^2$.

AR(1): $X_t = \phi_1 X_{t-1} + W_t$, $\gamma(h) = \sigma_w^2 \phi_1^{|h|}/(1 - \phi_1^2)$.
$f(\nu) = \frac{\sigma_w^2}{1 - 2\phi_1 \cos(2\pi\nu) + \phi_1^2}$.

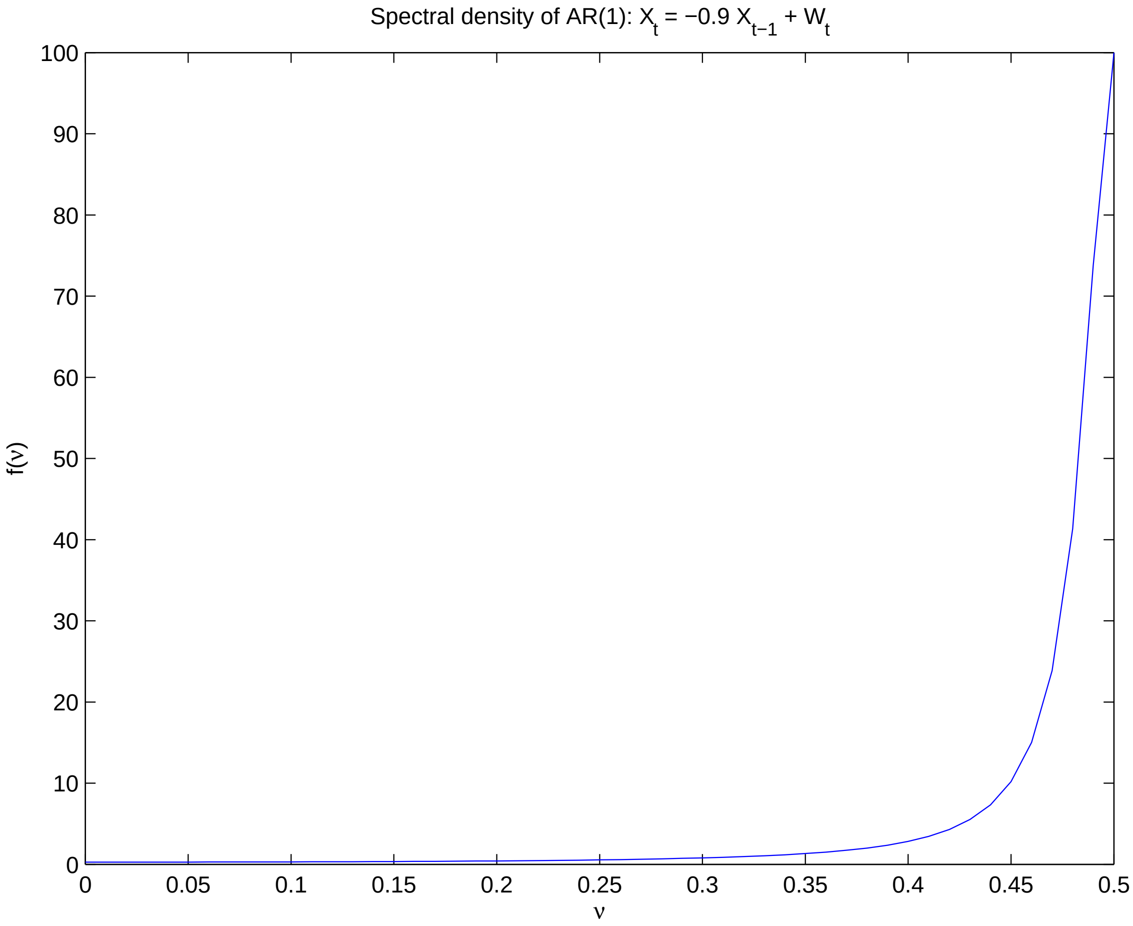
If $\phi_1 > 0$ (positive autocorrelation), spectrum is dominated by low frequency components—smooth in the time domain.
If $\phi_1 < 0$ (negative autocorrelation), spectrum is dominated by high frequency components—rough in the time domain.

Example: AR(1)



Spectral density of AR(1): X_t = +0.9 X_{t−1} + W_t
f(ν)
ν
100
90
80
70
60
50
40
30
20
10
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

Example: AR(1)



Spectral density of AR(1): X_t = −0.9 X_{t−1} + W_t
f(ν)
ν
100
90
80
70
60
50
40
30
20
10
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

Example: MA(1)

$$X_t = W_t + \theta_1 W_{t-1}.$$

$$\gamma(h) = \begin{cases} \sigma_w^2(1+\theta_1^2) & \text{if } h = 0, \\ \sigma_w^2\theta_1 & \text{if } |h| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} f(\nu) &= \sum_{h=-1}^{1} \gamma(h) e^{-2\pi i \nu h} \\ &= \gamma(0) + 2\gamma(1)\cos(2\pi\nu) \\ &= \sigma_w^2 \left(1 + \theta_1^2 + 2\theta_1 \cos(2\pi\nu)\right). \end{aligned}$$

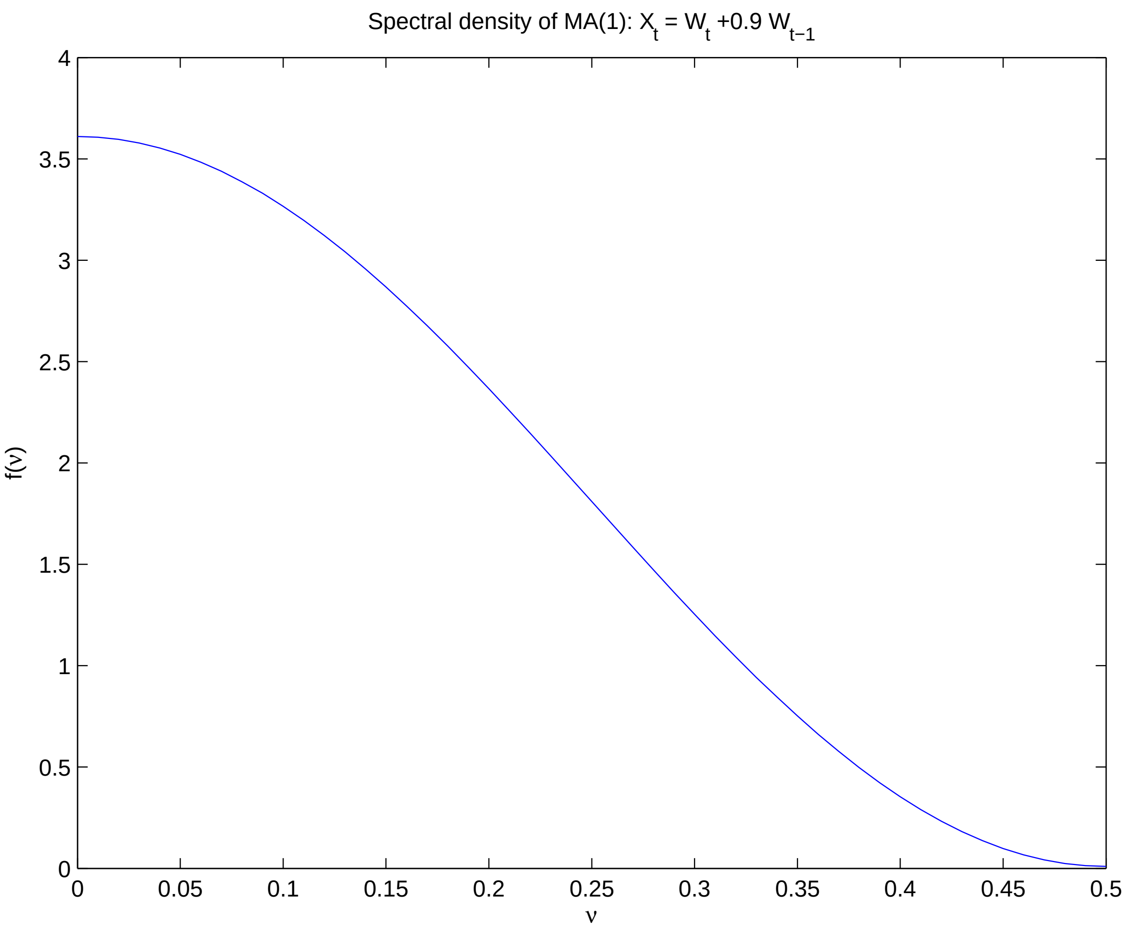
Example: MA(1)

$$X_t = W_t + \theta_1 W_{t-1}.$$

$$f(\nu) = \sigma_w^2 \left(1 + \theta_1^2 + 2\theta_1 \cos(2\pi\nu)\right).$$

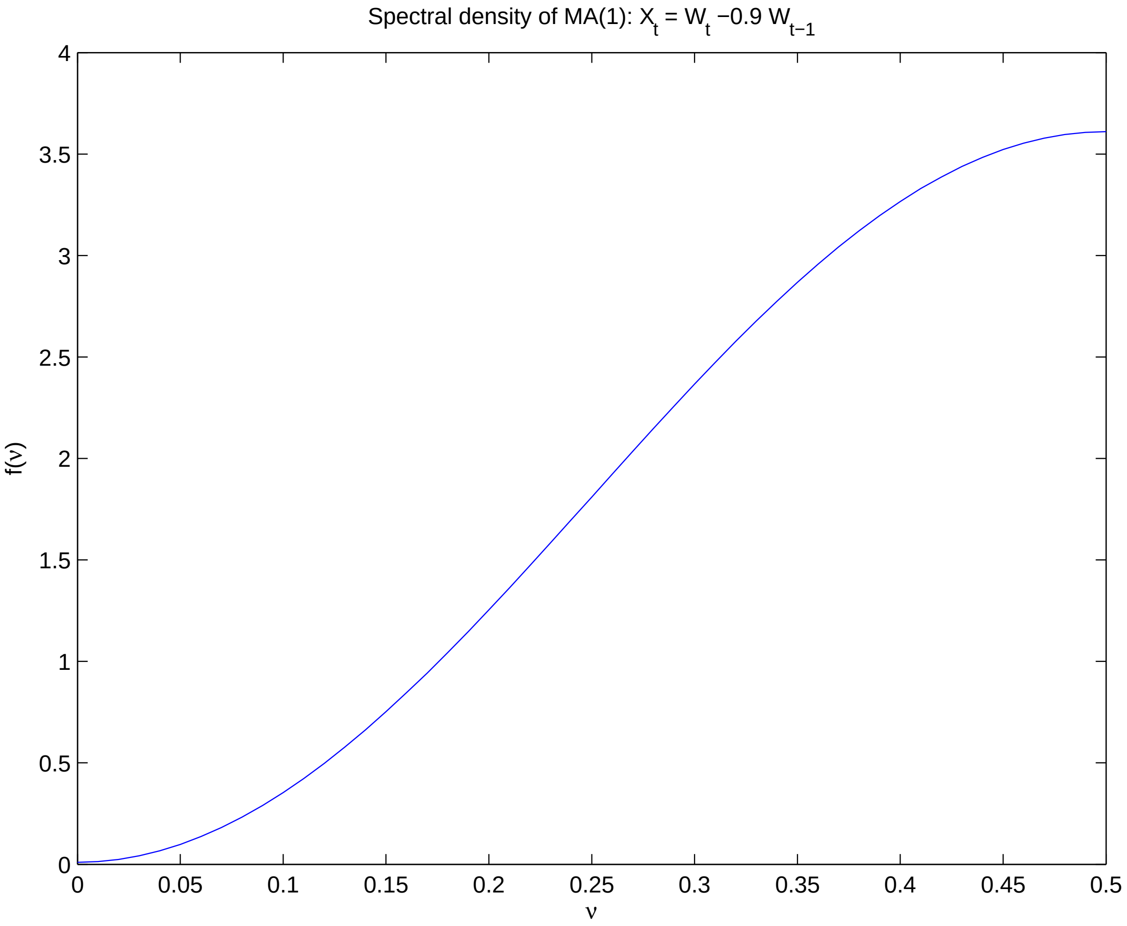
If $\theta_1 > 0$ (positive autocorrelation), spectrum is dominated by low frequency components—smooth in the time domain.
If $\theta_1 < 0$ (negative autocorrelation), spectrum is dominated by high frequency components—rough in the time domain.

Example: MA(1)



Spectral density of MA(1): $X_t = W_t +0.9\ W_{t-1}$
f(ν)
ν
4
3.5
3
2.5
2
1.5
1
0.5
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

Example: MA(1)



Spectral density of MA(1): $X_t = W_t - 0.9\, W_{t-1}$
f(ν)
ν
0
0.5
1
1.5
2
2.5
3
3.5
4
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

Review: A periodic time series

$$X_t = \sum_{j=1}^{k} (A_j \sin(2\pi\nu_j t) + B_j \cos(2\pi\nu_j t))$$

$$= \sum_{j=1}^{k} (A_j^2 + B_j^2)^{1/2} \sin(2\pi\nu_j t + \tan^{-1}(B_j/A_j)).$$

$$\mathrm{E}[X_t] = 0$$

$$\gamma(h) = \sum_{j=1}^{k} \sigma_j^2 \cos(2\pi\nu_j h)$$

$$\sum_h |\gamma(h)| = \infty.$$

Discrete spectral distribution function

For $X_t = A\sin(2\pi\lambda t) + B\cos(2\pi\lambda t)$, we have $\gamma(h) = \sigma^2\cos(2\pi\lambda h)$, and we can write

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i\nu h} dF(\nu),$$

where F is the discrete distribution

$$F(\nu) = \begin{cases} 0 & \text{if } \nu < -\lambda, \\ \frac{\sigma^2}{2} & \text{if } -\lambda \le \nu < \lambda, \\ \sigma^2 & \text{otherwise.} \end{cases}$$

The spectral distribution function

For any stationary $\{X_t\}$ with autocovariance γ, we can write

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i \nu h} dF(\nu),$$

where F is the *spectral distribution function* of $\{X_t\}$.

We can split F into three components: discrete, continuous, and singular.

If γ is absolutely summable, F is continuous: $dF(\nu) = f(\nu)d\nu$.

If γ is a sum of sinusoids, F is discrete.

The spectral distribution function

For $X_t = \sum_{j=1}^k (A_j \sin(2\pi\nu_j t) + B_j \cos(2\pi\nu_j t))$, the spectral distribution function is $F(\nu) = \sum_{j=1}^k \sigma_j^2 F_j(\nu)$, where

$$F_j(\nu) = \begin{cases} 0 & \text{if } \nu < -\nu_j, \\ \frac{1}{2} & \text{if } -\nu_j \le \nu < \nu_j, \\ 1 & \text{otherwise.} \end{cases}$$

Wold's decomposition

Notice that $X_t = \sum_{j=1}^{k} \left(A_j \sin(2\pi\nu_j t) + B_j \cos(2\pi\nu_j t) \right)$ is deterministic (once we've seen the past, we can predict the future without error).

Wold showed that every stationary process can be represented as

$$X_t = X_t^{(d)} + X_t^{(n)},$$

where $X_t^{(d)}$ is purely deterministic and $X_t^{(n)}$ is purely nondeterministic. (c.f. the decomposition of a spectral distribution function as $F^{(d)} + F^{(c)}$.)

Example: $X_t = A \sin(2\pi\lambda t) + \frac{\theta(B)}{\phi(B)} W_t$.

Autocovariance generating function and spectral density

Suppose X_t is a linear process, so it can be written
$X_t = \sum_{i=0}^{\infty} \psi_i W_{t-i} = \psi(B) W_t$.
Consider the autocovariance sequence,

$$\begin{aligned}\gamma_h &= \mathrm{Cov}(X_t, X_{t+h}) \\ &= \mathrm{E}\left[\sum_{i=0}^{\infty} \psi_i W_{t-i} \sum_{j=0}^{\infty} \psi_j W_{t+h-j}\right] \\ &= \sigma_w^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+h}.\end{aligned}$$

Autocovariance generating function and spectral density

Define the autocovariance generating function as

$$\gamma(B) = \sum_{h=-\infty}^{\infty} \gamma_h B^h.$$

Then,

$$\begin{aligned}\gamma(B) &= \sigma_w^2 \sum_{h=-\infty}^{\infty} \sum_{i=0}^{\infty} \psi_i \psi_{i+h} B^h \\ &= \sigma_w^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \psi_i \psi_j B^{j-i} \\ &= \sigma_w^2 \sum_{i=0}^{\infty} \psi_i B^{-i} \sum_{j=0}^{\infty} \psi_j B^j = \sigma_w^2 \psi(B^{-1}) \psi(B).\end{aligned}$$

Autocovariance generating function and spectral density

Notice that

$$\gamma(B) = \sum_{h=-\infty}^{\infty} \gamma_h B^h.$$

$$\begin{aligned} f(\nu) &= \sum_{h=-\infty}^{\infty} \gamma_h e^{-2\pi i \nu h} \\ &= \gamma\left(e^{-2\pi i \nu}\right) \\ &= \sigma_w^2 \psi\left(e^{-2\pi i \nu}\right) \psi\left(e^{2\pi i \nu}\right) \\ &= \sigma_w^2 \left|\psi\left(e^{2\pi i \nu}\right)\right|^2. \end{aligned}$$

Autocovariance generating function and spectral density

For example, for an MA(q), we have $\psi(B) = \theta(B)$, so

$$f(\nu) = \sigma_w^2 \theta\left(e^{-2\pi i \nu}\right) \theta\left(e^{2\pi i \nu}\right)$$
$$= \sigma_w^2 \left|\theta\left(e^{-2\pi i \nu}\right)\right|^2 .$$

For MA(1),

$$f(\nu) = \sigma_w^2 \left|1 + \theta_1 e^{-2\pi i \nu}\right|^2$$
$$= \sigma_w^2 \left|1 + \theta_1 \cos(-2\pi\nu) + i\theta_1 \sin(-2\pi\nu)\right|^2$$
$$= \sigma_w^2 \left(1 + 2\theta_1 \cos(2\pi\nu) + \theta_1^2\right) .$$

Autocovariance generating function and spectral density

For an AR(p), we have $\psi(B) = 1/\phi(B)$, so

$$f(\nu) = \frac{\sigma_w^2}{\phi\left(e^{-2\pi i\nu}\right)\phi\left(e^{2\pi i\nu}\right)}$$

$$= \frac{\sigma_w^2}{\left|\phi\left(e^{-2\pi i\nu}\right)\right|^2}.$$

For AR(1),

$$f(\nu) = \frac{\sigma_w^2}{\left|1 - \phi_1 e^{-2\pi i\nu}\right|^2}$$

$$= \frac{\sigma_w^2}{1 - 2\phi_1\cos(2\pi\nu) + \phi_1^2}.$$