Finite rooted tree. Each vertex $v$ defines a subtree rooted at $v$. Picking $v$ at random (uniformly) gives the random fringe subtree.
$T = \{ \text{ finite rooted trees } \}$, countable set.
$T_n : \text{ random tree, size } n.$
$F_n : \text{ random fringe subtree of } T_n.$

**Empirical Observation.** For most “natural” families of random trees,

$$F_n \to F \text{ (say), as } n \to \infty.$$ 

2. Maybe asymptotic cycling instead.
3. The point is

$$P(F = ) = \text{ asymptotic prop. of leaves in } T_n.$$ 

degree of root of $F = \text{ asymptotic distribution of out-degrees in } T_n$.

Informally, $F$ describes limits of all “local” properties of $T_n$. 
The Pessimist’s View of Life

1. You’re born; you have a random number of children at random times; you die.

2. Your children behave in the same way, independently of you.

The mathematical model is alternatively called the Crump-Mode-Jagers general continuous-time supercritical branching process.
Model. B.p. where each individual has $C$ children ($EC > 1$) at times $(\xi_1, \xi_2, \ldots, \xi_C)$ (arbitrary distribution) after own birth.

Standard facts.

1. Number born before $t \sim Z e^{\theta t}$ for a certain constant $\theta$.

2. Pick individual at random from those born before $T$; look at descendants born before $T$. As $T \to \infty$ this “random family tree” has the following limit.

Start the b.p. with 1 individual and watch for an exponential ($\theta$) time.

Point. Limits often exist for models of random trees with no b.p. structure.
Example where bare-hands calculations easy. **Trie** on 5 strings

.10010...
.01100...
.00011...
.11001...
.10110...

A natural model for a random trie $T_m$ on $m$ strings is to take all bits independent uniform.

For a given string $v$, it’s easy to calculate

(Binomial dist!)

\[
P(\text{v is vertex of random trie })
\]

\[
P( \text{fringe subtree at } v \text{ has } r \text{ leaves }).
\]

random trie has a “self-reducibility” property:
given the subtree at $v$ has exactly $r$ leaves, it is distributed as the random trie on $r$ strings.

There is an asymptotic fringe dist. $\mathcal{F}$ (as $m \to \infty$ and $\log_2 m$ modulo (1) $\to$ constant) which must be of the form

$$P(\mathcal{F} = \cdot) = \sum_r a_r P(T_r = \cdot)$$

for constants $(a_r)$. 
By analogy with \((S_n - a_n)/b_n \to Y\), ask:

“What distributions \(\pi\) on \(T\) occur as limits of random fringe subtrees \(\mathcal{F}_n\)?”

Answer: \(\pi\) occurs iff \(\pi Q = \pi\), where \(Q = Q(t, t^*)\) is the matrix counting first-generation subtrees.

\[
\begin{align*}
Q(t, \quad) &= 2. \\
Q(t, \quad) &= 1. \\
Q(t, \quad) &= 1.
\end{align*}
\]

Call such \(\pi\)’s fringe distributions. Given such a \(\pi\), define a Markov transition matrix by

\[
P_{\pi}(t, t^*) = \pi(t^*)Q(t^*, t)/\pi(t).
\]

Run this chain with initial dist. \(\pi \ldots\)
Analogy: weak limits (e.g. Brownian motion) may have lots more structure that the approximating processes do.]

**Abstract b.p. with arbitrary types.**

Type space $S$. Individual, type $s$, has random number/type of children.

$$R(s, \cdot) = E(\# \text{ offspring with type } \in \cdot).$$

Given a p.m. $\nu$ on $S$, define random tree $\mathcal{F}$ to be family tree of descendants of individual, type $\sim \nu$. Suppose finite.

**Fact.** $\mathcal{F}$ has a fringe dist. iff $\nu R = \nu$.

**Empirical observation.** (N.B. selection bias!) In all examples where asymptotic fringe dist. is known, it has description as abstract b.p. with simple type-space, even though underlying trees are not b.p.’s.
Harder example: **greedy undirected tree**.

\( n \) vertices, one distinguished (root). Start with no edges. Repeat
“add edge, chosen at random (uniformly) from set of all edges whose addition would not create a cycle”

**Fact.** Asymptotic fringe dist. is abstract b.p. with type space \((0, \infty)\) and invariant dist. \( \nu \).
Type \( s \) individual has:
- Poisson(\( \lambda(s) \)) offspring of type \( s \)
- Poisson (rate \( \rho(s^*) \)) process of offspring of types \( s^* < s \)

So asymptotic proportion of leaves is
\[
\int_0^\infty \exp(-\lambda(s)) - \int_0^s \rho(s^*)ds^* \nu(ds) \\
\approx 0.408.
\]
More precisely, let $L_n = (\text{random})$ proportion of vertices of $T_n$ which are leaves: then

$$EL_n \to 0.408\ldots$$

Guess: really $L_n \to 0.408\ldots$ in probability.

Not easy to use Chebyshev’s inequality here, but following abstract result applies.
Random trees $\mathcal{T}_n$, random fringe subtrees $\mathcal{F}_n$. Suppose asymptotic fringe $\mathcal{F}$ exists. This says
\[ E\phi(\mathcal{F}_n) \to E\phi(\mathcal{F}) \] (all bounded $\phi$).

**Theorem.** In order that
\[ \phi(\mathcal{F}_n) \to_p \phi(\mathcal{F}) \] (all bounded $\phi$),
it suffices that the “ancestor chain”
$s \to$ type of ancestor of $s$
\[ R^*(s, ds^*) = \nu(ds^*)R(s^*, ds)/\nu(ds) \]
has trivial invariant $\sigma$-field.

**Remarks.** 1. For CMJBP, ancestor chain is renewal process, and Choquet-Deny theorem applies.
2. Otherwise, seek to verify by coupling.
3. Proof: show $\text{dist}(\mathcal{F})$ is extreme amongst all fringe distributions.