A Strong Duality Principle for Total Variation and Equivalence Couplings

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I. Stochastic processes
Theorem (Ernst-Kendall-Roberts-Rosenthal, 2019)

For any $\theta_1, \theta_2 \in \mathbb{R}$, one can construct a probability space supporting
- a Brownian motion $B^{\theta_1} = \{B_{t}^{\theta_1}\}_{t \geq 0}$ with drift $\theta_1$,
- a Brownian motion $B^{\theta_2} = \{B_{t}^{\theta_2}\}_{t \geq 0}$ with drift $\theta_2$, and
- a random time $T$ with $T > 0$ almost surely,

such that $B_{t}^{\theta_1} = B_{t}^{\theta_2}$ for all $0 \leq t \leq T$ almost surely.
In words:

- BM with drift “starts out as” a BM without drift.
- BMs with drift are all “locally equivalent” at time zero.
- The drift of a BM cannot be detected, if it is only observed up to an adversarially-chosen time.

Explicit construction based on Itô excursion theory.
Definition
Say that a pair of Borel probability measures \((P, P')\) on \(D([0, \infty); \mathbb{R})\) has the germ coupling property (GCP) if one can construct a probability space supporting

- a stochastic process \(X = \{X_t\}_{t \geq 0}\) with law \(P\),
- a stochastic process \(X' = \{X'_t\}_{t \geq 0}\) with law \(P'\), and
- a random time \(T\) with \(T > 0\) almost surely,

such that \(X_t = X'_t\) for all \(0 \leq t \leq T\) almost surely.

Know that \((W^{\theta_1}, W^{\theta_2})\) has the GCP for all \(\theta_1, \theta_2 \in \mathbb{R}\), where \(W^\theta\) denotes the law of BM with drift \(\theta \in \mathbb{R}\).

Say \(P\) has the Brownian GCP if \((P, W^0)\) has the GCP.

Which other pairs have the GCP?
II. Some vignettes
• \( \Omega \) a Polish space,
• \( \Delta := \{(x, x) \in \Omega \times \Omega : x \in \Omega \} \) the diagonal in \( \Omega \times \Omega \),
• \( P, P' \) two Borel probability measures on \( \Omega \), and
• \( \Pi(P, P') \) the space of all couplings of \( P \) and \( P' \).

Then (folklore):

\[
\sup_{A \in \mathcal{B}(\Omega)} |P(A) - P'(A)| = \min_{\tilde{P} \in \Pi(P, P')} (1 - \tilde{P}(\Delta)).
\]
$Q = P \land P'$
Q
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$\tilde{Q} + \gamma M \otimes M'$
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\(P, P'\) two Borel probability measures on \(\Omega\), and
\(\Pi(P, P')\) the space of all couplings of \(P\) and \(P'\).

Then (folklore):
\[
\sup_{A \in \mathcal{B}(\Omega)} \left| P(A) - P'(A) \right| = \min_{\tilde{P} \in \Pi(P, P')} \left(1 - \tilde{P}(\Delta)\right).
\]
\( \Omega := S^\mathbb{N} \) the space of sequences for a finite set \( S \),
\( E_0 := \bigcup_{n \in \mathbb{N}} \{(x, x') \in \Omega \times \Omega : (x_n, x_{n+1}, \ldots) = (x'_n, x'_{n+1}, \ldots)\} \) the equivalence relation of eventual equality,
\( T := \cap_{n \in \mathbb{N}} \sigma(x_n, x_{n+1}, \ldots) \) the tail \( \sigma \)-algebra,
\( P, P' \) two Borel probability measures on \( \Omega \), and
\( \Pi(P, P') \) the space of all couplings of \( P \) and \( P' \).

Then (Griffeath 1974, Pitman 1976, Goldstein 1978):

\[
\sup_{A \in T} |P(A) - P'(A)| = 0 \text{ if and only if } \min_{\tilde{P} \in \Pi(P, P')} (1 - \tilde{P}(E_0)) = 0.
\]
\( \Omega := S^\mathbb{N} \) the space of sequences for a finite set \( S \),
\( \theta : \Omega \rightarrow \Omega \) the left-shift operation,
\( E_{\mathbb{Z}} := \bigcup_{n \in \mathbb{Z}} \{(x, x') \in \Omega \times \Omega : \theta^n(x) = x'\} \) the equivalence relation of shift-equivalence,
\( \mathcal{I}_{\mathbb{Z}} = \{ A \in \mathcal{B}(\Omega) : \theta^{-1}(A) = A \} \) the shift-invariant \( \sigma \)-algebra,
\( P, P' \) two Borel probability measures on \( \Omega \), and
\( \Pi(P, P') \) the space of all couplings of \( P \) and \( P' \).

Then (Aldous-Thorisson 1993):

\[
\sup_{A \in \mathcal{I}_{\mathbb{Z}}} |P(A) - P'(A)| = 0 \text{ if and only if } \min_{\tilde{P} \in \Pi(P, P')} (1 - \tilde{P}(E_{\mathbb{Z}})) = 0.
\]

Also have generalizations to sufficiently regular group and semigroup actions (Thorisson 1996, Georgii 1997).
\[
\sup_{A \in \mathcal{B}(\Omega)} |P(A) - P'(A)| = \min_{\tilde{P} \in \Pi(P,P')} (1 - \tilde{P}(\Delta)).
\]
\[
\sup_{A \in \mathcal{T}} |P(A) - P'(A)| = \min_{\tilde{P} \in \Pi(P,P')} \left(1 - \tilde{P}(E_0)\right).
\]
\[\sup_{A \in \mathcal{I}_Z} |P(A) - P'(A)| = \min_{\tilde{P} \in \Pi(P,P')} (1 - \tilde{P}(E_Z)).\]
\[ \sup_{A \in G} |P(A) - P'(A)| \equiv \min_{\tilde{P} \in \Pi(P,P')} (1 - \tilde{P}(E)). \]
Many probability settings lead to the \textit{E-coupling problem}

$$\inf_{\tilde{P} \in \Pi(P, P')} (1 - \tilde{P}(E)),$$

In general this problem is hard to solve and there are not many general-purpose tools available.

On the other hand, the \textit{G-total variation problem}

$$\sup_{A \in G} |P(A) - P'(A)|$$

is typically easy to analyze for probabilists.

These optimization problems are closely related! In fact, we’ll see that they are often dual, in the sense of mathematical optimization.
I. Stochastic processes

II. Some vignettes

III. Problem statement

IV. Results
III. Problem statement
Notation:

- $(\Omega, \mathcal{F})$ standard Borel space,
- $\mathcal{P}(\Omega, \mathcal{F})$ space of probability measures on $(\Omega, \mathcal{F})$,
- $\Pi(P, P')$ space of couplings of $P, P' \in \mathcal{P}(\Omega, \mathcal{F})$,
- $E$ equivalence relation on $\Omega$, and
- $\mathcal{G}$ sub-$\sigma$-algebra of $\mathcal{F}$. 
Definition
Say $E$ is \textit{measurable} if $E \in \mathcal{F} \otimes \mathcal{F}$. Say $(E, \mathcal{G})$ is \textit{strongly dual} if $E$ is measurable and if we have

$$\sup_{A \in \mathcal{G}} |P(A) - P'(A)| = \min_{\tilde{P} \in \Pi(P, P')} (1 - \tilde{P}(E)).$$

for all $P, P' \in \mathcal{P}(\Omega, \mathcal{F})$.

Roughly speaking, $(\Delta, \mathcal{B}(\Omega)), (E_0, \mathcal{T})$, and $(E_{\mathbb{Z}}, \mathcal{I}_{\mathbb{Z}})$ are strongly dual.

Which pairs $(E, \mathcal{G})$ are strongly dual?
If $E$ is given, then there is a natural choice of $G$:

**Lemma (AQJ)**

If $(E, G)$ is strongly dual for some $G$, then $(E, E^*)$ is strongly dual, where $E^*$ is the $E$-invariant $\sigma$-algebra

$$E^* := \{ A \in \mathcal{F} : \forall (x, x') \in E (x \in A \iff x' \in A) \}$$

Say that $E$ is strongly dualizable if $(E, E^*)$ is strongly dual.
Connection to optimal transport?

Note that the $E$-coupling problem

$$
\inf_{\tilde{P} \in \Pi(P,P')} (1 - \tilde{P}(E)).
$$

is exactly a Monge-Kantorovich problem with cost function

$$
c(x, x') = 1 - \mathbb{1}\{(x, x') \in E\}.
$$

In words: cost 0 to move within an equivalence class, and cost 1 to move between equivalence classes.
Classical Monge-Kantorovich theory (Rachev-Rüschendorf 1998, Villani 2009) requires topological regularity: $\Omega$ is a Polish space, $\mathcal{F}$ is its Borel $\sigma$-algebra, and $c$ is lower semi-continuous.

In our setting, this requires $E$ to be closed in $\Omega \times \Omega$; in this case, Kantorovich duality and some standard tricks can show that $E$ is strongly dualizable.

However, most interesting equivalence relations, from the point of view of probability, are $F_\sigma$ (countable union of closed) in $\Omega \times \Omega$. 
IV. Results
Some useful reductions:

It is easy to show that we always have \textit{weak dualizability}, that

$$\sup_{A \in E^*} \left| P(A) - P'(A) \right| \leq \inf_{\tilde{P} \in \Pi(P, P')} (1 - \tilde{P}(E)),$$

for all $P, P' \in \mathcal{P}(\Omega, \mathcal{F})$. The difficult part is showing the reverse inequality and that the inf is attained.

We say that $E$ is \textit{quasi-strongly dualizable} if for all $P, P' \in \mathcal{P}(\Omega, \mathcal{F})$ the following are equivalent:

\begin{itemize}
  \item $P(A) = P'(A)$ for all $A \in E^*$
  \item There exists $\tilde{P} \in \Pi(P, P')$ and $N \in \mathcal{F} \otimes \mathcal{F}$ with $\tilde{P}(N) = 0$ and $(\Omega \times \Omega) \setminus E \subseteq N$.
\end{itemize}

Then $E$ is strongly dualizable if and only if it is measurable and quasi-strongly dualizable.
Some basic descriptive set theory:

A measurable space \((S, \mathcal{S})\) is called a \textit{standard Borel space} if there exists a Polish topology \(\tau\) on \(\Omega\) such that \(\mathcal{S} = \mathcal{B}(\tau)\).

An equivalence relation \(E\) on a standard Borel space \((\Omega, \mathcal{F})\) is called \textit{smooth} if there exists a standard Borel space \((S, \mathcal{S})\) and a measurable function \(\phi : (\Omega, \mathcal{F}) \to (S, \mathcal{S})\) such that \((x, x') \in E\) is equivalent to \(\phi(x) = \phi(x')\).

Roughly speaking, \(E\) is smooth if and only if the quotient \(\Omega/E\) can be given a natural standard Borel structure.
Lemma (AQJ)

The following are equivalent:

(i) \(E\) is smooth.

(ii) \(E^*\) is countably generated.

(iii) \(E \in E^* \otimes E^*\).

The equivalence between (i) and (ii) is classical, but the equivalence with (iii) appears to be novel.
Theorem (AQJ)

Every smooth equivalence relation is strongly dualizable.

Consequence: All equivalence relations with \( G_\delta \) (countable intersection of open) equivalence classes are strongly dualizable.

Idea of proof: Do the folklore coupling in \((\Omega, E^*)\) then “smooth things over” with conditional expectations.
\( v = \mu \land \mu' \)
$v \circ (i,i)^{-1}$
$\tilde{Q}(\cdot \times \cdot \mid E^*) \approx \mu(\cdot \mid E^*)\mu'(\cdot \mid E^*)$
\( \tilde{Q} + \gamma M \otimes M' \)
\( \tilde{Q} + \gamma MM' \)
However, many equivalence relations of interest are not smooth.

Instead, we have the following closure result:

**Theorem (AQJ)**

_A countable increasing union of strongly dualizable equivalence relations is strongly dualizable._

Idea of proof: Apply strong duality, and iterate.
$\tilde{\mathcal{P}} \, ?$

$E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$

$P$

$P'$
$\tilde{P}_1( \cdot \cap E_1)$
\[ \tilde{P}_3( \cdot \cap E_3 ) \]
\[ P \subseteq E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots \]
\( \tilde{P}_1( \cdot \cap E_1 ) \)
\[ \sum_{n=1}^{2} \tilde{P}_n (\cdot \cap E_n) \]
\[ \sum_{n=1}^{3} \tilde{P}_n( \cdot \cap E_n ) \]
\[ \sum_{n=1}^{\infty} \tilde{P}_n \left( \cdot \cap E_n \right) + \gamma M \otimes M' \]
\[ \sum_{n=1}^{\infty} \tilde{P}_n (\cdot \cap E_n) + \gamma M \otimes M' \]
This shows strong dualizability for all *hypersmooth* (countable union of smooth) equivalence relations, and this covers most “reasonable” equivalence relations occurring in probability.
Thank you!
References


S. Hummel and A. Q. Jaffe. Constructing maximal germ couplings of Brownian motion with drift. *arxiv pre-print*.


