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GAMMA DENSITIES, STEP BY STEP

You have seen that the time till the rth arrival in a Poisson process has the gamma (r, λ) distribution, where $\lambda > 0$ is the intensity of the process. The density is given by

$$f(t) = \frac{\lambda^r}{(r-1)!} t^{r-1} e^{-\lambda t}, \qquad t \geq 0$$

The shape parameter r is an integer here, but the gamma density with parameters r and λ can be defined for any $r > 0$ and $\lambda > 0$. For example, gamma densities with half-integer r appear as densities of squares or sums of squares of independent standard normal variables, as you will see later in the course.

To define the gamma density for general $r > 0$, not necessarily an integer, look at the formula above notice that the only thing that gives trouble when r is not an integer is the factorial in the denominator. That factorial will be replaced by a sort of continuous extension of the factorial function to all non-negative reals. For now just assume that r and λ are two positive numbers and that there is a constant $c(r, \lambda)$ that makes the density integrate to 1.

Then the gamma denisty with parameters $r > 0$ and $\lambda > 0$ is defined by

$$f(t) = c(r, \lambda) t^{r-1} e^{-\lambda t}, \qquad t \geq 0$$

The parts that involve the variable t are really rather simple – a power and an exponential. Let's figure out how to work with this density.

1. Is it a gamma? Suppose you have the formula for a density, and you're trying to recognize whether it's a gamma density or not. First check that the variable takes only non-negative values. Then look at the main functional form of the density, that is, the part of the density formula that involves the variable. Suppose the variable is being denoted by t, as above. The density is a gamma density if the functional form is

$$constant_1 \; t^{power} e^{-constant_2 \cdot t}$$

where the power is greater than -1 and both the constants are positive.

Here are some densities. Which are gamma densities and which aren't? Assume that all the functions below are defined as stated on the non-negative reals and are 0 elsewhere.

(i) $f(x) = (x^2 e^{-x})/2$
(ii) $g(u) = \text{constant } e^{-u}$
(iii) $f(u) = \text{constant } e^{-u^2}$
(iv) $f(t) = \text{constant } e^{-t/2}/\sqrt{t}$

The density in Example (i) clearly has the right form. It doesn't matter where you write the constant, because the product will be the same. You decide about the others.

2. If it's a gamma, then which gamma is it? Once you have identified a gamma density you must figure out its parameters in order to be able to specify the distribution completely. The conventional parameters are $r > 0$ and $\lambda > 0$ chosen so that the density is a constant times

$$t^{r-1}e^{-\lambda t}$$

In Example (i) above, the parameters are $r = 3$ and $\lambda = 1$. What are the parameters in the other densities which you identified as gamma?

3. What's the constant that makes the density integrate to 1? Once you have identified the parameters r and λ, you may want to work out the constant that makes the density integrate to 1 (though often this is unnecessary, because you can do plenty of work with this density without ever evaluating that constant explicitly). Because the density has to integrate to 1,

$$\begin{aligned} 1 &= c(r,\lambda)\int_0^\infty t^{r-1}e^{-\lambda t}dt \\ &= c(r,\lambda)\frac{1}{\lambda^r}\int_0^\infty t^{r-1}e^{-t}dt \qquad \text{by change of variable} \\ c(r,\lambda) &= \frac{\lambda^r}{\int_0^\infty t^{r-1}e^{-t}dt} = \frac{\lambda^r}{\Gamma(r)} \end{aligned}$$

where $\Gamma(r)$ is the integral in the denominator. It is important to note that for every r, $\Gamma(r)$ is a positive number. It's not a density. It's called "gamma (r)" because it's the gamma function of mathematics. For our purposes, it's just the number that gives the density its name. The gamma density with parameters $r > 0$ and $\lambda > 0$ is

$$f(t) = \frac{\lambda^r}{\Gamma(r)}t^{r-1}e^{-\lambda t}, \qquad t \geq 0$$

Notice that this is the identical to the formula we had for the gamma density when r was an integer, apart from the fact that $(r-1)!$ has been replaced by $\Gamma(r)$.

4. What's the numerical value of $\Gamma(r)$? If the new formula for the gamma density is correct, it has to be true that $\Gamma(r) = (r-1)!$ for positive integer values of r. For non-integer r it's not very easy to get a numerical value for $\Gamma(r)$ (except for half-integer r, which we will see later in the course). Let us examine a property of Γ which makes it easier to get at its numerical value.

You know that the definition is

$$\Gamma(r) = \int_0^\infty t^{r-1}e^{-t}dt$$

If it is an extension of the factorial function then there should be a nice multiplicative relation between $\Gamma(r+1)$ and $\Gamma(r)$. After all, $n! = n \times (n-1)!$. We will integrate by parts, using the fact that $e^{-t} = \frac{d}{dt}(-e^{-t})$.

$$\begin{aligned} \Gamma(r+1) &= \int_0^\infty t^r e^{-t}dt \\ &= t^r(-e^{-t})|_0^\infty - \int_0^\infty rt^{r-1}(-e^{-t})dt \;=\; 0 + r\Gamma(r) \;=\; r\Gamma(r) \end{aligned}$$

This is **the most important computational property:**

$$\Gamma(r+1) = r\Gamma(r) \qquad \text{for all } r > 0$$

Apply this to integer values of r:

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty e^{-t}dt = 1 \\
\Gamma(2) &= 1\Gamma(1) = 1 \\
\Gamma(3) &= 2\Gamma(2) = 2 \times 1 = 2! \\
\Gamma(4) &= 3\Gamma(3) = 3 \times 2! = 3!
\end{aligned}$$

and so on. Use induction to show that $\Gamma(r) = (r-1)!$ for integer values of r. So our old formula for the gamma density with integer r was correct.

In Example (i) we identified the gamma $(3, 1)$ density. The constant of integration is $1^3/\Gamma(3) = 1/2! = 1/2$.

We will now use this property to find simple formulas for the expectation and SD of gamma random variables. First, an easy preliminiary.

5. Gamma integrals from the gamma density. Since the total integral of the gamma density is 1, we have

$$\int_0^\infty t^{r-1}e^{-\lambda t}dt = \frac{\Gamma(r)}{\lambda^r}$$

You know this already from the formula for the constant of integration.

6. Moments of the gamma distribution. A *moment* of the distribution of X is $E(X^s)$ for some positive power s. If X has the gamma (r, λ) distribution,

$$\begin{aligned}
E(X^s) &= \int_0^\infty t^s \frac{\lambda^r}{\Gamma(r)} t^{r-1}e^{-\lambda t}dt \\
&= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty t^{r+s-1}e^{-\lambda t}dt \\
&= \frac{\lambda^r}{\Gamma(r)} \cdot \frac{\Gamma(r+s)}{\lambda^{r+s}} \\
&= \frac{\Gamma(r+s)}{\Gamma(r)} \cdot \frac{1}{\lambda^s}
\end{aligned}$$

In the special case $s = 1$,

$$E(X) = \frac{\Gamma(r+1)}{\Gamma(r)} \cdot \frac{1}{\lambda} = \frac{r\Gamma(r)}{\Gamma(r)} \cdot \frac{1}{\lambda} = \frac{r}{\lambda}$$

This is the same as the formula we derived in the case where r is an integer, when X is the waiting time till the rth arrival in a Poisson process with rate λ.

In the special case $s = 2$ you should check that

$$E(X^2) = \frac{(r+1)r}{\lambda^2}$$

and hence that

$$Var(X) = \frac{r}{\lambda^2}, \qquad SD(X) = \frac{\sqrt{r}}{\lambda}$$

These are also the same as the formulas that we derived for integer values of r.

SUMMARY

• For any $r > 0$ and $\lambda > 0$, the gamma density with parameters r and λ is defined by

$$f(t) = \frac{\lambda^r}{\Gamma(r)} t^{r-1} e^{-\lambda t}, \qquad t \geq 0$$

The constant of integration involves the number $\Gamma(r)$ which is defined to make the density integrate to 1.

• If a random variable T has the gamma (r, λ) distribution, then

$$E(T^s) = \frac{\Gamma(r+s)}{\Gamma(r)} \cdot \frac{1}{\lambda^s} \qquad E(T) = \frac{r}{\lambda} \qquad SD(T) = \frac{\sqrt{r}}{\lambda}$$

• The constant of integration in the density involves the number $\Gamma(r)$ which is not very easy to compute numerically for general r. But $\Gamma(r+1) = r\Gamma(r)$ for all r.

This result implies that

$$\Gamma(r) = (r-1)! \qquad \text{for integer } r$$

We will show in Section 4.4 that $\Gamma(1/2) = \sqrt{\pi}$. Apply the result again to see that

$$\begin{aligned} \Gamma\Big(\frac{3}{2}\Big) &= \frac{1}{2} \cdot \sqrt{\pi} \\ \Gamma\Big(\frac{5}{2}\Big) &= \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} \\ \Gamma\Big(\frac{7}{2}\Big) &= \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} \end{aligned}$$

and so on.