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Percolating paths through random points

- Focus on one particular set of problems (which look easy)

- Digression to different views of "big picture".

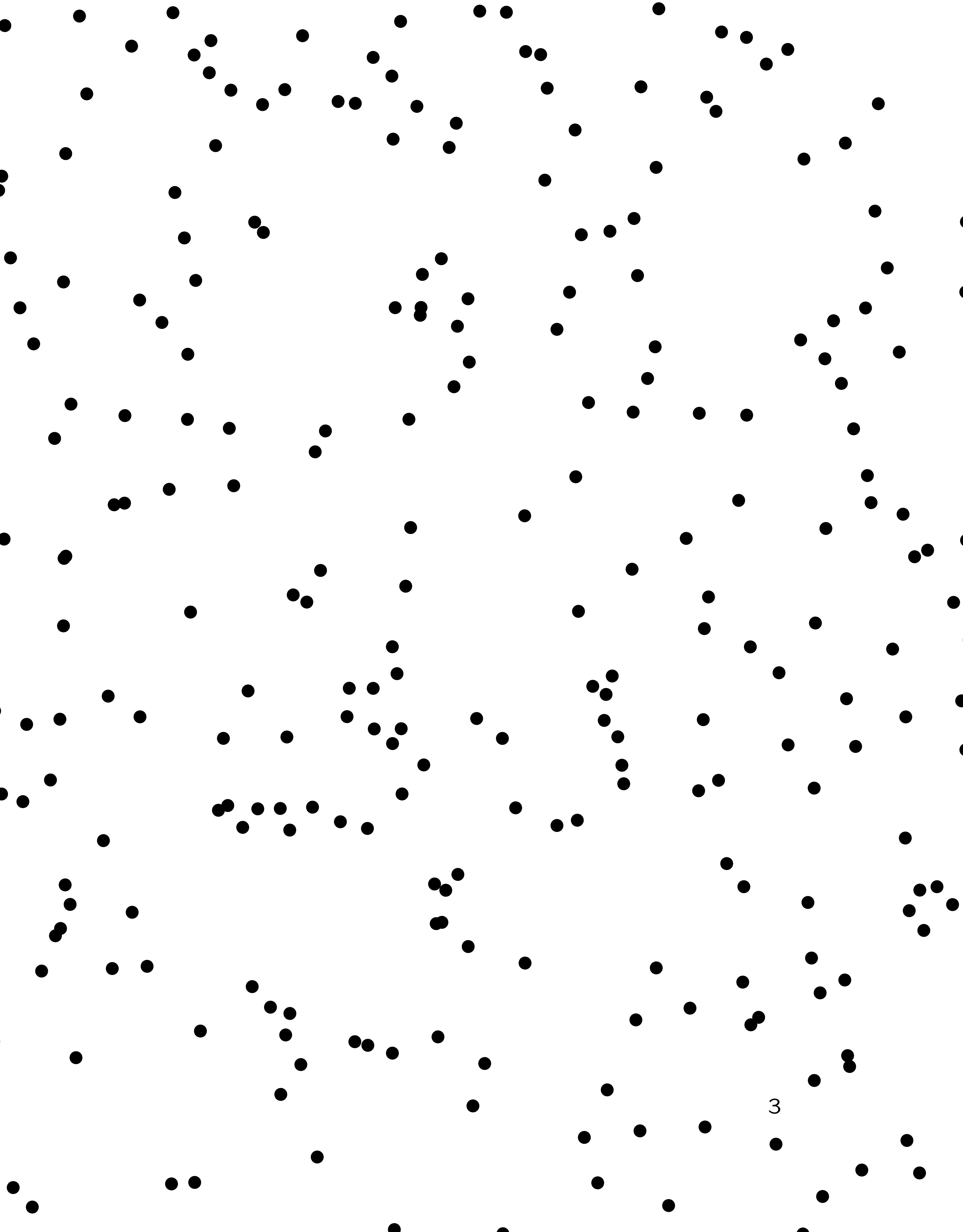
Big Picture

Random variable X_n associated with some "size n" random structure. Seek to study EX_n. Suppose
(i) can't do useful explicit calculations within size-n model
(ii) know order-of-magnitude, say order n.
Guess there is some limit constant c

$$n^{-1} EX_n \to c. \quad (1)$$

Two well-known techniques one can use to try to prove (1):

- subadditivity
- weak convergence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Topic of this talk. Take a Poisson point process (PPP) of rate 1 in $\mathbb{R}^d$ for $d \geq 2$. There should be some number whose intuitive interpretation is

"smallest possible average edge length in a path through an infinite subset of points of the PPP".

Analog to critical value in continuum percolation, where a rigorous definition is easy.

We'll discuss 4 possible formalizations.

1. Short paths from the origin. For each $m \geq 1$ define a r.v.

T_m = length of shortest path $0, \xi_1, \xi_2, \ldots, \xi_m$ through m distinct points of the PPP.

Guess: $T_m/m \to$ constant.

But is this easy to prove?

2. Paths across a diagonal. For $s > 0$ consider the cube $[0, s]^d$ with 0 and $\mathbf{s}$ as diagonally opposite vertices. For a path π: $0, \xi_1, \xi_2, \ldots, \xi_m, \mathbf{s}$ through distinct points of the PPP in $[0, s]^d$, write

$$m(\pi) = \text{number of points}$$

$$\ell(\pi) = \text{length of path}$$

and then define a r.v.

$$W_s = \min_\pi \frac{\ell(\pi)}{m(\pi)}$$

(minimum of average edge-length in a path).

Guess: $W_s \to$ constant as $s \to \infty$.

This definition designed for help with subadditivity.

3. Cycles through a given proportion of points.

Poissonized version of a result going back to Beardwood-Halton-Hammersley (1959) on "the Euclidean TSP":

Write $N(s)$ for number of points of the PPP in the cube $[0, s]^d$. Define $L_s(1) :=$

$$\frac{\text{length of shortest cycle through all } N(s) \text{ points}}{N(s)}$$

(minimum of average edge-length in a tour).

BHH proved (subadditivity argument) $L_s(1) \to c(1)$.

We consider a variation: Define $L_s(\delta) :=$

$$\frac{\text{length of shortest cycle through some } \lceil \delta N(s) \rceil \text{ points}}{\lceil \delta N(s) \rceil}$$

(minimum of average edge-length in a sparse cycle).

Guess: $L_s(\delta) \to c(\delta)$ as $s \to \infty$.

Is this easy to prove by subadditivity?

Guess:

the function $\delta \to c(\delta)$ is increasing;

the limit $c(0+)$ is the limit constant in Formalizations 1 and 2.

Seek definition directly on $\mathbb{R}^d$, as with continuum percolation.

4. Invariant paths on $\mathbb{R}^d$. Consider pair $(\mathcal{X}, \mathcal{E})$ where $\mathcal{X}$ is a locally finite point set in $\mathbb{R}^d$ and $\mathcal{E}$ is set of edges (x_i, x_j) with $x \in \mathcal{X}$, these edges forming a collection of doubly-infinite paths. Formalize space S of such pairs – marked point process. Consider a translation-invariant probability measure μ on S under which the points form a rate-1 PPP. There there exist constants $\delta(\mu), \ell(\mu)$ such that, writing $\mathcal{V}$ for end-vertices of $\mathcal{E}$,

$$E \left| \mathcal{V} \cap [0, s]^d \right| = \delta(\mu)\ s^d$$

$$E \left(\text{length of } \mathcal{E} \cap [0, s]^d \right) = \delta(\mu)\ell(\mu) s^d.$$

Via Palm theory, interpret
$\delta(\mu) =$ proportion of the Poisson points which are in some path
$\ell(\mu) =$ average edge-length within paths.

Define $\bar{c}(\delta) := \inf\{\ell(\mu) : \quad \delta(\mu) = \delta\}$

Guess: $\bar{c}(\delta) = c(\delta)$ (from formalization 3).

Easy to prove via weak convergence?

Which of these guesses are in fact easy to prove?

Recall how subadditivity is used in Beardwood-Halton-Hammersley. Same ideas work to prove

$$L_s(\delta) \to c(\delta) \text{ as } s \to \infty.$$

Moreover there are two cheap tricks:

(i) use proportion δ points in some subsquares, 0 in others;
(ii) use proportion δ_1 points in some subsquares, proportion δ_2 in others
which show
(i) $\delta \to c(\delta)$ is weakly increasing;
(ii) $\delta c(\delta)$ is convex.
This implies: either
(a) $c(\delta)$ is strictly increasing on $0 < \delta < 1$;
or (b) $c(\delta)$ is constant on some $0 < \delta < \delta_0$.

How to relate "paths across a diagonal" to this?

If we know a limit constant exists for W_s, easy to show limit $= c(0+)$.

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One can give general result on "optimal cost/reward ratios" in subadditive settings. The trick is: for constant γ the criterion

$$E\,\min\{\ell(\pi) - \gamma m(\pi) : \ \pi \text{ path 0 to } \mathbf{s}\} \geq 0 \ \forall s$$

determines a critical value γ_0 which is the limit

$$W_s := \min\{\ell(\pi)/m(\pi) : \ \pi \text{ path 0 to } \mathbf{s}\} \to \gamma_0.$$

One can invent many other problems which can be solved this way ....

Short paths from the origin. T_m = length of shortest path through m distinct points of the PPP. Natural approach:

Let's suppose $T_m/m \to c^*$ where (easy) $c^* \leq c(0+)$.

Then the **Conjecture** $c^* = c(0+)$ is equivalent to:

there exist m-step paths from the origin, with length $\sim c^* m$, which stay inside ball of radius $o(m)$ (**sublinear growth**).

Maybe proof requires more sophisticated "percolation" techniques.

Invariant paths on $\mathbb{R}^d$.

easy to justify via **local weak convergence**, which looks at a window around a randomly-chosen origin in the cube $[0, s]^d$.

Letting $s \to \infty$ and considering a subsequential weak limit gives a translation-invariant distribution on points-and-paths.

Summary of "percolating paths through random points"

Easy to prove equivalence of
(2) Paths across a diagonal
(3) Cycles through a given proportion of points
(4) Invariant paths on $\mathbb{R}^d$
and that $c(0+) > 0$ (comparison with branching RW).

Open Problems

- (1) Short paths from the origin?
- $c(\delta)$ strictly increasing?
- $c(\delta) - c(0+) \asymp \delta^\alpha$ for some α, maybe $\alpha = 1/3$?
- Monte Carlo study of $c(\delta)$?
- $\mathrm{var}(T_m) \asymp m^{2/3}$, or just $o(m)$?

The invariant measure on (collections of0 infinite paths fits theme **Stochastic analysis and non-classical random processes.** Can we do calculations with this type of random object?

$\exists$ lots of scattered work on discrete infinite random graphical structures of different kinds ...... In particular there is a "mean-field" model where one can do explicit calculations.

In our Euclidean setting, no hope for explicit calculation on $c(\delta)$. But maybe

(i) study strict monotonicity of $c(\delta)$

(ii) let $\delta \to 0$, do spatial rescaling; guess limit is some continuum self-avoiding path – related to SLE ???