The Stretch - Length Tradeoff in Geometric Networks: Worst-case and Average case Study.

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xxx Working notes.

Three main things remain to do.
(1) Either include Tamar’s argument [now in separate document] for Theorem 5, or replace by some better construction for worst-case upper bound.
(2) Put in the average-case lower bound calculus calculations needed in section 6.
(3) General version of the worst-case lower bound method in section 7.3.

A fourth thing one might contemplate:
(4) Is there a way to use Proposition 7 to get an upper bound in the worst case?

Minor things:
(z) Picture for Lemma 8.
(z) Numerics for the value of $L_k$ in (24) for small $k$.
(z) Graph of $\Psi_{\text{ave}}$ to complement Figure 10 (remember the Delaunay point!)
(z) General polishing, checking calculations, making notation consistent.

1 Introduction

This paper describes results and open problems falling between the topic of geometric spanner networks (see the recent monograph [8]) and the author’s work motivated by analysis of route-length efficiency of real-world transportation networks (see the overview paper [3]). Let us first describe each of those topics separately (see [3, 8] for more details) and then move on to the specifics of this paper.

1.1 Geometric spanner networks

This topic concerns design of networks on arbitrary sets of vertices in the plane (or higher dimensions). Here a default assumption is that an edge \((v, w)\) can only be a line segment between two of the given vertices. The interpretation of “size” of the network is sometimes as number of edges and sometimes as network length (sum of Euclidean edge lengths). Similarly, the interpretation of in-network distance between two vertices is sometimes taken as minimum number of edges of a route between them (hop length) and sometimes as shortest total length (sum of Euclidean edge lengths) of a route between them (route length). How well the network provides short routes is usually measured by the stretch, defined as

\[
\text{stretch} = \max_{v \neq w} \frac{\text{route-length from } v \text{ to } w}{d(v, w)} \geq 1
\]

where \(d(v, w)\) denotes straight line (Euclidean) distance. A network with stretch \(S\) is called a \(S\)-spanner. This topic emphasizes algorithms: both algorithms for constructing spanners, and the use of spanners in algorithms for computational geometry problems.

1.2 Route-length efficiency

Here we start by considering descriptive statistics for real-world networks such as road or rail networks linking cities. As before, the total length of the network is a reasonable summary statistic for “size”. For each pair of cities, consider

\[
R(v, w) = \frac{\text{route-length from } v \text{ to } w}{d(v, w)} - 1.
\]

So \(R(v, w) = 0.2\) means that route-length is 20% longer than straight-line distance. Note that (aside from the “-1” convention) the stretch equals
the maximum of $R(v, w)$ over all pairs. We are interested in the “typical” value of $R(v, w)$, so using a maximum isn’t so natural. For instance, in the U.K. rail network there is no very direct route between Oxford and Cambridge, but we don’t want this one case to have undue influence on the value of a “route length efficiency” statistic for the network as a whole. One might consider the overall average $R_{\text{ave}}$ over all pairs $(v, w)$, but for reasons explained in [3], for our basic “route length efficiency” summary statistic we prefer a statistic $R$ defined as follows. For each distance $d$, set $\rho(d)$ as the average of $R(v, w)$ over pairs at distance approximately $d$; then let $R$ be the maximum of $\rho(d)$ as $d$ varies. Thus we summarize a network via two statistics: $R$ and a “normalized network length” statistic $L$ defined at (1) below. Within this setup, we can start with an empirical question

find the values of $(L, R)$ for different real-world networks

and then ask

how close to optimal is a given real-world network, in the sense of how close is the actual $R$ to the minimum possible value of $R$ over all possible networks on the given cities with the same value of $L$?

For further theoretical study we use the random model; $n$ cities whose positions are uniform random in a square of area $n$. In this model, we can ask

What is the optimal trade-off between $L$ and $R$? Can we say anything about the qualitative or quantitative structure of networks attaining the optimum? Are there simple network constructions that are close to optimal?

1.3 The setting of this paper

In this paper we adopt a setting intermediate between the two settings above.

(a) The statistic $R$ is analytically intractible; our results in [3] for $R$ depend on Monte Carlo estimation of its value. To obtain analytic bounds we switch back to considering stretch, or more precisely the “stretch $−1$” statistic

$$S = \max_{v \neq w} R(v, w).$$

(b) In geometric spanner networks one almost always studies worst-case results; we want to study in parallel worst-case and average-case (i.e. the random model above) results.
(c) In the random model, network length scales as order \( n \), so it is natural to define \textit{normalized network length} as

\[
L = n^{-1} \times \text{(network length)}.
\]  

(1)

In order to facilitate comparison between average-case and worst-case results, we will take “worst-case” to refer to an arbitrary configuration \( x = (x_1, \ldots, x_n) \) of cities in the square of area \( n \), and then define normalized network length \( L \) as above. Intuitively, given \( n \) cities in some region \( A \), this is like taking normalized network length to be \( (\text{network length}) / \sqrt{n \times \text{area}(A)} \).

In contrast, in the setting of geometric spanner networks, one often compares the length of a network to the length of the minimum spanning tree, so one is implicitly taking normalized network length as \( (\text{network length}) / (\text{length of MST}) \).

(d) Our notion of “network” is the general notion suggested by envisaging actual road networks, and need not consist only of straight city-city edges. See Figure 1. The left diagram shows a network on 4 cities which is connected (a driver can switch roads at the junction where they cross), the center diagram shows that other junctions (Steiner points) may be created, and the right diagram (envisaged as part of a larger network) shows that roads need not go through cities at all.

![Figure 1. Illustration of possible networks.](image)

As the reader will have noticed, we are using concrete terminology for mathematical objects: \textit{cities} for the specified points in the plane, \textit{roads} for edges (which are required to be line segments), and \textit{junctions} for non-city points where roads meet.

(e) In this paper we focus on one issue, the trade-off between normalized length \( L \) and stretch \( S + 1 \). Though we use explicit constructions to obtain bounds, we are not concerned with algorithmic implementation. Instead, we study the two functions \( \Psi_{\text{worst}} \), \( \Psi_{\text{ave}} \) defined, in the \( n \to \infty \) limit, by

\[
L = \Psi_{\text{worst}}(s) \text{ is the minimum normalized length of a network required to have stretch } 1 + s, \text{ in the worst case}
\]
and analogously in the average case. Getting explicit values for these functions seems impossible analytically and very challenging numerically. Here’s what we will do in this paper. The sections are, to a large extent, independent.

- Prove existence of these limit functions (section 2).
- Prove that their $s \to \infty$ limits are equal to (rather than greater than) the Steiner tree values (section 3).
- Derive upper bounds on $\Psi_{\text{worst}}(s)$ from elementary constructions where one first lays down a regular network of roads without paying attention to city positions, and then adds local links from cities to the network (section 4).
- Derive upper bounds on $\Psi_{\text{ave}}(s)$ from constructions (similar to the $\Theta$-graphs in geometric spanner networks: [8] Chapter 4) in which from every city and every cone of given angle there exists a road leaving the city within the cone (section 5).
- Derive lower bounds on $\Psi_{\text{ave}}(s)$ for small $s$, based on stochastic geometry calculations in the spirit of the lower bound arguments in [1] (section 6).
- Derive lower bounds on $\Psi_{\text{worst}}(s)$ based on proofs of “local optimality” (xxx need better phrase) for specific networks on specific configurations (section 7).

An interesting theoretical question that is (perhaps) amenable to analytic study is the scaling behavior in the $s \to 0$ limit. That is, in the spirit of “universality” in statistical physics, one can speculate that there exists an exponent $\alpha$ such that

$$L = \Psi(s) \asymp s^{-\alpha} \text{ as } s \to 0$$

where the value of $\alpha$ does not depend on any detailed assumptions in the model (worst-case or average-case; whether or not Steiner points are allowed) but instead depends only on the the fact we’re working in two-dimensional space.

xxx say what we prove here
xxx outside the asymptotic domain our numerical bounds are rather poor; scope for future research!
1.4 What is already known?

Granted that the limits defining \( \Psi_{\text{ave}}(s) \) and \( \Psi_{\text{worst}}(s) \) exist (which is easy provided we admit the possibility the limits might be \( +\infty \) – see section 2), it is obvious that

(i) each function is non-increasing in \( s \);
(ii) \( \Psi_{\text{ave}}(s) \leq \Psi_{\text{worst}}(s) \).

The most relevant result in [8] is perhaps Theorem 15.2.16, which says that for small \( s > 0 \) one can construct \((1 + s)\)-spanners such that (amongst other properties) the network length is bounded by \( O(s^{-4} \times \text{length of MST}) \). It is well-known and elementary (see e.g. [10] section 2.2) that in the worst case the length of MST is \( O(n) \), so the Theorem mentioned above implies

\[
\Psi_{\text{worst}}(s) = O(s^{-4}) \quad \text{as} \quad s \to 0
\]

and in particular that \( \Psi_{\text{worst}}(s) < \infty \) for all \( s > 0 \).

Another remarkable result [5] is that the Delaunay triangulation is always a \( t \)-spanner for \( t = \frac{2\pi}{3\cos \pi/6} \approx 2.42 \). Because the length of a Delaunay triangulation is not \( O(n) \) in the worst case, this does not help us bound \( \Psi_{\text{worst}} \). But in the random model, a classical result ([7] page 113) shows the limit normalized length of the Delaunay triangulation equals \( \frac{32}{3\pi} = 3.40\ldots \)

So we get a numerical bound

\[
\Psi_{\text{ave}}(1.42) \leq 3.40. \tag{3}
\]

Curiously, (2) and (3) are the only results about our functions \( \Psi_{\text{worst}} \) and \( \Psi_{\text{ave}} \) that we have been able to read off from results in the geometric spanner network literature. We should emphasize that that literature focusses on more algorithmic issues, and it is quite likely that methods in that literature could be modified to give further results about our functions.

Finally, it seems intuitively certain that (i) can be improved to

each function \( \Psi(s) \) is continuous and strictly decreasing \( \tag{4} \)

but we do not know any proof.
2 Existence of the limit functions $\Psi$

In this section we use a subadditivity argument to prove existence of the limit functions $\Psi$. Note that in the simplest kind of subadditivity argument [10], a big square is divided into small subsquares, and optimal solutions on subsquares are used to construct some near-optimal solution on the big square. We argue in the opposite direction: use an optimal solution on the big square to construct near-optimal solutions on subsquares. This leads to the “superadditive” inequalities (6, 8).

Fix $0 < s < \infty$ and let $a_n$ be the worst-case value (over configurations $x = (x_1, \ldots, x_n)$ of cities in the square of area $n$) of the length of the shortest network on $x$ with stretch $\leq 1 + s$. We shall prove existence of the limit

$$\Psi_{\text{worst}}(s) := \lim_{n} n^{-1} a_n. \tag{5}$$

We will first argue

$$\frac{a_n}{n} \leq \frac{a_{nk^2}}{nk^2} + \frac{4}{\sqrt{n}}, \quad n \geq 1, \quad k \geq 2. \tag{6}$$

Fix $n$ and $k$. Let $x$ be a configuration in the area-$n$ square attaining $a_n$. Take $k^2$ copies of this configuration, and translate each to construct a configuration $x^*$ of $nk^2$ cities in the square of area $nk^2$. So there is a network on $x^*$ with $S \leq s$ and with length $\leq a_{nk^2}$. Add to this network the four boundary edges of each of the $k^2$ subsquares (so we get two copies of each edge interior to the big square). We now have a network $N^*$ whose length $c_{n,k}$ satisfies $c_{n,k} \leq a_{nk^2} + 4n^{1/2}k^2$. Consider the restriction of this network to one of the subsquares. The length of the restricted network may depend on the subsquare, but there must be at least one subsquare $Q$ for which this length of the restricted network $N_Q$ is less than the average $c_{n,k}/k^2$. Routes in the network $N^*$ between cities of $Q$ might go outside $Q$, but replacing these external segments by the boundary edges of $Q$ can only shorten the route length, so $N_Q$ has stretch at most $1 + s$. But $N_Q$ defines (by translation) a network on the original configuration $x$, and so $a_n \leq c_{n,k}/k^2$, which gives (6).

Deducing existence of a limit from (6) is one of many variants of routine “subadditivity” arguments, as follows. First note that $(a_n)$ is increasing; indeed

$$a_{n+1} \geq a_n \sqrt{\frac{n+1}{n}}$$

by adding an arbitrary city to the configuration attaining $a_n$ and rescaling. Next define

$$\gamma = \lim_{n} \inf a_n/n \leq \infty$$

7
and use monotonicity to show

\[ \gamma = \lim \inf_k \frac{a_{nk^2}}{nk^2}, \quad \text{for each fixed } n. \]

Then (6) shows

\[ a_n/n \leq \gamma + 4n^{-1/2} \]

and so \( \lim sup_n a_n/n \leq \gamma \), meaning that indeed \( \lim a_n/n = \gamma \).

This argument shows \( \Psi_{\text{worst}}(s) \leq \infty \) exists. To show that \( \Psi_{\text{worst}}(s) < \infty \) for all \( s > 0 \) requires more effort. As already mentioned at (2), it follows from known but rather intricate constructions; alternatively it follows from the more elementary constructions in our section 4.

For the random model we use the same construction with a Poissonized number of random points. Fix \( s \) again. Let \( b_n \) be the expectation of the length of the shortest network with stretch \( \leq s + 1 \) over \( n \) uniform random cities in the unit square. So \( n^{1/2}b_n \) is the corresponding expectation in the area-\( n \) square. We shall prove existence of the limit

\[ \Psi_{\text{ave}}(s) := \lim_n \frac{n^{1/2}b_n}{n}. \quad (7) \]

Write \( N(t) \) for a random variable with Poisson\((t)\) distribution and write

\[ \beta_t = \mathbb{E}b_{N(t)}. \]

Take a Poisson point process (rate 1 per unit area) of cities on the whole plane. Now \( t^{1/2}\beta_t \) is the expectation of the length of the shortest network with stretch \( \leq s + 1 \) on the Poisson cities in an area-\( t \) square. Consider partitioning a square of area \( tk^2 \) into \( k^2 \) subsquares of area \( t \). Repeating the argument for (6), now using a random subsquare, gives an inequality analogous to (6):

\[ \frac{\beta_t}{t^{1/2}} \leq \frac{\beta_{tk^2}}{t^{1/2}k} + \frac{4}{t^{1/2}}, \quad 0 < t < \infty, \ k \geq 2. \quad (8) \]

Using the fact that \( \beta_t \) is increasing in \( t \), we can repeat the “subadditivity” argument to show existence of the limit

\[ \lim_t t^{-1/2}\beta_t = \gamma^* \leq \infty. \quad (9) \]

The “average case better than worst case” inequality \( n^{1/2}b_n \leq a_n \) and the fact \( \Psi_{\text{worst}}(s) < \infty \) now easily implies \( \gamma^* < \infty \).
To finish we need a routine “dePoissonization” argument to show that (8) and monotonicity of $b_n$ imply
\[ \lim_n n^{-1/2} b_n = \gamma^*. \] (10)

First fix $\varepsilon > 0$ and consider $t_n/n \to 1 + \varepsilon$. Then
\[ \beta t_n \geq b_n \mathbb{P}(N(t_n) \geq n) = b_n(1 - o(1)) \]
so
\[ \limsup_n n^{-1/2} b_n \leq \limsup_n n^{-1/2} \beta t_n = (1 + \varepsilon)^{1/2} \gamma^* \]
and the upper bound for (10) follows. Next, the following property of the Poisson distribution
\[ \max_{i \geq n} \frac{\mathbb{P}(N((1-\varepsilon)n) = i)}{\mathbb{P}(N(n) = i)} \to 0 \]
implies
\[ \mathbb{E} b_{N((1-\varepsilon)n)} \mathbb{1}(N((1-\varepsilon)n) \geq n) = o(\mathbb{E} b_{N(n)}) = o(n^{1/2}) \]
and so
\[ \beta_{N((1-\varepsilon)n)} \leq b_n + o(n^{1/2}) \]
implying
\[ \liminf_n n^{-1/2} b_n \geq \liminf_n n^{-1/2} \beta_{N((1-\varepsilon)n)} = (1 - \varepsilon)^{1/2} \gamma^* \]
and the lower bound for (10) follows.

2.1 Poisson process interpretation of $\Psi_{\text{ave}}$

xxx hard to know how much to say here

The argument above interprets $\Psi_{\text{ave}}(s)$ as a $n \to \infty$ limit of the random $n$-city model. By standard “soft” arguments which we will not give here (see e.g. [2] section 3.5 for details in a somewhat similar setting) we can give an “exact” interpretation of $\Psi_{\text{ave}}(s)$ in terms of a Poisson (rate 1) point process of cities on the infinite two-dimensional plane. Consider a network $\mathcal{N}_{\infty}$ on such cities whose distribution $\mu$ is translation invariant. Associated with $\mu$ are two numbers: the stretch, say $\text{stretch}(\mu)$, and the normalized length (mean length-per-unit area), say $\text{len}(\mu)$, which is well-defined by translation invariance (of course these numbers might be $+\infty$). Then
\[ \Psi_{\text{ave}}(s) = \inf \{ \text{len}(\mu); \mu \text{ is translation invariant, } \text{stretch}(\mu) \leq 1 + s \}. \] (11)
3 Short networks and the Steiner constants

Write \( z^n = (z_1, \ldots, z_n) \) for a configuration of city positions in the square of area \( n \). Write ST\((z^n)\) for the Steiner tree (i.e. minimum length connected network) on \( z^n \), and for any network \( \mathcal{N} \) write \( \text{len}(\mathcal{N}) \) for its total length. By an easy “superadditive” argument similar to that in section 2, there exists a limit constant for worst-case normalized Steiner tree length:

\[
c_{\text{worst}} := \lim_{n \to \infty} \text{sup}_{z^n} n^{-1} \text{len}(\text{ST}(z^n)).
\] (12)

It is known [4] that \( c_{\text{worst}} \leq 0.995 \) and that (by considering the hexagonal lattice) \( c_{\text{worst}} \geq (3/4)^{1/4} = 0.9306 \). Clearly we must have \( \Psi_{\text{worst}}(s) \geq c_{\text{worst}} \), for all \( s \), and this inequality must persist in the limit (which exists by monotonicity): \( \lim_{s \to \infty} \Psi_{\text{worst}}(s) \geq c_{\text{worst}} \).

Turning to the average-case setting, it follows from the general theory of subadditive Euclidean functionals (cf. [10], [11]) that there exists a limit constant \( c_{\text{ave}} \) such that

\[
n^{-1} \text{len}(\text{ST}(Z_1, Z_2, \ldots, Z_n)) \to c_{\text{ave}} \quad \text{in } L^1 \] (13)

where the \( (Z_i) \) are independent uniform random in the area-\( n \) square. As above, we clearly have \( \lim_{s \to \infty} \Psi_{\text{ave}}(s) \geq c_{\text{ave}} \). It is natural to guess (but not obvious) that these limit inequalities are really equalities. This guess is correct, as an immediate corollary of the following estimate for arbitrary city configurations.

**Proposition 1** There exists a function \( \delta(s) \leq \infty \) with \( \lim_{s \to \infty} \delta(s) = 0 \), and a function \( K(s) < \infty \), such that for all \( 0 < s < \infty \), all \( n \geq K(s) \) and all city configurations \( z^n \) in the area-\( n \) square, there exists a network \( \mathcal{N} \) connecting cities \( z^n \) such that

\[
\text{stretch}(\mathcal{N}) \leq 1 + s; \quad n^{-1}(\text{len}(\mathcal{N}) - \text{len}(\text{ST}(z^n))) \leq \delta(s).
\]

**Corollary 2**

\[
\lim_{s \to \infty} \Psi_{\text{worst}}(s) = c_{\text{worst}} \quad \text{and} \quad \lim_{s \to \infty} \Psi_{\text{ave}}(s) = c_{\text{ave}}.
\]

The idea of the proof is to partition of the area-\( n \) square into rectangles containing at most \( K \) cities, and then use a crude construction (Lemma 3) of networks on \( K \) cities. We will set up some notation, state the lemma, give the reduction of the Proposition to the lemma, and then prove the lemma.
Fix $K \geq 0$. Let $A$ be a rectangle; write $\partial A$ for its boundary, so that $\text{len}(\partial A)$ is its boundary length. Let $y_1, \ldots, y_K$ be an arbitrary configuration of $K$ cities in $A$. Consider a network $N = N(A)$ in $A$ which includes the boundary $\partial A$ and links the cities. For such a network define

$$\text{stretch}^*(N) = \max_{y \neq y'} \frac{\text{route-length from } y \text{ to } y'}{d(y, y')}$$

(14)

where $y$ and $y'$ run over the cities and over points of $\partial A$.

**Lemma 3** Let $\hat{t}$ be the Steiner tree on the cities $y_1, \ldots, y_K$ and (possibly) other cities outside $A$. Let $t$ be the intersection of $\hat{t}$ with $A$. There there exists a network $N$ in $A$ such that

$$\text{stretch}^*(N) \leq \rho(K); \quad \text{len}(N) - \text{len}(ST(z^n) \cap A) \leq 2 \text{len}(\partial A)$$

where $\rho(K) < \infty$ depends only on $K$ and is nondecreasing in $K$.

**Proof of Proposition 1.** Fix $K$ and $n > K$. We use a simple and natural decomposition (e.g. called multidimensional search tree or $k - d$ tree in computational geometry [9]). Split the square $[0, n^{1/2}]^2$ into two rectangles using a vertical line through the city with median $x$-coordinate (if $n$ is odd) or a vertical line separating the two median $y$-coordinate cities (if $n$ is even). In either case, each rectangle has at most $n/2$ cities in its interior. Separately for each rectangle, split into two rectangles using horizontal lines through the median $y$-coordinate(s). Now (end of stage 1) we have 4 rectangles, each with at most $n/4$ cities in its interior. Continue recursively for $L$ stages, where $L$ is the smallest integer such that $n4^{-L} \leq K$, to get a partition into $4^L$ rectangles, each with at most $K$ cities in its interior. Write $A$ for a generic rectangle in this partition.

Given a configuration $z^n$ in $[0, n^{1/2}]^2$, apply Lemma 3 (where $\hat{t}$ is the Steiner tree on $z^n$) to each $A$ and the cities inside $A$ to obtain a network $N(A)$ satisfying

$$\text{stretch}^*(N(A)) \leq \rho(K); \quad \text{len}(N(A)) - \text{len}(\text{ST}(z^n) \cap A) \leq 2 \text{len}(\partial A).$$

(15)

Then consider the network $N$ on the cities $z^n$ obtained as the union of networks $N(A)$. Note that the bound $\rho(K)$ on $\text{stretch}^*(N(A))$ does not depend on $A$. For any pair of cities $z_i, z_j$, we can define a route in $N$ between them by considering the points $v_1, v_2, v_3, \ldots$ at which a straight line between them intersects boundaries of successive rectangles $A_1, A_2, A_3, \ldots$, and within each such rectangle $A$ use the shortest route in $N(A)$ between these
boundary points (or the cities themselves). It follows that \( \text{stretch}^*(\mathcal{N}) \leq \max_A \text{stretch}^*(\mathcal{N}(A)) \leq \rho(K) \). Note that the intermediate rectangles may contain no cities of \( z^n \), explaining why we must allow \( K = 0 \) in Lemma 3.

To bound \( \text{len}(\mathcal{N}) \), imagine a stage 0 in which the edges of the external boundary \( \partial_0 \) of \([0, n^{1/2}]^2\) are added. At stage 1, the length of roads added equals \( 2n^{1/2} \), and inductively at stage \( j \) the length of roads added equals \( 2^j n^{1/2} \). Because each segment of these added roads (except the external boundary) is part of the boundary of exactly 2 of the final rectangles \( A \),

\[
\sum_A \text{len}(\partial A) = \text{len}(\partial_0) + 2 \sum_{j=1}^{L} 2^j n^{1/2} = 2n^{1/2} (2 + \sum_{j=1}^{L} 2^j) = 2^{L+1} n^{1/2}.
\]

By definition of \( L \) we have \( n4^{-(L-1)} > K \), giving \( 2^L \leq 2n^{1/2}K^{-1/2} \), and so

\[
\sum_A \text{len}(\partial A) \leq 8nK^{-1/2}. \tag{16}
\]

So

\[
\text{len}(\mathcal{N}) \leq \sum_A \text{len}(\mathcal{N}(A)) \\
\leq \sum_A (\text{len}(\text{ST}(z^n) \cap A) + 2\text{len}(\partial A)) \text{ by (15)} \\
= \text{len}(\text{ST}(z^n)) + 2 \sum_A \text{len}(\partial A).
\]

Combining with (16),

\[
R(\mathcal{N}) \leq \rho(K); \quad n^{-1}(\text{len}(\mathcal{N}) - \text{len}(\text{ST}(z^n))) \leq 16K^{-1/2}.
\]

This establishes Proposition 1 for \( K(s) := \max\{K : \rho(K) \leq 1 + s\} \) and \( \delta(s) := 16K^{-1/2}(s) \).

**Proof of Lemma 3.** We may suppose \( A \) is an \( a_1 \times a_2 \) rectangle, where \( a_1 \leq a_2 \). The network \( \mathcal{N} \) will consist of

- \( t \) (the intersection of \( t \) with \( A \))
- the boundary \( \partial A \) of \( A \)
- extra edges, of total length at most \( \text{len}(\partial A) \).

Set \( m = \lceil a_2/a_1 \rceil \) and partition \( A \) into \( m + 1 \) similar \( a_1 \times \frac{a_2}{m+1} \) rectangles by using \( m \) equally spaced roads of length \( a_1 \). So the total length of these added roads is \( ma_1 \leq a_2 \leq \frac{1}{2} \text{len}(\partial A) \). So the network \( \mathcal{N}_0 \) consisting of \( t \)
and \( \partial A \) and these extra roads has \( \text{len}(\mathcal{N}_0) - \text{len}(t) \leq \frac{3}{2} \text{len}(\partial A) \). It is easy to check that this network \( \mathcal{N}_0 \) (without using the edges of \( t \)) satisfies

\[
\max_{y \neq y' \in \partial A} \frac{\text{route-length from } y \text{ to } y'}{d(y, y')} \leq 2.
\]

In particular, the \( K = 0 \) case of Lemma 3 holds with \( \rho(0) = 2 \).

Now consider the case \( K \geq 1 \). To cover the possibility that \( \hat{t} \) and hence \( t \) is entirely in the interior of one of the subrectangles of \( A \), add to \( \mathcal{N}_0 \) a road to the boundary from the city closest to the boundary. This road has length at most \( \frac{1}{2} a_1 \leq \frac{1}{8} \text{len}(\partial A) \), and the resulting network \( \mathcal{N}_1 \) has length \( \text{len}(\mathcal{N}_1) - \text{len}(t) \leq \frac{13}{8} \text{len}(\partial A) \).

Now set

\[
\eta := \frac{3}{8} \text{len}(\partial A) K + \left( \frac{K}{2} \right).
\]

Let \( \mathcal{N} \) be the network \( \mathcal{N}_1 \) augmented as follows: for each city within distance \( \eta \) from the boundary, add a road from the city to the closest boundary point; for each pair of cities within distance \( \eta \) of each other, add a road directly linking them. From the definition of \( \eta \), the extra length added in this stage is at most \( \frac{3}{8} \text{len}(\partial A) \), and so \( \mathcal{N} \) satisfies the length requirement

\[
\text{len}(\mathcal{N}) - \text{len}(t) \leq 2 \text{len}(\partial A)
\]

in Lemma 3.

It remain to bound \( \text{stretch}^*(\mathcal{N}) \). We quote a simple bound on Steiner tree length (given for squares in [1] Lemma 10; the extension to rectangles is straightforward).

**Lemma 4** Under the assumptions of Lemma 3,

\[
\text{len}(t) \leq C_1(K) \text{len}(\partial A)
\]

where \( C_1(K) \) depends only on \( K \).

So \( \text{len}(\mathcal{N}) \leq (2 + C_1(K)) \text{len}(\partial A) \) and then

\[
\frac{\text{len}(\mathcal{N})}{\eta} \leq \frac{8}{3} (2 + C_1(K))(K + \left( \frac{K}{2} \right)), \tag{17}
\]

To bound \( \text{stretch}^*(\mathcal{N}) \) we need to treat several cases for the pairs \((y, y')\) in (14). We have already obtained an upper bound of 2 for the case where both points are on the boundary. If the two points are at distance \( \geq \eta \) apart then,
because route length is at most network length, \( \frac{\ell(y,y')}{d(y,y')} \leq \frac{\text{len}(\mathcal{N})}{\eta} \). If the two points are cities within distance \( \eta \) then \( \frac{\ell(y,y')}{d(y,y')} = 1 \). The only remaining case is a city \( y \) within distance \( \eta \) from the boundary, and a boundary point \( y' \) within distance \( \eta \) from \( y \). In this case, by using the edge from \( y \) to the closest boundary point and then following the boundary we find (the worst case is near a corner) \( \frac{\ell(y,y')}{d(y,y')} \leq 3 \). So

\[
\text{stretch}^*(\mathcal{N}) \leq \max(3, \frac{\text{len}(\mathcal{N})}{\eta})
\]

and by (17) we have proved Lemma 3.
4 Upper bounds via a “freeways and access roads” construction

We will describe an elementary construction using parallel “freeways” in different directions, with “access roads” linking cities to nearby freeways. We will show that, for fixed $0 < s < 1$, the construction gives a $(1 + s)$-spanner with total length bounded by the quantity $\Psi^*(s)$ below, which is therefore an upper bound on $\Psi_{\text{worst}}(s)$.

**Theorem 5** For $0 < s < 1$ set

$$\phi_s = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{1+s}\right)$$

$$\Psi^*(s) = \frac{2\left\lceil \frac{\pi}{\phi_s} \right\rceil \sqrt{(1 + \lceil \frac{1}{s} \rceil) \tan \phi_s}}{\sin \phi_s}.$$  

Then $\Psi_{\text{worst}}(s) \leq \Psi^*(s)$.

In particular, as $s \to 0$ we have $\phi_s \sim \sqrt{2} s$ and then $\Psi^*(s) \sim 2^{1/4} \pi s^{-5/4}$, so

$$\Psi_{\text{worst}}(s) = O(s^{-5/4}) \text{ as } s \to 0.$$  

We will prove Theorem 5 in section 4.2. To show the idea and get simple explicit bounds, we first work with the square grid.

4.1 Constructions based on a square grid of roads

**Proposition 6**

$$\Psi_{\text{worst}}(1) \leq 4$$  

(19)

$$\Psi_{\text{worst}}\left(\frac{1}{2}\right) \leq 4\sqrt{2}$$  

(20)

$$\Psi_{\text{worst}}\left(\sqrt{2} - 1\right) \leq 4\sqrt{3}.$$  

(21)

**Proof.** XXX notation $s$

Fix $0 < s_\infty < \infty$ and choose $s = s(n) \to s_\infty$ such that $n^{1/2}/s(n)$ is an integer $m = m(n)$. First construct a network of grid roads which partition the region $[0, n^{1/2}]^2$ into $m^2$ squares of side-length $s$. These grid roads (including the boundary of $[0, n^{1/2}]^2$) have total length

$$n^{1/2} \times 2(m + 1) \sim 2n/s_\infty.$$
Next, for each city construct a north-south (N-S) and an east-west (E-W) road through the city and across the square containing the city. These access roads have total length $2sn$.

We now study the network $N^1_n$ thus constructed. We have already seen that

$$n^{-1} \text{len}(N^1_n) \to 2(s_\infty + \frac{1}{s_\infty})$$

so we need to bound the stretch. Note that in a right angle triangle with side-lengths $a, b, c = \sqrt{a^2 + b^2}$ we have

$$\frac{a+b}{c} \leq \sqrt{2}.$$

Thus to show that a city-pair $(i, j)$ has $R(i, j) \leq \sqrt{2} - 1$ it is enough to show that (supposing w.l.o.g. that city $j$ is to the south-west of city $i$) there is a route from $i$ to $j$ using only southward and westward roads.

![Figure 2. All the grid roads and some of the access roads.](image)

But, consulting Figure 2, this is clearly true in the three cases
(i) the two cities are in the same square (as $a$ and $b$)
(ii) the two cities are in different rows and different columns (as $a$ and $c$).
(iii) the two cities are in adjacent squares (as $a$ and $d$).
So it remains to consider the final case
(iv) the two cities are in squares in the same column (say) separated by some number $k \geq 1$ of squares.

The remainder of the argument rests upon being able to recognize, within case (iv), which city positions $(v, w)$ maximize $R(v, w)$. In the context of the square grid, these ‘worst situations’ are intuitively clear, and we will
state them without proof (xxx we will give proofs in the harder setting of the next section). It turns out (see Figure 3, left diagram) that the worst situation in case (iv) is where $k = 1$, this intervening square contains no cities, and the two cities are (arbitrarily close to) the centers of the north and the south edges of the intervening square (as $e$ and $f$). In this situation $R(e, f) = 1$, so this is an upper bound for case (iv). Thus the networks $\mathcal{N}^1_n$ have stretch($\mathcal{N}^1_n$) $- 1 \leq 1$. Consulting (4.1), we can choose $s_\infty = 1$ so that $\text{len}(\mathcal{N}^1_n) \sim 4n$, establishing (19).

![Figure 3](image)

**Figure 3.** Grid roads and interior roads. The access roads are not useful in this worst situation.

Now consider the networks $\mathcal{N}^2_n$ obtained from $\mathcal{N}^1_n$ by adding, for each square, the N-S and the E-W interior roads through the center of the square (and across the square). Now the case (iv) worst situation is where (as $g$ and $h$ in Figure 3, center diagram) the two cities are (arbitrarily close to) a quarter of the way along the north and the south edges of the intervening square. In this situation $R(g, h) = 1/2$, so this is an upper bound for case (iv). That is, stretch($\mathcal{N}^2_n$) $- 1 \leq 1/2$. The total extra network length is $2n/s$, so $n^{-1} \text{len}(\mathcal{N}^2_n) \rightarrow 2(s_\infty + \frac{2}{s_\infty})$. Choosing $s_\infty = \sqrt{2}$ gives $n^{-1} \text{len}(\mathcal{N}^2_n) \rightarrow 4\sqrt{2}$ and establishes (20).

Finally consider the networks $\mathcal{N}^3_n$ obtained from $\mathcal{N}^1_n$ by adding, for each square, two N-S and two E-W interior roads partitioning the square into nine equal subsquares. Here the case (iv) worst situation is where (as $e$ and $f$ in Figure 3, right diagram) the two cities are (arbitrarily close to)
half of the way along the north and the south edges of the intervening square. In this situation $R(e,f) = 1/3$, so this is an upper bound for case (iv). But here $1/3$ is less than the bound $\sqrt{2} - 1$ from the other cases. So $\text{stretch}(N^3_n) - 1 \leq \sqrt{2} - 1$. The total extra network length (relative to $N^1_n$) is $4n/s$, so $n^{-1}\text{len}(N^3_n) \to 2(s_\infty + \frac{3}{s_\infty})$. Choosing $s_\infty = \sqrt{3}$ gives $n^{-1}\text{len}(N^3_n) \to 4\sqrt{3}$ and establishes (21).

4.2 Proof of Theorem 5

xxx proof
5 Upper bounds by putting a road in every cone

We first show (Proposition 7) that one can achieve a given stretch $1 + s$ by insisting that the network contain roads from each city within each cone of appropriate base angle $\theta_s$. This idea is quite similar to the notion of $\Theta$-graph in [8] section 4.1. In section 5.1 we show how, in the random model, it is easy to construct networks with the desired property whose expected length can be calculated; this leads to bounds on $\Psi^{\text{ave}}(s)$, stated in Proposition 9. It seems plausible that one can get bounds on $\Psi^{\text{worst}}(s)$ in a similar way, adapting other methods from [8], but we have not investigated this question carefully.

Given a point $z$ in the plane and angles (relative to $x$-axis, as usual) $\phi$ and $\theta$, write cone($z, \phi, \phi + \theta$) for the cone bounded by the two rays from $z$ at angles $\phi$ and $\phi + \theta$ mod $2\pi$. Fix $0 < \theta < \pi$. Consider a network consisting only of city-to-city roads. Call a network $\theta$-dense if for each city $z$ and each $\phi$, if there exists another city in cone($z, \phi, \phi + \theta$), then there exists a road from $z$ to some city in that cone. One can find versions of Proposition 7 below for finite configurations, but it is simpler (and sufficient for our purposes) to work under the assumption

for each city $z$ and each $\phi$, the cone($z, \phi, \phi + \theta$) contains another city

which of course cannot hold for any finite configuration but does hold for the Poisson process on the infinite plane.

Proposition 7 Consider a locally finite configuration on the plane satisfying (22). Every $\theta$-dense network on the configuration has stretch $\leq \frac{1}{\cos \frac{\theta}{2}}$.

Proof. Fix two cities, w.l.o.g. at $(0,0)$ and $(x_0,0)$. We first show that the Proposition can be reduced to the following lemma.

Lemma 8 For some $-\theta < \phi < 0$, there exists a route from $(0,0)$ to $(x_0,0)$ such that the slope of each segment lies in the range $[\phi, \theta + \phi]$.

Each segment of the Lemma 8 route can be visualized as one edge of a triangle whose other edges are at angles $\phi$ and $\theta + \phi$, So the length of the route is upper bounded by the length of the path using these other edges of each triangle (this is a path in the plane, not a route in the network). The length of this path is the length of the path in the plane from $(0,0)$ to $(x_0,0)$ using a line of slope $\phi$ followed by a line of slope $\theta + \phi$. The length of this path is maximized (as $\phi$ varies) when $\phi = \theta/2$ in which case the length equals $x_0/\cos(\theta/2)$, establishing the Proposition 7 bound on stretch.
Proof of Lemma 8. Fix $\phi \in (-\theta, 0)$. Define a lower route from the city $(0,0)$ to some point on the line $\{(x,y) : -\infty < y < \infty\}$ via the simple procedure: $v_0 = (0,0)$, and inductively

from $v_i$, follow the road to $v_{i+1}$, where $v_{i+1}$ is chosen so that the angle of the segment $(v_i, v_{i+1})$ is the lowest possible value in $[\phi, \phi + \theta]$ amongst all roads from $v_i$.

At each step there is some possible choice, by assumption (22) and the assumption of $\theta$-dense. Stop the route where it crosses the line $\{(x,y) : -\infty < y < \infty\}$.

Define the analogous upper route using the maximum possible angle at each step. It is easy to check that the upper route lies (weakly) above the lower route. In particular, the routes are stopped at two points $(x_0, y_{\text{R lower}})$ and $(x_0, y_{\text{R upper}})$ where $y_{\text{R lower}} \leq y_{\text{R upper}}$. These are eastward routes, but we can also define the analogous westward routes, which start at city $(x_0, 0)$ and are stopped at points $(0, y_{\text{L lower}})$ and $(0, y_{\text{L upper}})$ where $y_{\text{L lower}} \leq y_{\text{L upper}}$. The roads in these routes are constrained to have angles in the same interval $[\phi, \phi + \theta]$ as in the eastward routes, xxx oriented eastwards.

To establish the lemma it is enough to show one of the eastward routes meets one of the westward routes at some point (23) because then the route from $(0,0)$ to $(x_0,0)$ (switching between eastward and westward routes at the meeting point) satisfies the conclusion of the lemma. Clearly (23) holds in the following cases:

(i) $0 \in [y_{\text{R lower}}, y_{\text{R upper}}]$
(ii) $0 \in [y_{\text{L lower}}, y_{\text{L upper}}]$
(iii) $y_{\text{upper}} \leq 0$ and $y_{\text{lower}} \leq 0$
(iv) $y_{\text{lower}} \leq 0$ and $y_{\text{upper}} \leq 0$.

There remain two symmetric cases; w.l.o.g. we take the case
(v) $y_{\text{upper}}^R < 0$ and $y_{\text{lower}}^L > 0$.

The argument so far uses a fixed $\phi$; now we exploit our freedom to choose $\phi$. Rewrite $y_{\text{lower}}^L$ as $y_{\text{upper}}^L$ and rewrite $y_{\text{upper}}^R$ as $y_{\text{upper}}^R$. We are working in the case: there exists $\phi_0 \in (-\theta, 0)$ such that $y_{\text{lower}}^L(\phi_0) > 0$ and $y_{\text{upper}}^R(\phi_0) < 0$.

Consider the eastward lower route for a given $\phi$. The route has some lowest angle, say $\phi \geq \phi$. As $\phi$ increases, the route does not change (and so $y_{\text{upper}}^R(\phi)$ does not change) until $\phi$ reaches $\phi^*$, at which stage $y_{\text{upper}}^R(\phi)$ may change but can only increase.

By considering $\phi$ arbitrarily close to 0, either there is a road from $(0,0)$ to $(x_0,0)$ (xxx in which case the result is trivial) or else $y_{\text{upper}}^R(\phi) > 0$. It follows that there exists some $\phi^* \in [\phi_0, 0)$ such that $y_{\text{upper}}^R(\phi^*) \leq 0$ but $y_{\text{upper}}^R(\phi^* + \epsilon) \geq 0$.
for all sufficiently small \( \varepsilon > 0 \). Now consider the two eastward lower routes for \( \phi^* \) and for \( \phi^* + \varepsilon \). The westward upper route for \( \phi^* \) must meet one of those eastward routes, so the conclusion of the lemma holds for \( \phi^* \).

xxx picture???

5.1 An upper bound on \( \Psi_{\text{ave}}(s) \)

As in section 2.1 we work with the Poisson process of cities on the infinite plane. There are several ways one might try to use Proposition 7; we will just treat one of the simplest. Fix \( k \geq 2 \). For \( 0 \leq i \leq k - 1 \) define a network \( \mathcal{N}_i \) by:

for each city \( z \), create a road to its closest neighbor city in \( \text{cone}(z, i\pi/k, (i + 1)\pi/k) \) and a road to its closest neighbor city in \( \text{cone}(z, \pi + i\pi/k, \pi + (i + 1)\pi/k) \).

Network \( \mathcal{N}_i \) has a certain normalized length (mean length per unit area) \( L_k \), calculated below. Construct a network \( \mathcal{N} \) as the union of \( \mathcal{N}_i \) over \( 0 \leq i \leq k - 1 \). Its normalized length equals \( kL_k \), by rotational symmetry of the Poisson point process. And it is clearly \( \theta \)-dense for \( \theta = 2\pi/k \), so Proposition 7 bounds its stretch by \( 1/\cos(\pi/k) \). In other words, using (11)

\[
\Psi_{\text{ave}}(s) \leq kL_k \leq \frac{2}{1 - 1/4} s^{-3/4} + o(1) \text{ as } s \to 0.
\]

Proof of Proposition 9. We need only calculate the normalized length \( L_k \) of network \( \mathcal{N}_0 \). Write \( 0 \) for the origin. Consider a position measured in polar coordinates as \( (r, \omega) \) with \( 0 < \omega < \pi/k \). So \( (r, \omega) \in \text{cone}(0, 0, \pi/k) \) and \( 0 \in \text{cone}((r, \omega), \pi, \pi + \pi/k) \). Suppose there are cities at \( 0 \) and at \( (r, \omega) \), with other cities at the points of a Poisson process. Define

\[
p(r, \omega) = \mathbb{P}((r, \omega) \text{ is nearest city to } 0 \text{ in } \text{cone}(0, 0, \pi/k))
\]
\[
p_1(r, \omega) = \mathbb{P}((r, \omega) \text{ is nearest city to } 0 \text{ in } \text{cone}(0, 0, \pi/k)
\]
\[\text{and } 0 \text{ is nearest city to } (r, \omega) \text{ in } \text{cone}((r, \omega), \pi, \pi + \pi/k)).\]
We assert
\[ L_k = \int_0^\infty \int_0^{\pi/k} r \left[ 2p(r,\omega) - p_1(r,\omega) \right] \, d\omega \, dr. \quad (25) \]

To argue this, first consider only roads \((v_L, v_R)\), written so that the \(x\)-coordinate of \(v_L\) is less than the \(x\)-coordinate of \(x_R\), and for which each city is the closest neighbor of the other city in the relevant cone. Since the density of possible positions of \((v_L, v_R)\) has intensity 1 (xxx explain) the normalized length will be
\[ \int_0^\infty \int_0^{\pi/k} r \left[ p_1(r,\omega) \right] \, d\omega \, dr. \]

If instead we consider only roads \((v_L, v_R)\) where \(v_R\) is the nearest neighbor to \(v_L\) in its cone but not conversely, then the normalized length of such roads is
\[ \int_0^\infty \int_0^{\pi/k} r \left[ p(r,\omega) - p_1(r,\omega) \right] \, d\omega \, dr. \]

By symmetry, the opposite possibility – \(v_L\) is the nearest neighbor to \(v_R\) in its cone but not conversely – makes the same contribution. Summing these three contributions gives (25).

To write formulas for \(p(\cdot)\) and \(p_1(\cdot)\), recall that the probability that the Poisson process assigns no cities to a region \(A\) equals \(\exp(-\text{area}(A))\). For \(p(\cdot)\), the relevant region is the finite cone \(0CE\) in Figure 4, which has area \(\pi r^2 / 2k\), and so
\[ p(r,\omega) = \exp\left( -\frac{\pi r^2}{2k} \right). \quad (26) \]

For \(p_1(\cdot)\), the relevant region is the entire region \(0ABCDEFG\) in Figure 4. The area of this region can be represented as
\[ \text{area of cone } 0CE, \text{ plus area of cone } DGA, \text{ minus area of parallelogram } 0BDF. \]

The parallelogram has height \(r \sin \omega\) and base \(r \cos \omega - r \frac{\sin \omega}{\tan \pi/k}\) and hence has area
\[ r^2 \left( \cos \omega - \frac{\sin \omega}{\tan \pi/k} \right) \sin \omega. \]

So the area of \(0ABCDEFG\) equals
\[ \frac{\pi r^2}{2k} + \frac{\pi r^2}{2k} - r^2 \left( \cos \omega - \frac{\sin \omega}{\tan \pi/k} \right) \sin \omega \]
and finally
\[ p_1(r,\omega) = \exp\left( -r^2 \frac{\pi}{k} - \cos \omega \sin \omega + \frac{\sin^2 \omega}{\tan \pi/k} \right) \]
\[ (27) \]
Returning to formula (25), because \( \int_0^\infty r^2 \exp(-ar^2) \, dr = \frac{1}{4} \pi^{1/2} a^{-3/2} \), we can integrate out \( r \) to get

\[
L_k = \frac{1}{4} \pi^{1/2} \int_0^{\pi/k} \left( 2\left(\frac{\pi}{2k}\right)^{-3/2} - \left[ \frac{\pi}{k} \cos \omega \sin \omega + \frac{\sin^2 \omega \tan \pi/k}{\tan \pi/k} \right]^{-3/2} \right) \, d\omega
\]

\[
= \sqrt{2k} - \frac{1}{4} \pi^{1/2} \int_0^{\pi/k} \left[ \frac{\pi}{k} - \cos \omega \sin \omega + \frac{\sin^2 \omega \tan \pi/k}{\tan \pi/k} \right]^{-3/2} \, d\omega
\]

which is formula (24).

\[
\frac{\pi}{k}
\]

\[
F
\]

\[
E
\]

\[
G
\]

\[
A
\]

\[
B
\]

\[
D = (r, \omega)
\]

\[
C
\]

\[
\text{Figure 4.}
\]

xxx numerics for small k
6 Lower bounds in long networks; average-case analysis

Turning to lower bounds, for $\Psi_{ave}$ we start by giving (Lemma 11) a reformulation of the interpretation (11) in terms of a Poisson point process on the infinite plane. In (11) we required the distribution $\mu$ of the network to be translation invariant; by applying a random rotation $\Theta$ (uniform on $(0, 2\pi)$) we may suppose also that $\mu$ is isotropic. Recall $\text{len}(\mu)$ denotes normalized length. Consider the number

$$\text{intersect}(\mu) = \text{mean number of intersections of network edges with the } x\text{-axis per unit length}$$

There is a general formula (xxx citation???) that for any isotropic translation invariant network,

$$\text{len}(\mu) = \kappa \times \text{intersect}(\mu); \quad \kappa = \frac{\pi}{2}. \quad (29)$$

Intuitively, the fact this is true for some constant $\kappa$ follows from linearity of expectation, and for the network consisting of horizontal lines separated by unit distance (normalized length = 1), rotated by $\theta$, the intersection rate is $|\sin \theta|$, so the randomly-rotated isotropic version has $\text{intersect}(\mu) = \mathbb{E} |\sin \Theta| = 2/\pi$, identifying the constant as $\kappa = \pi/2$. This argument is analogous to the intuitive explanation of the Buffon’s needle formula – see e.g. [6] Example 2.2.3.

We next define a certain interval process consisting of overlapping intervals of the real line. Fix $s > 0$. Consider a pair of points $z_1, z_2$ in the plane, where $z_1$ is above the $x$-axis and $z_2$ is below the $x$-axis. Consider the ellipse with foci $z_1, z_2$ defined by

$$\{ z : ||z - z_1|| + ||z - z_2|| = (1 + s)||z_2 - z_1|| \}$$

where $|| \cdot ||$ is Euclidean distance. This ellipse intersects the $x$-axis in a certain interval which we denote by $I_s(z_1, z_2)$. Now take a rate-1 Poisson process of cities in the plane, and define the interval process to consist of all intervals $I_s(z_1, z_2)$ such that $z_1$ is a city of the Poisson process above the $x$-axis and $z_2$ is a city of the Poisson process below the $x$-axis. Each point of the $x$-axis is covered by infinitely many long intervals, but what will concern us is the short intervals. Suppose we can associate with the interval process a process $M$ of marks such that

(i) $M$ is stationary (translation-invariant)

(ii) each interval $I_s(z_1, z_2)$ of the interval process contains at least one mark.

By stationarity there is a number rate($M$) giving the mean number of marks per unit length.
Lemma 11 $\Psi_{\text{ave}}(s) \geq \frac{\pi}{2} \inf_{\mathcal{M}} \text{rate}(\mathcal{M})$ where $\inf_{\mathcal{M}}$ denotes infimum over all mark processes as above.

Proof. Consider an isotropic translation invariant network $\mathcal{N}$ on Poisson cities with stretch at most $1 + s$. For each pair $(z^1, z^2)$ of cities, the shortest route between them must cross the $x$-axis somewhere in the interval $I(z^1, z^2)$. Thus the process $\mathcal{M}_0$ of points where network roads cross the $x$-axis satisfies (i) and (ii), and then by (11) and (29) the optimal $\mathcal{N}$ has

$$\Psi_{\text{ave}}(s) = \text{len}(\mathcal{N}) = \frac{\pi}{2} \text{rate}(\mathcal{M}_0).$$

xxx To continue: let $\mathcal{M}_*$ be a process which puts a mark at the start of each interval which does not overlap any shorter interval. Then $\inf_{\mathcal{M}} \text{rate}(\mathcal{M}) \geq \text{rate}(\mathcal{M}^*)$ and we can lower bound $\text{rate}(\mathcal{M}^*)$ by the argument below.
xxx old material, not edited.

Fix \( r > 0 \). Consider a pair of points \( z^1, z^2 \) where \( z^1 = (x_1, y_1) \) is above the \( x \)-axis (so \( y_1 > 0 \)) and \( z^2 = (x_1 + x_2, -y_2) \) is below the \( x \)-axis (so \( y_2 > 0 \); note \(-\infty < x_2 < \infty \) is the difference between the two \( x \)-coordinates). Consider the ellipse with foci \( z^1, z^2 \) defined by

\[
\{ z : ||z - z^1|| + ||z - z^2|| = (1 + r)||z^2 - z^1|| \}
\]

where \( ||\cdot|| \) is Euclidean distance. This ellipse intersects the \( x \)-axis in a certain interval which we denote by \( \text{cut}_r(x_1, x_2, y_1, y_2) \). Note that functionals such as

(i) length of \( \text{cut}_r(x_1, x_2, y_1, y_2) \)

(ii) \( ||z^2 - z^1|| \)

are shift-invariant, in the sense that they depend on \((x_2, y_1, y_2)\) but not on \(x_1\).

Consider for a moment point-pairs \( z^1, z^2 \) which are centered by requiring \( x_1 = 0 \). Consider some nonnegative function \( \psi_r(z^1, z^2) = \psi_r(0, x_2, y_1, y_2) \) such that

\[
\text{Leb}^3\{(x_2, y_1, y_2) : \psi_r(0, x_2, y_1, y_2) \leq b \} < \infty \text{ for all } b < \infty
\]

where \( \text{Leb}^3 \) is the natural 3-dimensional Lebesgue measure on \((x_2, y_1, y_2)\). For instance, the particular functionals above have this property. Extend \( \psi_r \) to general pairs \( z^1, z^2 \) by translation invariance.

Given \((x_2, y_1, y_2)\) define

\[
v(x_2, y_1, y_2) = \text{Leb}^4\{(\hat{x}_1, \hat{x}_2, \hat{y}_1, \hat{y}_2) : \text{cut}_r(\hat{x}_1, \hat{x}_2, \hat{y}_1, \hat{y}_2) \text{ intersects } \text{cut}_r(0, x_2, y_1, y_2) \text{ and } \psi_r(\hat{x}_1, \hat{x}_2, \hat{y}_1, \hat{y}_2) < \psi_r(0, x_2, y_1, y_2) \}
\]

\[
+ \text{Leb}^2\{(\hat{x}, \hat{y}_2) : \text{cut}_r(0, \hat{x}_2, \hat{y}_1, \hat{y}_2) \text{ intersects } \text{cut}_r(0, x_2, y_1, y_2) \text{ and } \psi_r(0, \hat{x}_2, \hat{y}_1, \hat{y}_2) < \psi_r(0, x_2, y_1, y_2) \}
\]

\[
+ \text{Leb}^2\{(\hat{y}_1) : \text{cut}_r(\hat{x}, x_2, \hat{y}_1, \hat{y}_2) \text{ intersects } \text{cut}_r(0, x_2, y_1, y_2) \text{ and } \psi_r(\hat{x}, x_2, \hat{y}_1, \hat{y}_2) < \psi_r(0, x_2, y_1, y_2) \}
\]

Then consider

\[
\int \int \int (1 - v(x_2, y_1, y_2))^+ \, dx_2 dy_1 dy_2. \tag{30}
\]

This depends on the choice of functional \( \psi_r \). We want to make it large, so we define

\[
G(r) = \sup \text{ of } (30) \text{ over choices of } \psi_r.
\]

One can see that \( G(r) \uparrow \infty \) as \( r \downarrow 0 \). We want a “good” lower bound on \( G(r) \) as \( r \downarrow 0 \). Obviously we can get a lower bound by choosing a particular functional such as (i) or (ii) and doing the calculation . . . .
7 Lower bounds on $\Psi_{\text{worst}}$ based on local optimality

Consider, for instance, the “square grid” configuration of cities at the points \{(i, j); -\infty < i, j < \infty\}. The usual “square lattice” network (roads between city pairs \((v, w)\) at distance 1) has normalized length = 2 and stretch = $\sqrt{2}$.

It is natural to conjecture this network is optimal, in the following sense.

**Conjecture 12** If a network on the square grid configuration has stretch $\leq \sqrt{2}$ then it normalized length is at least 2.

If true, this would imply $\Psi_{\text{worst}}(\sqrt{2} - 1) \geq 2$. Similarly, any result of the type

A particular network $N_0$ on a particular configuration is optimal, in the sense that any other network $N$ with stretch($N$) $\leq$ stretch($N_0$) = $s_0$ has normalized length $L(N) \geq L(N_0) = \ell_0$

would imply $\Psi_{\text{worst}}(s_0 - 1) \geq \ell_0$. However, we are unable to prove any result of this type. Instead, we can only prove weaker results of the following type. Consider the “alternate diagonals” network on the square grid, shown in Figure 5.

*Figure 5. The “alternate diagonals” network.*

By inspection, this network has normalized length = $\sqrt{2}$ and satisfies

route-length from $v$ to $w$ is $\leq \sqrt{2}$ for each city pair $(v, w)$ at distance 1.

(31)

We can prove this network is optimal with respect to those properties.

**Proposition 13** Any network on the square grid configuration satisfying (31) has normalized length $\geq \sqrt{2}$.

We call (31) a “local” condition, because it’s not the stretch.
Corollary 14 \(\Psi_{\text{worst}}(\sqrt{2} - 1) \geq \sqrt{2}\).

Proof of Proposition 13. Take some network. Consider a route through cities \(\ldots \rightarrow (-2,0) \rightarrow (-1,0) \rightarrow (0,0) \rightarrow (1,0) \rightarrow (2,0) \rightarrow \ldots\) using minimum-length routes between each successive pair of cities. As we traverse this route, we might backtrack, meaning that the \(x\)-coordinate of position might decrease, but discarding any backtracking segments leaves a (maybe disconnected) non-backtracking route \(((x,y(x)), -\infty < x < \infty)\). Call this “horizontal” route \(H_0\). We can define a measure on \(H_0\) by:

\[
\text{a line segment in } H_0 \text{ from } (x_1,y(x_1)) \to (x_2,y(x_2)) \text{ has measure } x_2 - x_1.
\]

Repeat for routes \(H_j\) through \(\ldots \rightarrow (-2,j) \rightarrow (-1,j) \rightarrow (0,j) \rightarrow (1,j) \rightarrow (2,j) \to \ldots\). Note that assumption (31) implies that routes \(H_j\) are disjoint as \(j\) varies, except possibly at isolated points. Let \(\mu_H\) be the sum over \(j\) of the measures on \(H_j\). It is clear that \(\mu_H\) has “density 1”, in the sense that

\[
\frac{\mu_H(A)}{\text{area}(A)} \to 1 \text{ as area}(A) \to \infty.
\]

Repeat the construction with vertical routes \(V_i\) through \(\ldots \rightarrow (i,-2) \rightarrow (i,-1) \rightarrow (i,0) \rightarrow (i,1) \rightarrow (i,2) \to \ldots\) to define a measure \(\mu_V\) which also satisfies (32).

Now write \(L\) for length measure on the edges of the network. Consider a point \((x,y)\) on a road segment at angle \(\theta\). By the disjointness property, this point is in at most one \(H_j\), in which case the density \(d\mu_H/dL\) at the point equals \(|\cos \theta|\), and in at most one \(V_i\), in which case the density \(d\mu_V/dL\) at the point equals \(|\sin \theta|\). It follows that

\[
\frac{d(\mu_H + \mu_V)}{dL}(x,y) \leq |\cos \theta| + |\sin \theta|.
\]

But always \(|\cos \theta| + |\sin \theta| \leq \sqrt{2}\), so for any region \(A\)

\[
\mu_H(A) + \mu_V(A) \leq \sqrt{2} \ L(A)
\]

and then (32) implies

\[
L(A) \geq (\sqrt{2} - o(1)) \ \text{area}(A) \text{ as area}(A) \to \infty.
\]

That is, normalized network length is at least \(\sqrt{2}\).
7.1 Another bound from hexagons

Here we show how the argument above can be applied to the hexagonal configuration of cities (Figure 7).

**Proposition 15** Let \( \mathcal{N} \) be a network on the hexagonal configuration such that

\[
\frac{\text{route-length from } v \text{ to } w}{d(v, w)} \leq \sqrt{3} \quad \text{for all (Euclidean) nearest-neighbor pairs } (v, w).
\]

Then its normalized length is at least \( 2^{-1/3/4} \).

**Corollary 16** \( \Psi_{\text{worst}}(\sqrt{3} - 1) \geq 2^{-1/3/4} = 1.14 \ldots \)

**Proof of Proposition 15.** Consider the hexagonal configuration with \( \ell = \) distance between nearest neighbors. The density of cities (number per unit area) is

\[
\rho(\ell) = 4 \cdot 3^{-3/2} \ell^{-2}.
\]

Figure 6 shows four adjacent cities \( ABCD \) in one hexagon. In that figure we see

\[
\frac{\text{len}(AZB)}{d(A, B)} = \frac{\text{len}(DZC)}{d(D, C)} = \sqrt{3}
\]

and it is easy to check the optimality property:

If \( \pi_1 \) and \( \pi_2 \) are paths in the plane from \( A \) to \( B \) and from \( C \) to \( D \) respectively, and if \( \max(\frac{\text{len}(\pi_1)}{d(A, B)}, \frac{\text{len}(\pi_2)}{d(D, C)}) \leq \sqrt{3} \), then the paths cannot meet except possibly at \( Z \).

![Figure 6](image)

**Figure 6.** An optimality property.

Now consider the minimum-length route in \( \mathcal{N} \) through a “slope \( \pi/6 \) staircase” like \( abcdef \ldots \) in Figure 7. By assumption (38) and the optimality property above, this route does not meet the corresponding route through the next staircase \( ghi j k l \ldots \) except at isolated points. As in the previous
section, each path segment on such a route is at some angle $\theta$ to the slope $= \pi/6$ line; put a measure on the non-backtracking parts of the route with density $\cos \theta$ w.r.t. length measure $L$ on the segment. Repeating for each slope $= \pi/6$ staircase gives a measure $\mu_{\pi/6}$ on network edges, which has the property

$$\frac{\mu_{\pi/6}(A)}{\lambda_{\pi/6}(a)} \to 1 \text{ as area}(A) \to \infty$$

where $\lambda_{\pi/6}$ is length measure on parallel slope $= \pi/6$ lines through the staircases. The orthogonal distance between such lines equals $3\ell/2$ (this is easiest to see with the slope $= \pi/2$ lines, where the distance is the average of $d(c,d)$ and $d(b,k)$), so

$$\frac{\lambda_{\pi/6}(A)}{\text{area}(A)} \to \frac{2}{3\ell} \text{ as area}(A) \to \infty$$

and thus

$$\frac{\mu_{\pi/6}(A)}{\text{area}(A)} \to \frac{2}{3\ell} \text{ as area}(A) \to \infty. \quad (36)$$

![Hexagonal configuration](Image)

**Figure 7.** The hexagonal configuration.

Repeat the construction with staircases like $cdkl\ldots$ with slope $= -\pi/6$ to get a measure $\mu_{-\pi/6}$ on the associated routes; repeat again with staircases like $jkde\ldots$ with slope $= -\pi/2$ to get a measure $\mu_{\pi/2}$. These measures also satisfy (36). (Note each adjacent pair of cities is in two staircases, of different slopes). The analog of (33) is that, at a point $(x,y)$ on a road
segment at angle $\theta$,
\[
\frac{d(\mu_{\pi/2} + \mu_{\pi/6} + \mu_{-\pi/6})}{dL}(x,y) \leq |\cos(\theta - \pi/2)| + |\cos(\theta - \pi/6)| + |\cos(\theta + \pi/6)|
\]
because the point is in at most one route for each of the three slopes. But
\[
|\cos(\theta - \pi/2)| + |\cos(\theta - \pi/6)| + |\cos(\theta + \pi/6)| \leq 2
\]
and so
\[
(\mu_{\pi/2} + \mu_{\pi/6} + \mu_{-\pi/6})(A) \leq 2L(A).
\]
Use (36) to see
\[
L(A) \geq (\ell^{-1} - o(1)) \text{area}(A) \text{ as area}(A) \to \infty. \quad (37)
\]
Our normalization convention is that cities have density 1, that is $\rho(\ell) = 1$ at (35), so $\ell^{-1} = 2^{-1}3^{3/4}$ is the lower bound.

7.2 The triangular lattice

We sketch the minor modification which uses the triangular lattice (Figure 8).

Figure 8. The triangular configuration.

**Proposition 17** Let $\mathcal{N}$ be a network on the triangular configuration such that
\[
\frac{\text{route-length from } v \text{ to } w}{d(v,w)} \leq \frac{1}{2} + \sqrt{\frac{3}{4}} \text{ for all (Euclidean) nearest-neighbor pairs } (v,w).
\]
\[
(38)
\]
Then its normalized length is at least $2^{-1/2}3^{3/4}$. 

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Corollary 18. \( \Psi_{\text{worst}}(\sqrt{\frac{3}{4}} - \frac{1}{2}) \geq 2^{-1/2}3^{3/4} = 1.61 \ldots \)

Outline proof of Proposition 17. We indicate changes in the previous argument. The density of cities is now

\[
\rho(\ell) = 2 \cdot 3^{-1/2} \ell^{-2}.
\]  

(39)

Figure 9 shows four cities \( ABCD \) in the triangular configuration. In that figure we see

\[
\frac{\text{len}(AZD)}{d(A,D)} = \frac{\text{len}(BZD)}{d(B,C)} = \frac{1}{2} + \sqrt{\frac{3}{4}}
\]

and it is easy to check the optimality property:

if \( \pi_1 \) and \( \pi_2 \) are paths in the plane from \( A \) to \( D \) and from \( C \) to \( B \) respectively, and if \( \max(\frac{\text{len}(\pi_1)}{d(A,D)} : \frac{\text{len}(\pi_2)}{d(B,C)}) \leq \frac{1}{2} + \sqrt{\frac{3}{4}} \), then the paths cannot meet except possibly at \( Z \).

Figure 9. An optimality property.

As before, there is a measure \( \mu_0 \) on routes through cities like \( abckl \) on “slope = 0 routes, and measures \( \mu_{\pi/3} \) and \( \mu_{-\pi/3} \) associated with slopes \( \pi_3 \) (like \( hicde \)) and \( -\pi/3 \). These satisfy

\[
\frac{d(\mu_0 + \mu_{\pi/3} + \mu_{-\pi/3})}{dL}(x, y) \leq |\cos(\theta)| + |\cos(\theta - \pi/3)| + |\cos(\theta + \pi/3)| \leq 2.
\]

The orthogonal distance between parallel lines is \( \ell \sqrt{3/4} \), and repeating the argument for (37) leads to

\[
L(A) \geq (3^{1/2} \ell^{-1} - o(1)) \text{area}(A) \quad \text{as} \quad \text{area}(A) \to \infty.
\]

Taking \( \rho(\ell) = 1 \) in (39), the lower bound on normalized length is \( 3^{1/2} \ell^{-1} = 2^{-1/2}3^{3/4} \).
7.3 xxx

We could repeat the arguments above in a less symmetric context. Take the square grid configuration. Take some finite set of integer pairs \((a_1, b_1), \ldots, (a_k, b_k)\). For each \(i\) consider the line through \((0,0)\) and \((a_i, b_i)\) and the parallel lines through \((0,1), (0,2), \ldots\). Any choices lead to some lower bound.
8 Final remarks

z. Figure 10 shows the numerical values of the upper and lower bounds on $\Psi_{\text{worst}}$ that we have proved. Plainly there is scope for numerical improvements!

![Graph](image)

**Figure 10.** Lower $\circ$ and upper $\bullet$ bounds on $\Psi_{\text{worst}}$, from Proposition 6 and Corollaries 14, 16, 18. The dashed lines are the lower and upper bounds on the asymptote $c_{\text{worst}}$ from section 3. The plot has $s$ on the vertical axis.

xxx do graphs of $\Psi_{\text{ave}}$ too.

z. Roughly speaking, given any scheme for assigning edges to $n$ cities, in the $n \to \infty$ limit the values of stretch in the worst case is the same as in the average case, because the worst case configuration for $n$ will occur somewhere in a random instance of size $n' \gg n$.

z. xxx why different methods in ave and worst case? Partly convenience, partly natural calculations with Poisson process.
References


