Introduction to Time Series Analysis. Lecture 16.

1. Review: ARIMA and Seasonal ARMA

2. Seasonal ARIMA models

3. Spectral Analysis

Review: Integrated ARMA Models: ARIMA(p,d,q)

For $p, d, q \geq 0$, we say that a time series $\{X_t\}$ is an **ARIMA (p,d,q) process** if $Y_t = \nabla^d X_t = (1 - B)^d X_t$ is ARMA(p,q). We can write

$$\phi(B)(1 - B)^d X_t = \theta(B) W_t.$$

Review: Pure seasonal ARMA Models

For $P, Q \geq 0$ and $s > 0$, we say that a time series $\{X_t\}$ is an **ARMA(P,Q)$_s$ process** if $\Phi(B^s)X_t = \Theta(B^s)W_t$, where

$$\Phi(B^s) = 1 - \sum_{j=1}^{P} \Phi_j B^{js},$$

$$\Theta(B^s) = 1 + \sum_{j=1}^{Q} \Theta_j B^{js}.$$

Pure seasonal ARMA Models

The ACF and PACF for a seasonal ARMA(P,Q)$_s$ are zero for $h \neq si$. For $h = si$, they are analogous to the patterns for ARMA(p,q):

Model:	**ACF:**	**PACF:**
AR(P)$_s$	decays	zero for $i > P$
MA(Q)$_s$	zero for $i > Q$	decays
ARMA(P,Q)$_s$	decays	decays

Multiplicative seasonal ARMA Models

For $p, q, P, Q \geq 0$ and $s > 0$, we say that a time series $\{X_t\}$ is a **multiplicative seasonal ARMA model** (ARMA(p,q)$\times$(P,Q)$_s$) if $\Phi(B^s)\phi(B)X_t = \Theta(B^s)\theta(B)W_t$.

If, in addition, $d, D > 0$, we define the **multiplicative seasonal ARIMA model** (ARIMA(p,d,q)$\times$(P,D,Q)$_s$)

$$\Phi(B^s)\phi(B)\nabla_s^D \nabla^d X_t = \Theta(B^s)\theta(B)W_t,$$

where the *seasonal difference operator of order* D is defined by

$$\nabla_s^D X_t = (1 - B^s)^D X_t.$$

Multiplicative seasonal ARMA Models

Notice that these can all be represented by polynomials

$$\Phi(B^s)\phi(B)\nabla_s^D\nabla^d = \Xi(B), \qquad \Theta(B^s)\theta(B) = \Lambda(B).$$

But the difference operators imply that $\Xi(B)X_t = \Lambda(B)W_t$ does not define a stationary ARMA process (the AR polynomial has roots on the unit circle). And representing $\Phi(B^s)\phi(B)$ and $\Theta(B^s)\theta(B)$ as arbitrary polynomials is not as compact.

How do we choose p, q, P, Q, d, D?

First difference sufficiently to get to stationarity. Then find suitable orders for ARMA or seasonal ARMA models for the differenced time series. The ACF and PACF is again a useful tool here.

Spectral Analysis

Idea: decompose a stationary time series $\{X_t\}$ into a combination of sinusoids, with random (and uncorrelated) coefficients.

Just as in Fourier analysis, where we decompose (deterministic) functions into combinations of sinusoids.

This is referred to as 'spectral analysis' or analysis in the 'frequency domain,' in contrast to the time domain approach we have considered so far.

The frequency domain approach considers regression on sinusoids; the time domain approach considers regression on past values of the time series.

A periodic time series

Consider $X_t = A\sin(2\pi\nu t) + B\cos(2\pi\nu t)$, where A, B are uncorrelated, mean zero, variance σ^2.

Writing $C^2 = A^2 + B^2$ and $\tan\phi = B/A$, we can think of this as

$$\begin{aligned} X_t &= C\cos\phi\sin(2\pi\nu t) + C\sin\phi\cos(2\pi\nu t) \\ &= C\sin(2\pi\nu t + \phi). \end{aligned}$$

That is, $A^2 + B^2$ determines the amplitude, and B/A determines the phase of X_t.

A periodic time series

For $X_t = A\sin(2\pi\nu t) + B\cos(2\pi\nu t)$, we have

$$\begin{aligned}
\mu_t &= \mathrm{E}[X_t] = 0 \\
\gamma(t, t+h) &= \mathrm{Cov}(X_t, X_{t+h}) \\
&= \sin(2\pi\nu t)\sin(2\pi\nu(t+h)) + \cos(2\pi\nu t)\cos(2\pi\nu(t+h)) \\
&= \cos(2\pi\nu t - 2\pi\nu(t+h)) \\
&= \cos(2\pi\nu h).
\end{aligned}$$

So $\{X_t\}$ is a stationary time series. (But notice that it does not satisfy $\sum_h |\gamma(h)| < \infty$.)

An aside: Some trigonometric identities

$$
\begin{aligned}
\tan\theta &= \frac{\sin\theta}{\cos\theta}, \\
\sin^2\theta + \cos^2\theta &= 1, \\
\sin(a+b) &= \sin a\cos b + \cos a\sin b, \\
\cos(a+b) &= \cos a\cos b - \sin a\sin b.
\end{aligned}
$$

A periodic time series

The random sinusoid $X_t = A\sin(2\pi\nu t) + B\cos(2\pi\nu t)$, with uncorrelated A, B, has sinusoidal autocovariance, $\gamma(h) = \cos(2\pi\nu h)$.

The autocovariance of the sum of two uncorrelated time series is the sum of their autocovariances (recall HW2). Thus, the autocovariance of a sum of random sinusoids is a sum of sinusoids with the corresponding frequencies:

$$
\begin{aligned}
X_t &= \sum_{j=1}^{k} \left(A_j \sin(2\pi\nu_j t) + B_j \cos(2\pi\nu_j t)\right), \\
\gamma(h) &= \sum_{j=1}^{k} \sigma_j^2 \cos(2\pi\nu_j h),
\end{aligned}
$$

where A_j, B_j are all uncorrelated, mean zero, and $\mathrm{Var}(A_j) = \mathrm{Var}(B_j) = \sigma_j^2$.

A periodic time series

$$X_t = \sum_{j=1}^{k} \left(A_j \sin(2\pi\nu_j t) + B_j \cos(2\pi\nu_j t) \right), \quad \gamma(h) = \sum_{j=1}^{k} \sigma_j^2 \cos(2\pi\nu_j h).$$

Thus, we can represent $\gamma(h)$ using a Fourier series. The coefficients are the variances of the sinusoidal components.

The *spectral density* is the continuous analog: the Fourier transform of γ.

(The analogous *spectral representation* of a stationary process X_t involves a *stochastic integral*—a sum of discrete components at a finite number of frequencies is a special case. We won't consider this representation in this course.)

Spectral density

If a time series $\{X_t\}$ has autocovariance γ satisfying

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty,$$

then we define its **spectral density** as

$$f(\nu) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h}$$

for $-\infty < \nu < \infty$.

Spectral density: Some facts

1. The series $\sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h}$ is absolutely summable.
   This is because $|e^{i\theta}| = |\cos\theta + i\sin\theta| = (\cos^2\theta + \sin^2\theta)^{1/2} = 1$, and because of the absolute summability of γ.

2. f is periodic, with period 1.
   This is true since $e^{-2\pi i \nu h}$ is a periodic function of ν with period 1.

   Thus, we can restrict the domain of f to $-1/2 \leq \nu \leq 1/2$. (The text does this.)

Spectral density: Some facts

3. f is even (that is, $f(\nu) = f(-\nu)$).
   To see this, write

$$f(\nu) = \sum_{h=-\infty}^{-1} \gamma(h) e^{-2\pi i \nu h} + \gamma(0) + \sum_{h=1}^{\infty} \gamma(h) e^{-2\pi i \nu h},$$

$$\begin{aligned} f(-\nu) &= \sum_{h=-\infty}^{-1} \gamma(h) e^{-2\pi i \nu (-h)} + \gamma(0) + \sum_{h=1}^{\infty} \gamma(h) e^{-2\pi i \nu (-h)}, \\ &= \sum_{h=1}^{\infty} \gamma(-h) e^{-2\pi i \nu h}, + \gamma(0) + \sum_{h=-\infty}^{-1} \gamma(-h) e^{-2\pi i \nu h} \\ &= f(\nu). \end{aligned}$$

4. $f(\nu) \geq 0$.

Spectral density: Some facts

5. $\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i\nu h} f(\nu)\, d\nu.$

$$
\begin{aligned}
\int_{-1/2}^{1/2} e^{2\pi i\nu h} f(\nu)\, d\nu &= \int_{-1/2}^{1/2} \sum_{j=-\infty}^{\infty} e^{-2\pi i\nu(j-h)} \gamma(j)\, d\nu \\
&= \sum_{j=-\infty}^{\infty} \gamma(j) \int_{-1/2}^{1/2} e^{-2\pi i\nu(j-h)}\, d\nu \\
&= \gamma(h) + \sum_{j\neq h} \frac{\gamma(j)}{2\pi i(j-h)} \left(e^{\pi i(j-h)} - e^{-\pi i(j-h)} \right) \\
&= \gamma(h) + \sum_{j\neq h} \frac{\gamma(j)\sin(\pi(j-h))}{\pi(j-h)} = \gamma(h).
\end{aligned}
$$

Example: White noise

For white noise $\{W_t\}$, we have seen that $\gamma(0) = \sigma_w^2$ and $\gamma(h) = 0$ for $h \neq 0$.

Thus,

$$f(\nu) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h}$$
$$= \gamma(0) = \sigma_w^2.$$

That is, the spectral density is constant across all frequencies: each frequency in the spectrum contributes equally to the variance. This is the origin of the name *white noise*: it is like white light, which is a uniform mixture of all frequencies in the visible spectrum.

Example: AR(1)

For $X_t = \phi_1 X_{t-1} + W_t$, we have seen that $\gamma(h) = \sigma_w^2 \phi_1^{|h|}/(1-\phi_1^2)$. Thus,

$$
\begin{aligned}
f(\nu) &= \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i\nu h} = \frac{\sigma_w^2}{1-\phi_1^2} \sum_{h=-\infty}^{\infty} \phi_1^{|h|} e^{-2\pi i\nu h} \\
&= \frac{\sigma_w^2}{1-\phi_1^2} \left(1 + \sum_{h=1}^{\infty} \phi_1^h \left(e^{-2\pi i\nu h} + e^{2\pi i\nu h} \right) \right) \\
&= \frac{\sigma_w^2}{1-\phi_1^2} \left(1 + \frac{\phi_1 e^{-2\pi i\nu}}{1-\phi_1 e^{-2\pi i\nu}} + \frac{\phi_1 e^{2\pi i\nu}}{1-\phi_1 e^{2\pi i\nu}} \right) \\
&= \frac{\sigma_w^2}{(1-\phi_1^2)} \frac{1-\phi_1 e^{-2\pi i\nu}\phi_1 e^{2\pi i\nu}}{(1-\phi_1 e^{-2\pi i\nu})(1-\phi_1 e^{2\pi i\nu})} \\
&= \frac{\sigma_w^2}{1-2\phi_1\cos(2\pi\nu)+\phi_1^2}.
\end{aligned}
$$