1. Local weak convergence of graphs/networks

- Stuff that's obvious when you think about it

- 4 non-obvious examples/results

2. The core idea in our probabilistic reformulation of special cases of the cavity method is: do exact calculation on some infinite random graph (tree-like, in practice). LWC provides link with the finite-n problem. Illustrate with

- mean-field TSP

- flow through a disordered network.

Aldous-Steele survey "The objective method ..." on my home page.

Some math infrastructure

Consider an abstract space S (complete separable metric space) with a notion of convergence $x_n \to x$. There is automatically a notion of convergence of probability measures on S (all reasonable definitions are equivalent).

$\mu_n \to \mu_\infty$ iff there exist S-valued random variables X_n such that

$$\text{dist}(X_n) = \mu_n; \quad P(X_n \to X_\infty) = 1.$$

This is called **weak convergence**.

Conceptual point: When you consider some new abstract space S, you don't need to think about what convergence of distributions means.

Most concrete case is $S = R^1$, where we have e.g. the central limit theorem

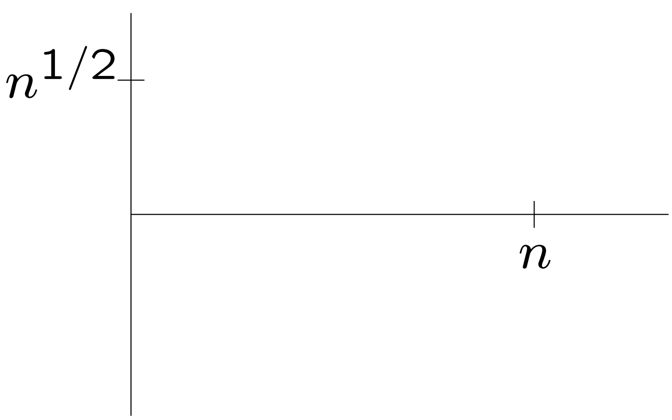
$$n^{-1/2} \sum_{i=1}^{n} \xi_i \ \xrightarrow{d}\ \text{Normal}(0,1)$$

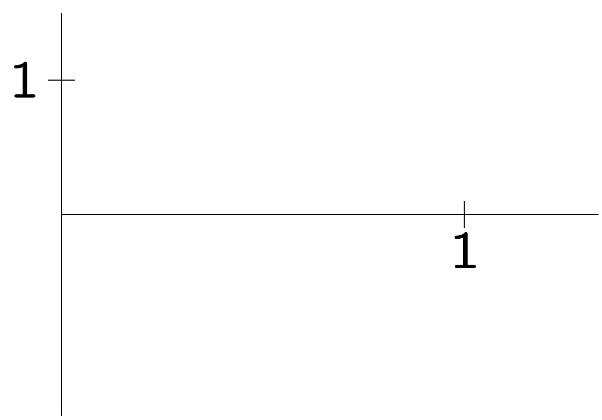
for i.i.d. (ξ_i) with $E\xi = 0$ and var $\xi = 1$.

Best known abstract case is

$$S = \{\text{continuous functions } [0,1] \to R\}$$

which allows one to formalize “rescaled random walk converges to Brownian motion”.

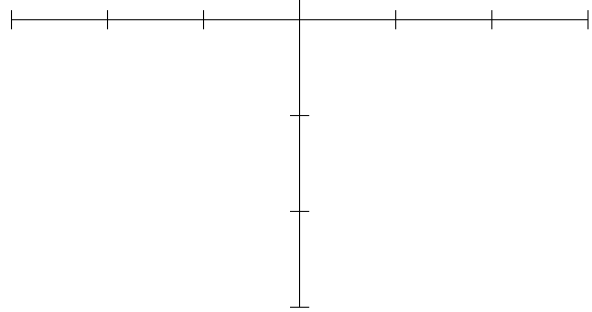




Another abstract case is

$$S = \{\text{locally finite point sets in } R^2\}$$

which allows one to formalize "n uniform random points in square of area n converges to the Poisson point process on R^2".



A **graph** has vertices and edges
A **network** is a graph whose edges have postive real lengths (default length = 1) and maybe extra structure indicated by numbers/labels on vertices/edges. Write G for a network.

Consider the abstract space

$$S = \{\text{locally finite rooted networks}\}.$$

What should convergence $G_n \to G_\infty$ mean?
Note: interesting case is where G_n is finite and G_∞ is infinite.

Window of radius r in G defines subgraph $G[r]$ of vertices within distance r from root, with edges **both** of whose endpoints are within window (a convention which turns out convenient).

Definition: $G_n \to G_\infty$ means that for each fixed generic $0 < r < \infty$, for large n there is graph-isomorphism between $G_n[r]$ and $G_\infty[r]$ such that edge-lengths of isomorphic edges converge as $n \to \infty$ (and also other labels converge).

Given n-vertex network (deterministic or random) let U_n be uniform random vertex. Write $G_n[U_n]$ for G_n rooted at U_n.

Definition. If $G_n[U_n] \stackrel{d}{\to}$ some G_∞, call this **local weak convergence (LWC)** of G_n to G_∞.

Formalizes the idea: for large n the local structure of G_n near a typical vertex is approximately the local structure of G_∞ near the root.

Intuition: in models where degree distribution is bounded in probability as $n \to \infty$ we expect LWC to some limit infinite network.

Obvious examples

1a G_n: geometric graph (all edges of length $\leq c$) on n random points in square of area n

G_∞: geometric graph on Poisson point process (rate 1) on R^2 with point at origin.

1b As above with complete graphs.

2 G_n: discrete cube $C^d_m \subset Z^d$ with i.i.d. (independent random) edge-lengths.

G_∞; all Z^d with i.i.d. edge-lengths.

3a G_n: Erdos-Renyi random graph $\mathcal{G}(n, c/n)$

G_∞: tree of Galton-Watson branching process with Poisson(c) offspring.

3b G_n: random r-regular graph

G_∞: infinite degree-r tree.

3c G_n: random graph model designed as "random subject to degree distribution approximately a prescribed distribution $(p(i), i \geq 0)$

G_∞: Galton-Watson tree with offspring distribution $\tilde{p}(i) \propto (i+1)p(i+1)$

4 G_n: de Bruijn graph on $n = 2^b$ binary strings

G_∞: infinite tree with in-degree 2 and out-degree 2.

100100

001100 110010

100101

011001

100110

101100 110011

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5 G_n: Simple random walk with n steps

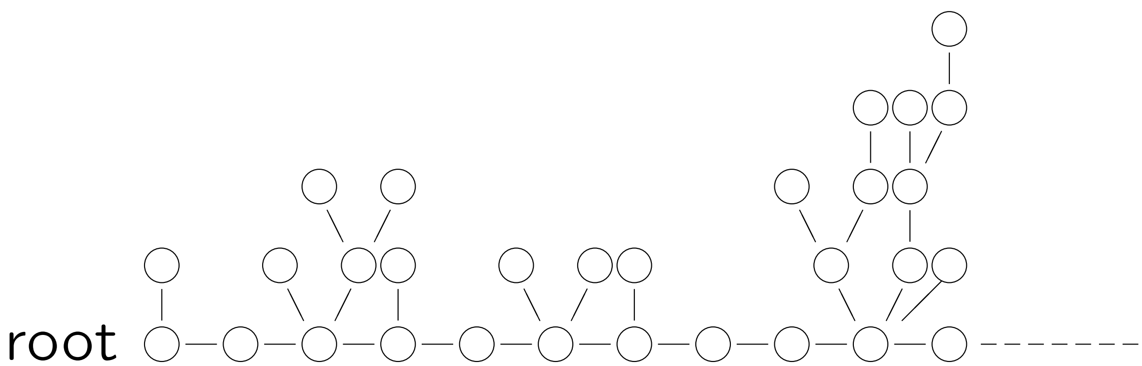
G_∞: 2-sided infinite simple RW.

(represent as linear graph with edge-marks ± 1).

6a-z: For many models of random n-vertex trees one can explicitly describe G_∞. For instance

G_n: uniform random tree on n labeled vertices

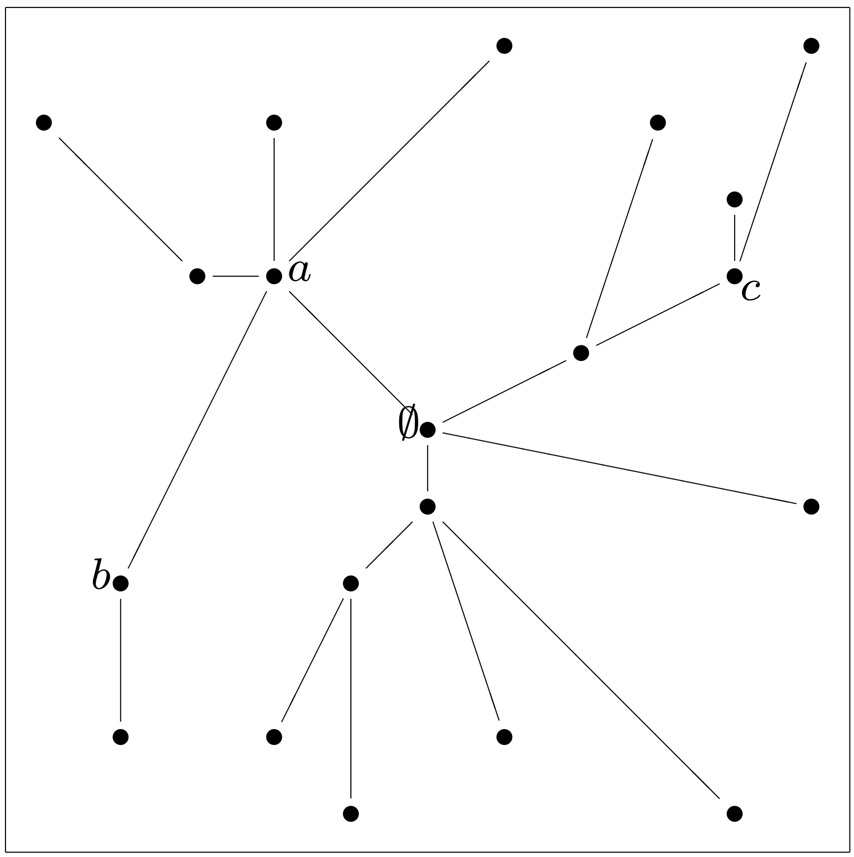
G_∞: infinite path from root; i.i.d. "bushes" are Galton-Watson trees with Poisson(1) offspring.



Remark: qualitative behavior similar in most models: semi-infinite path with i.i.d. finite bushes, whose mean size is infinite. Bush at root gives limit of subtree defined by random vertex in original rooted tree.

7 G_n: complete graph on n vertices; edge-lengths random, independent Exponential(mean n) distribution.

G_∞: the PWIT (Poisson weighted infinite tree)



Distances $0 < \xi_1 < \xi_2 < \xi_3 < \ldots$ from a vertex to its **near neighbors** (indicated by lines) are successive points of a Poisson (rate 1) process on $(0, \infty)$. Continue recursively.

Q: So what's the use of knowing LWC . . . ?

A: not much, but it's a start . . . .

Let's mention 4 results/examples <u>not</u> related to cavity method.

Result A: According to the graph-theoretic definition of **planar graph**, the infinite binary tree is a planar graph. but this seems silly to a probabilist, because probabilistic models (random walk, percolation, interacting particles) behave quite differently on trees than on Z^2. The class of random networks defined as

($*$)LWC limits of finite random planar graphs

provides a more natural formalization of "random infinite planar graphs". Benjamini-Schramm (2001) show that on graphs (*) with bounded degree, RW is recurrent. Suggests many other questions . . . .

Result B: Particular models of random planar n-vertex graphs include

- uniform random triangulations (Angel-Schramm 2003)

- uniform random quadrangulations (Chassaing-Schaeffer 2004).

In each case there is a LWC limit which may be called the **uniform infinite planar triangulation/quadrangulation**.

xxx pictures

Result C: Because of the 'uniform random rooting" in the definition

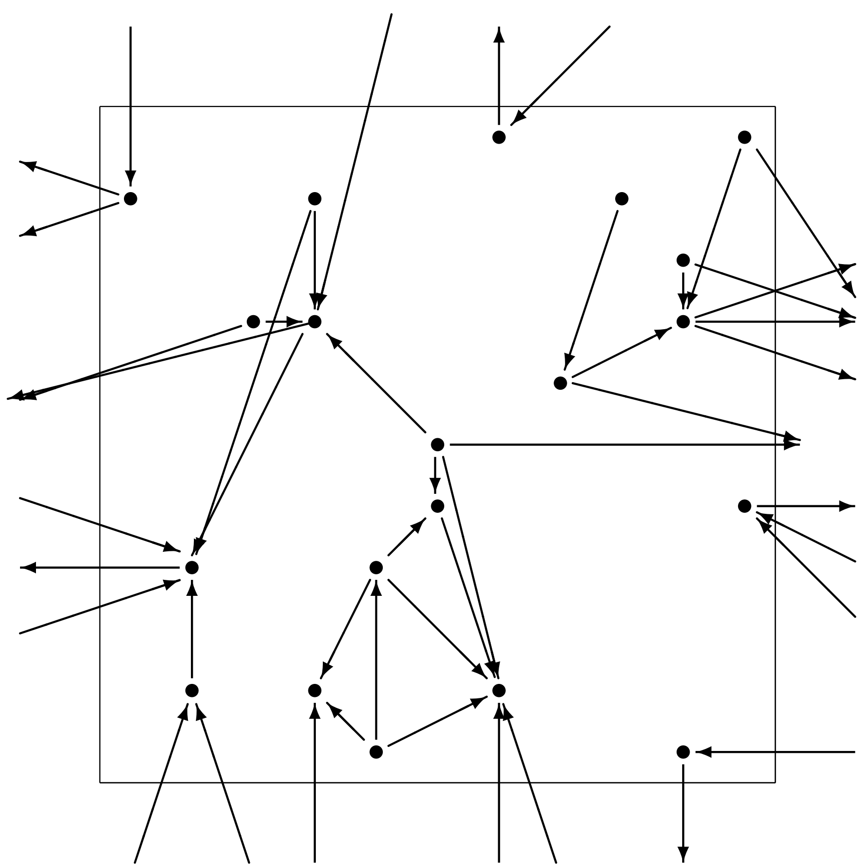
Definition. If $G_n[U_n] \stackrel{d}{\to}$ some G_∞, call this **local weak convergence (LWC)** of G_n to G_∞

a random infinite network G_∞ which is a LWC limit is not entirely arbitrary, but has a property interpretable as "each vertex is equally likely to be the root" (**stationary** or **involution invariant** or **unimodular**).

Not obvious (but true: Aldous-Lyons, in preparation) that any random infinite network with this property really is some LWC limit.

(This is technically useful in extending obvious results in finite setting to the infinite setting)

Result D: A tractable complex network model (Aldous 2003/4) designed to have a LWC limit within which explicit formulas can be calculated (giving $n \to \infty$ asymptotics for finite-n models).



Q: So what's the use of knowing LWC . . . ?
One goal is to prove that solution of CO problem on G_n converges to solution of CO problem on G_∞. Not always true, of course!

Example E: Suppose edge-lengths are distinct. Then G_n has a unique MST (minimum spanning tree). Also we can define the (wired) minimum spanning forest (MSF) on an infinite network G_∞.

Lemma: If $G_n \to G_\infty$ (LWC), if (technical condition on G_∞), then

$$(G_n, \text{MST}(G_n)) \to (G_\infty, \text{MSF}(G_\infty)) \text{ (LWC)} .$$

In particular, on the PWIT one can calculate

$$E(\text{length of MSF per vertex})$$

$$= \tfrac{1}{2} E(\text{length of MSF-edges at root})$$

$$= \zeta(3) = \sum_{j=1}^{\infty} j^{-3}$$

and re-derive result of Frieze (1985) that in complete graph with random edge lengths model

$$n^{-1} E(\text{length of MST}) \to \zeta(3).$$

xxx java picture

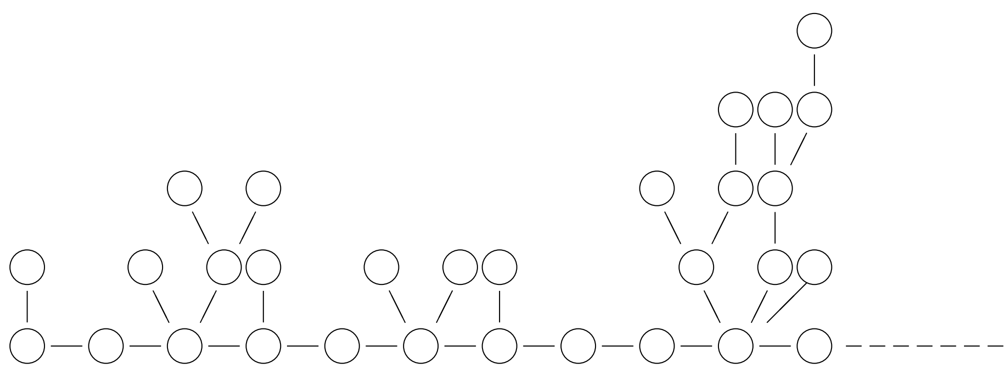
Example F: (in Aldous-Steele survey). Uniform random tree on n vertices. Put i.i.d. positive weights on edges.

$M_n :=$ weight of max-weight partial matching

Then $n^{-1}EM_n \to EM_\infty$ where

$$M_\infty = \frac{1}{2}(\text{weight of edge at root})$$

in max-weight matching on limit infinite tree



Explicitly, for Exponential(1) edge-weights we get $EM_\infty \approx 0.2396$ where the limit equals

$$\int_0^\infty se^{-s}ds \int_0^s c(e^{-y} - be^{-s}) \exp(-ce^{-y} - ce^{-(s-y)})\, dy$$

where $c \approx 0.7146$ is the strictly positive solution of $c^2 + e^{-c} = 1$ and $b = \frac{c^2}{c^2+2c-1} \approx 0.5433$.

LWC and the cavity method

Recall model

G_n: complete graph on n vertices; edge-lengths random, independent Exponential(mean n) distribution.

Write L_n for length of TSP tour under this model. Mezard-Parisi (1980s) used replica/cavity methods to argue

$$n^{-1} E L_n \to c \approx 2.04.$$

We have explicit program to make rigorous – but can't carry through two of the technical steps.

Here are two relaxations of TSP for n-vertex network.

(M2F): minimum 2-factor. Minimize total length of a 2-factor, that is an edge-set in which each vertex has degree 2. That is, a union of cycles which spans all vertices.

(MA2F): minimum almost 2-factor. Minimize total length over edge-sets $\mathcal{E}_n$ such that

$$n^{-1}|\{v: \text{ degree}(v) \neq 2\}| \to 0.$$

Proposition (Frieze 2004) In this model,

$$n^{-1}(EL_n - EL'_n) \to 0$$

where L_n is TSP length and L'_n is M2F length.

Missing Proposition Want to know

$$n^{-1}(EL_n - EL_n^{\prime prime}) \to 0$$

where L_n is TSP length and $L_n^{\prime prime}$ is MA2F length.

xxx java slide; hand write Zs

Central part of method – which I'll explain only superficially – is to do analysis of TSP on the (infinite) PWIT. each edge e of PWIT splits it into two subtrees. There are random variables $Z^1(e), Z^2(e)$, measurable functions of the subtrees, such that

$$e \in \text{TSP-path iff length}(e) < Z^1(e) + Z^2(e)$$

(another "missing proposition" in proving this) from which one can calculate mean length of TSP-path edges.

Q: How do we go back from the PWIT to the finite-n model?

More math infrastructure

A measurable function $f(Y_1, Y_2, \ldots)$ of some infinite collection of r.v.'s can be approximated arbitrary closely by continuous functions $f_k(Y_1, \ldots, Y_k)$ of finitely many of the r.v.'s.

So on the PWIT we can define an edge-set $\mathcal{E}_r$ such that
(i) the edges of $\mathcal{E}_r$ at a vertex v are determined by the restriction of the PWIT to the window of radius r around v; (ii) $\delta_r := P($ some edge at $v \in \mathcal{E}_r \triangle \{) \to 0$ as $r \to \infty$.

Using LWC of the finite-n model to the PWIT, we can apply the same rule to a window of radius r around a vertex v, and define edge-sets $\mathcal{E}_{r,n}$ such that

$$\limsup_n P(\text{degree}(v) \text{ in } \mathcal{E}_{r,n} \neq 2) \leq 2\delta_r$$

xxx and similarly the edge-lengths xxx.

This constructs an almost-2-factor of G_n whose cost-per-vertex converges to the c given in the PWIT analysis.