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A game-theoretic analysis of rank-order mechanisms for user-generated content ☆

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Abstract

We investigate the widely-used *rank-order* mechanism for displaying user-generated content, where contributions are displayed on a webpage in decreasing order of their ratings, in a game-theoretic model where strategic contributors benefit from attention and have a cost to quality. We show that the lowest quality elicited by this rank-order mechanism in *any* mixed-strategy equilibrium becomes optimal as the available attention diverges. Additionally, these equilibrium qualities are higher, with probability tending to 1 in the limit of diverging attention, than those elicited by a more equitable proportional mechanism which distributes attention in *proportion* to the positive ratings a contribution receives, but the proportional mechanism elicits a greater number of contributions than the rank-order mechanism.

JEL classification: C72; D80

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1. Introduction

There is a proliferation of user-contributed content on the Web, and a multitude of instances where user-contributed content adds significant value to websites. The product reviews written by users on Amazon, for instance, are a very valuable component of the service that Amazon provides, while online question-and-answer sites such as Yahoo! Answers and StackOverflow, or sites aggregating service reviews such as Yelp owe almost all their utility to contributions from users. But while there is a large amount of user-contributed content online, not all of it is of the same *quality*—some content is excellent, while some is mediocre and some is outright bad.

Many websites attempt to *rank* content according to its quality, using thumbs-up/thumbs-down style ratings by viewers—this is the case, for example, with comments on Yahoo! News, reviews on Amazon, and posts on Reddit. These websites display higher quality contributions more prominently by placing them near the top of the page and pushing lower quality ones to the bottom. Since content displayed near the top of the page is more likely to be viewed by a user, ranking good content higher leads to a better user experience. But there is also another aspect to displaying better content more prominently: it potentially provides an *incentive* to produce high quality content that might appeal to a contributor's desire for attention. In other words, how contributions are displayed as a function of their estimated quality constitutes a *mechanism* for allocating attention, which might affect the incentives of contributors and influence the quality of their contributions.

What can we understand, using a game-theoretic approach, about how the mechanism used to display content influences the quality of the contributions? In particular, how does the choice of mechanism influence quality when the number of potential viewers, and therefore the potential available attention, grows very large? The diverging attention regime is arguably the most important setting for user-generated content. First, these are the situations where delivering high quality content matters the most for viewer welfare. Second, the popular sites are the ones that draw the most attention-motivated contributors, as well as the ones that tend to attract contributions of varying quality. Indeed, tremendously large amounts of attention are not uncommon for popular content on the web; for instance, the most popular YouTube videos have been viewed over a hundred million times and even days-old 'trending' videos have hundred of thousands of views.

In this paper, we analyze two mechanisms that use viewer ratings to allocate attention to content—the widely used rank-order mechanism, where contributions are allocated positions on the page in decreasing order of their ratings, and a *proportional* mechanism [10,14], which distributes attention in proportion to the number of positive ratings. The rank-order mechanism is ubiquitous throughout the Web, while the proportional mechanism is a natural and more 'fair' alternative: if two contributions receive very similar numbers of votes, it only seems fair that they receive similar amounts of attention as well, but this need not hold in the rank-order mechanism. Furthermore, the proportional mechanism is a mechanism whose implementation is widely discussed in various online contexts such as online question-and-answer forums [14], resource allocation problems [3,25], network-rate control [17,18], online auctions [21], and scheduling [26]. What happens to equilibrium quality and participation in the rank-order mechanism as the amount of available attention diverges, and how does it compare against the more fair proportional mechanism?

Our contributions We analyze equilibrium behavior in the rank-order mechanism in a game-theoretic model where contributors are motivated by attention and have a cost of participation that increases with the quality of their content, as in [10]. Contributors are strategic agents who choose *both* whether to contribute, *and* the quality of their contribution to maximize their payoff.

Analyzing the rank-order mechanism is nontrivial because an agent's payoff depends on her choice of quality and other agents' choices of quality in a complicated way—the number of votes m_i a contributor i receives is a *random variable* in her quality q_i. The final payoff to i depends on the *rank* of the instantiation of this random variable amongst the m_{-i}, the numbers of votes received by other contributors, which are *also* random variables in the other agents' quality choices.

We first show that symmetric mixed strategy equilibria, where all agents of a given type τ participate with probability β_τ and randomly choose a quality from a common distribution $F_\tau(q)$ if contributing, always exist for the rank-order mechanism. We then investigate equilibrium behavior as the amount of attention increases, and show that the *lowest* quality that can arise in a mixed strategy equilibrium tends to the highest possible quality as the available attention diverges. We also show that if the number of potential contributors grows at a smaller rate than the total amount of attention, then it is possible to choose the number of votes used to rank contributions in such a way as to guarantee full participation. However, if the number of potential contributors grows at the same rate as the total amount of attention, then the participation rate goes to zero in the limit.

Next we consider the proportional mechanism. The proportional mechanism always has an equilibrium in which all agents of type τ participate with probability β_τ and choose a fixed quality q_τ upon participating. However, unlike the rank-order mechanism, this equilibrium quality only converges to the optimal quality if the number of potential contributors grows at the same rate as the number of viewers. In this case, the proportional mechanism also elicits full participation, but unlike the rank-order mechanism, the participation rate remains bounded away from zero even if the number of potential contributors grows at the same rate as the total amount of attention.

We then compare the equilibrium quality choices in the rank-order mechanism and the proportional mechanism and show that the probability an agent chooses higher quality in the rank-order mechanism than in the proportional mechanism tends to 1 as the amount of available attention diverges. However, the proportional mechanism leads to greater participation than the rank-order mechanism if the number of potential contributors grows at the same rate as the number of viewers.

Our results thus suggest that an interesting participation-quality tradeoff arises in comparing the rank-order and proportional mechanisms. The rank-order mechanism consistently elicits higher quality contributions, but the proportional mechanism elicits greater participation. This suggests that in circumstances under which it is most important to have a few exceptionally high quality contributions, the rank-order mechanism should be preferred, while in situations in which the goal is to elicit as much participation as possible, the proportional mechanism is better.

Related work There is a large literature in economics on using rank-order tournaments as incentive schemes (*e.g.* [5,11–13,19,20,22–24]). This literature analyzes the consequences of contests used to rank players such as employee compensation schemes which reward employees based on how their output compares to that of other employees. While agents in our model are also ranked on how their output compares to that of other agents, there are several important differences between our paper and this work. In our paper, an individual's observable output is the random number of users who vote positively on her contribution, but this type of framework is not captured by the assumptions made in existing work on rank-order tournaments. Also, not all agents need to participate in our model, but all employees must work in the economics literature. In addition, most existing work on rank-order tournaments focuses on equilibria in which all agents exert a deterministic level of effort in equilibrium,[1] whereas we extensively analyze mixed strategy equilibria. The focus on the limiting case we consider as well as the comparison between the rank-order and proportional mechanisms is also missing in this literature.[2]

There is also a growing body of research on human computing systems and user-contributed content, but relatively little of this work addresses the analysis and design of these systems from a game-theoretic perspective [7,14–16]. Jain et al. [14] study the question of designing incentives for online question-and-answer forums, and focus on incentives for participants to contribute their answers *quickly*, but do not address the issue of incentivizing high quality contributions. The most relevant paper to our work is [10], which introduces a model to address the quality of user-generated content. This paper shows that a simple mechanism which eliminates contributions that are not rated highly by all voters achieves optimal quality in the limit as the amount of available attention diverges. Our model has a few technical differences from that in [10], and is also used to instead address the problem of incentives in the widely used rank-order mechanism and compare those to incentives in the proportional mechanism. Papers on crowdsourcing on the web such as [1,4,6], and [9] have also not compared incentives in the rank-order and proportional mechanisms.

2. Model

Content Each unit of content, or contribution, has a *quality* q, where $q \in [0, 1)$ is the probability that a viewer will rate a contribution as 'good' or 'useful'.[3] The quality q of a contribution is not directly observable in our model, but it influences the number of positive votes the content receives.

Each contribution is rated by at least T viewers, and the mechanism then decides how much attention to reward to each agent on the basis of the results of the first T votes that each agent received. The exact number of votes T that is used is a parameter that can be chosen by the mechanism. We assume that viewers are not strategic, and simply provide this binary feedback non-strategically by truthfully indicating whether they found the contribution to be good or useful.

Given a contribution i with quality q_i, the number of positive votes it receives is a random variable. We let m_i denote the number of positive votes received by this contribution, and note that the distribution of m_i is binomial with parameters (T, q_i). We also let m_{-i} denote the vector of the numbers of positive votes received by other contributions.

Contributors There is a pool of potential contributors, or agents, of size K. Each agent can choose both whether she will contribute, as well as the *quality* of her contribution should she

[1] One very rare exception is [24], which briefly mentions that agents may have an incentive to exert random levels of effort, but does not conduct an extensive analysis of such equilibria.

[2] Work on Tullock functions such as [8] presents comparisons of proportional mechanisms and all-pay auctions. However, this work does not consider the more general rank-order tournaments in our paper, does not capture settings in which an individual's observable output is the random number of users who vote positively on her contribution, and also does not consider the limiting case in which the number of potential participants grows large.

[3] It is possible that there will be some fraction of the population that always votes down good content because, for example, some voters are impossible to please or there are malicious voters who intentionally vote down good content. This can be modeled by assuming that the largest possible quality an agent can produce is some $\gamma < 1$, and our results immediately extend to a model in which each unit of content has a quality $q \in [0, \gamma)$ for some $\gamma < 1$.

decide to participate. Agent i may use mixed strategies, and we denote the probability that agent i decides to contribute by β_i and the quality she chooses when she contributes by q_i. Since each potential contributor may use mixed strategies in deciding whether to contribute, the number of actual contributors is a random variable. We denote the instantiation of this random variable by k, and note that the distribution of k depends on K as well as $\beta_1, \ldots, \beta_K$.

Contributors are *strategic*: they choose both whether to contribute as well as the quality of their contribution strategically to maximize their expected payoffs given the potential costs and benefits from contributing. If an agent chooses not to contribute, then she incurs no cost but also receives no benefit, so her payoff is 0. Otherwise, the agent pays a cost reflecting the effort she expended to produce content, and obtains a benefit reflecting the attention she was able to get as a result.

The *cost* incurred by a contributor depends on the quality of her content and her type. We assume that each agent has some type $\tau \in \{1, \ldots, t\}$, where the realization of agent i's type, τ_i, is an independent random variable that takes on the value $\tau \in \{1, \ldots, t\}$ with probability π_τ. The cost of producing content of quality q for an agent of type τ is then $c_\tau(q)$, which is a convex, continuously differentiable, and increasing function of q for all τ. We also assume that $c'_\tau(0) = 0$, $c_\tau(0) > 0$, and $\lim_{q \to 1} c_\tau(q) = \infty$ for all τ. These last two conditions indicate that making a contribution takes more effort than not participating and producing perfect content is nearly impossible.[4]

The *benefit* derived by a contributor depends on the amount of attention she receives, which can depend both on the rating of her contribution and the number and ratings of other contributions. Let A denote the total amount of attention available. If contributor i is allocated a fraction $\alpha_i(m_i, m_{-i}) \geq 0$ of this attention, then her benefit from receiving this is $V(m_i, m_{-i}) = v(\alpha_i(m_i, m_{-i})A)$, where $v(a)$ is some continuously differentiable, strictly increasing, and (weakly) concave function satisfying $v(0) = 0$ such that $v'(a)$ remains bounded away from 0 for all a.[5]

A contributor i's *payoff* from generating content of quality q_i is the difference between her expected benefit and cost, $\pi(q_i, q_{-i}, \beta_{-i}) = E[V(m_i, m_{-i})|(q_i, q_{-i}, \beta_{-i})] - c_{\tau_i}(q_i)$, where $q_{-i} = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_k)$ denotes the quality choices of the other contributors, and β_{-i} denotes the participation probabilities of the remaining contributors. Note that the expectation in this payoff is taken over the random number of contributors k as well as the random variables m_i.

Solution concept Since agents' payoffs are symmetric, we focus throughout on symmetric equilibria. In a symmetric equilibrium, each agent follows a mixed strategy such that all agents of the same type participate with the same probability and follow the same strategy of quality choices conditional on participating. Formally, a *symmetric mixed strategy equilibrium* is a set of probabilities $\{\beta_\tau\}$ and a set of distributions $\{F_\tau\}$ over qualities q such that when every agent of type

[4] It is worth noting that while in our model, all qualities chosen will be in the interval [0, 1), this model is mathematically equivalent to a model in which agents may choose any quality level q in the non-negative reals and the probability that an agent receives a positive vote is some strictly increasing function $f(q)$ that satisfies $f(0) = 0$, $\lim_{q->\infty} f(q) = 1$, and the cost to an agent of type τ of contributing with quality q is the function $C_\tau(q) = c_\tau(f(q))$, where $c_\tau(.)$ is the original cost function for agents of type τ in our model. Thus the assumption that there is a maximum possible quality level is not needed for any of the results in our paper.

[5] If the agents did not know the exact value of A, but instead only knew its distribution, then the equilibrium analysis would stay the same except that $v(a)$ would be replaced with $E[v(a)]$.

τ contributes with probability β_τ, and chooses a quality drawn from the CDF $F_\tau(q)$ conditional on contributing, no agent can increase her expected payoff by deviating from this strategy.

Asymptotics First note that if $v(A) \le c_\tau(0)$ for all τ, *i.e.*, the total value of attention is smaller than the cost of producing zero quality content for all types, then no agents would want to participate. The case where no agents participate is not interesting, so we will assume that $v(A) > c_\tau(0)$ for at least one type τ for the remainder of this paper. This assumption guarantees that there is at least one type τ for which $\beta_\tau > 0$ in any symmetric equilibrium.

We will be particularly interested in the qualities and participation levels in equilibrium in the limiting case as $A \to \infty$. Since this diverging amount of attention comes from a diverging number of viewers, the number of potential contributors, K, as well as the number of viewers available to vote on contributions, T, can increase with A as well. We also assume throughout that $K \le A$.

We will sometimes write $K(A)$ and $T(A)$ to make explicit the possible dependence of K and T on A, and assume that as A diverges, $K(A)$ diverges and $T(A)$ can be chosen to diverge as well. This corresponds to the observation that as sites grow more popular (A increases), they attract more potential contributors ($K(A)$ increases), and more voters ($T(A)$ increases). We emphasize, though, that $T = T(A)$ is a parameter chosen by the mechanism, and may entail only using a subset of the available ratings (for instance, the first T votes) to rank contributions. Thus our formulation does not require that all agents receive the same numbers of votes. The number of votes T in our model is to be interpreted as some minimum number of votes that is received by all contributions.[6]

3. Rank-order mechanisms

We first consider *rank-order* mechanisms. A rank-order mechanism arranges contributions in decreasing order of the number of positive votes they receive and allocates more attention to contributions which are ranked higher. We formally define rank-order mechanisms below.

Definition 3.1 *(Rank-Order Mechanism $M_r(T,\alpha)$).* Let $\alpha_j(k) \ge 0$ be a sequence of numbers that is nonincreasing in j for all k and satisfies $\sum_{j=1}^{k} \alpha_j(k) = 1$, where the values of α_j *do not depend* on the qualities q for any j, although they can depend on the *number* of contributors k. Suppose there are k contributors and each contribution is voted on by T viewers. The rank-order mechanism ranks contributions in decreasing order of the number of positive votes received (with ties broken randomly) and awards the jth ranked contribution attention $\alpha_j A$.

Note that both α, which specifies the distribution of attention amongst the ranks, as well as T, the number of votes used to determine the rankings, are *parameters* of the mechanism that can be chosen to achieve desirable properties. We first state the following simple proposition.

Proposition 3.1. *Suppose $q_i > q_j$, and let the number of votes received by i and j be m_i and m_j respectively. Then* $\lim_{T\to\infty} Pr(m_i > m_j) = 1$.

[6] It is worth noting that all the substantive conclusions of the paper can be extended to an alternative model in which mechanisms use votes from all of the voters rather than just the first T votes; we use the current model because it allows for a cleaner presentation.

All proofs are in Appendix A. Thus a higher quality contribution receives a larger number of positive votes, and is therefore ranked higher, than a lower quality contribution in the limit as the number of votes T goes to infinity. We next show that the rank-order mechanism always has a symmetric mixed strategy equilibrium in which all agents of the same type participate with the same probability, and choose a quality from the same distribution if they decide to contribute.

Theorem 3.1. *For any values of A, K, T, and α, there exists a symmetric mixed strategy equilibrium in which all contributors of type τ participate with probability β_τ and choose a quality drawn from the same cumulative distribution function $F_\tau(q)$ conditional on contributing.*

In general, a mechanism can have multiple equilibria. A natural question to ask is whether any of these equilibria is '*bad*', in the sense of inducing low-quality equilibrium contributions. Our next result illustrates that as long as there is at least a small difference between the amount of attention allocated to contributions that are ranked differently, such bad Nash equilibria cannot exist. Specifically, if the number of votes used to rank the contributions becomes large as the amount of attention grows, contributors will choose quality arbitrarily close to 1 in the limit.

Theorem 3.2. *Suppose that $\alpha_j(k) - \alpha_{j+1}(k) = \Theta(k^{-2})$ for all $j \le k-1$, $\alpha_k(k) = 0$ for sufficiently large k, and $\lim_{A\to\infty} T(A) = \infty$. Then, for any $q^* < 1$, the probability an agent who contributes chooses quality $q > q^*$ goes to 1 as A goes to infinity.*

In a mixed strategy equilibrium, a contributor can draw any quality in the support of the equilibrium distribution: the theorem says that the *lowest* quality in this equilibrium distribution tends to the optimal quality as the amount of attention diverges.

It is worth noting that this result holds even in the case where $\liminf_{A\to\infty} \frac{K(A)}{A} > 0$ and the average amount of attention that will be given to any of the $K(A)$ agents is bounded away from infinity, meaning each agent, on average, only obtains a finite benefit from the mechanism. When each agent can, on average, only obtain a finite award from the mechanism, it might seem implausible that the mechanism could induce agents to choose quality arbitrarily close to 1 and thereby exert an infinite amount of effort. Nonetheless, this theorem shows that the rank-order mechanism is so powerful that it can induce agents to still produce arbitrarily high quality content in this case.[7]

This theorem uses the condition that $\alpha_j(k) - \alpha_{j+1}(k) = \Theta(k^{-2})$. It is worth noting that the result that contributors produce arbitrarily high quality content does not depend crucially on this assumption. If we instead replaced this assumption with the weaker assumption that $\alpha_j(k) - \alpha_{j+1}(k) = \Omega(k^{-2})$ for all $j \le k-1$, then the result would go through with a very similar proof. That is, there is no problem inducing high quality content as long there is at least some minimum difference between the attention allocated to content ranked differently.

Since the precise equilibrium strategies that agents use may vary with A, we will henceforth use $\beta_\tau(A)$ to denote the dependence of the equilibrium participation probability for agents of type τ, and $F_{\tau,A}(q)$ to denote the dependence of the equilibrium distribution from which contributors of type τ choose their quality, on the available attention A. To understand the intuition behind this result, first note that in any equilibrium a large fraction of participating contributors must produce content with quality close to q_A, where q_A is the minimum quality in the union of the supports

7 This results in part from the fact that not all agents will participate in equilibrium.

of $F_{\tau,A}(q)$ for all participating types τ: if only a small fraction of agents produce content with quality close to q_A, then a contributor who produces quality q_A will achieve a lower ranking than almost all other contributors, and will obtain almost no attention. Such an agent could achieve a higher expected payoff by not contributing, meaning this would not be an equilibrium.

But if a large fraction of participating contributors are producing content with quality close to q_A, then for large T, an agent can ensure that she will achieve a higher ranking than a significant number of additional contributors by producing content with quality $q = q_A + \epsilon$ for some small $\epsilon > 0$ (in contrast with choosing quality $q = q_A$). Thus if q_A is bounded away from 1, an agent could profitably deviate by producing content with quality $q = q_A + \epsilon$ instead of content with quality $q = q_A$ for some small $\epsilon > 0$. From this it follows that q_A must be close to 1 for large A.

Next we show that one can choose the number of individuals who vote on the content, T, in such a way to induce all contributors to participate in the limit, as long as the pool of potential contributors does not grow too quickly with the number of viewers.

Theorem 3.3. *Suppose that* $\lim_{A\to\infty} \frac{K(A)}{A} = 0$ *and* $\alpha_j(k) - \alpha_{j+1}(k) = \Theta(k^{-2})$ *for all* $j \le k-1$. *Then there exists a sequence* $\{T(A)\}_{A=1}^{\infty}$ *such that* $\lim_{A\to\infty} T(A) = \infty$ *and* $\beta_\tau = 1$ *in equilibrium for all* τ *for sufficiently large* A.

This result indicates that if $T(A)$ does not grow too quickly with A, then all contributors will participate for sufficiently large A. While this result makes use of a technical assumption that $\lim_{A\to\infty} \frac{K(A)}{A} = 0$, we believe this is the most critical case to ensure large participation, as this is the case where there is only a relatively small number of agents who may contribute.

While full participation can be achieved if the number of potential contributors becomes vanishingly small compared to the amount of attention, this is no longer the case if $\lim_{A\to\infty} \frac{K(A)}{A} = r$ for some $r > 0$ and the number of potential contributors grows at the same rate as the amount of attention. Nonetheless it is still possible to ensure that a large number of agents will participate in equilibrium when $\lim_{A\to\infty} \frac{K(A)}{A} = r$ for some $r > 0$. This is shown in the following theorem:

Theorem 3.4. *Suppose that* $\lim_{A\to\infty} \frac{K(A)}{A} = r$ *for some* $r > 0$ *and* $\alpha_j(k) - \alpha_{j+1}(k) = \Theta(k^{-2})$ *for all* $j \le k-1$. *Then* $\lim_{A\to\infty} \beta_\tau(A) = 0$ *for all* τ. *Furthermore, if* $\alpha_j(k) > \alpha_{j+1}(k)$ *for all* k *and all* $j \le k-1$, *then there exists a sequence* $\{T(A)\}_{A=1}^{\infty}$ *such that* $\lim_{A\to\infty} T(A) = \infty$ *and* $\lim_{A\to\infty} \sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) K(A) = \infty$.

This result indicates that if the number of potential contributors grows at the same rate as the amount of available attention, then only a vanishingly small fraction of agents will participate in equilibrium. Nonetheless, if $T(A)$ does not grow too quickly with A, it is still possible to ensure that a diverging number of agents will participate in equilibrium in expectation, as the result that $\lim_{A\to\infty} \sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) K(A) = \infty$ guarantees this.

Implementing ranking mechanisms Theorem 3.2 requires a mechanism designer to choose the values of α to satisfy $\alpha_j(k) - \alpha_{j+1}(k) = \Theta(k^{-2})$ and $\alpha_k(k) = 0$ to induce high quality. The needed difference in attention between two contributions can be achieved as follows: To increase the difference in attention between two contributions, one can show one contribution prominently more often than the other, and to decrease the difference one can show the two contributions in prominent positions roughly equally often. One can also easily ensure that $\alpha_k(k) = 0$ by simply never showing a contribution that is ranked last by the initial voting.

The analysis in Theorems 3.3 and 3.4 suggests that a mechanism designer can achieve diverging participation by choosing T so that $T(A)$ does not grow too quickly with A. This can also be achieved in practice by simply ignoring votes beyond the first T votes. Thus the type of restrictions on T and α used in these results should be easily implementable in practice.

Robustness of results While the types of restrictions on T and α that were used to obtain the results in Theorems 3.2 to 3.4 should be easily implementable in practice, it is also interesting to address the question of whether high quality contributions can be elicited without these assumptions. In particular, there are some online settings in which only the l best contributions receive attention because there is only room on the page to display l contributions. In this setting, we would have $\alpha_j(k) = 0$ for all $j > l$ and for all k, meaning the condition that $\alpha_j(k) - \alpha_{j+1}(k) = \Theta(k^{-2})$ that was used in Theorem 3.2 would not be satisfied. The question of whether the rank-order mechanism can elicit contributions of optimal quality in this setting is addressed in the following theorem.

Theorem 3.5. *Suppose there is some $l < \lim_{A\to\infty} K(A)$ such that $\alpha_j(k) = 0$ for all $j > l$ and for all k and $\alpha_j(k) > 0$ for all $j \le l$ and all k. Then for any $q^* < 1$, the probability that at least l agents make a contribution of quality $q \ge q^*$ goes to 1 in the limit as $A \to \infty$.*

Theorem 3.5 gives a slightly different type of result than that considered in Theorem 3.2, as this theorem does not illustrate that all agents who participate will make high-quality contributions. Instead the theorem illustrates that the probability that at least l agents make arbitrarily high quality contributions goes to 1. However, since only the l most highly-ranked contributions will ever receive attention in the setting considered in Theorem 3.5, this result ensures that users will only experience arbitrarily high-quality content in the setting considered in Theorem 3.5.

4. Proportional mechanism

Since all contributors produce similar qualities in equilibrium in the rank-order mechanism, it might seem unfair to give significantly greater amounts of attention to agents who receive higher numbers of positive votes. A natural alternative is to reward contributors in proportion to the number of positive votes they receive—in such a mechanism, two agents who both receive very similar numbers of positive votes also receive similar amounts of attention. However, while the proportional system might allocate attention in a more equitable manner, the mechanism also significantly changes incentives for agents to produce high-quality content. In this section, we analyze the equilibria of the proportional mechanism. First we formally define the proportional mechanism.

Definition 4.1 *(Proportional Mechanism $M_p(T)$).* Suppose k agents contribute, each contribution is voted on by T viewers, and each participating contributor i receives m_i positive votes. Then the proportional mechanism gives the ith contributor a share $\frac{m_i}{\sum_{j=1}^{k} m_j}$ of the available attention if $m_i > 0$ for some i. If $m_i = 0$ for all i, every contributor receives a share $\frac{1}{k}$ of the available attention.

Note that in the proportional mechanism, only the number of votes used by the mechanism, T, is available as a parameter of the mechanism that can be varied to achieve desirable incentives. We first prove a result on the nature of the equilibrium in the proportional mechanism.

Theorem 4.1. *For any values of A, K, and T, there exists a symmetric equilibrium to the proportional mechanism in which all agents of type τ participate with probability β_τ and choose the same quality q_τ conditional on contributing.*

Theorem 4.1 indicates that there are some differences between the nature of equilibria in the proportional and the rank-order mechanisms, as there exists an equilibrium in which agents do not randomize amongst their quality choices in the proportional mechanism, but such equilibria need not exist in the rank-order mechanism. Our next result further shows that this difference is even more stark—in the proportional mechanism, there will not exist *any* symmetric equilibrium in which agents randomize over their quality choices.

Theorem 4.2. *For any values of A, K, and T, there does not exist a symmetric mixed strategy equilibrium of the form $(\beta_\tau, F_\tau(q))$ where $F_\tau(q)$ is not a point mass for all τ, i.e., in which agents of some type τ randomize over qualities.*

Next we investigate asymptotic equilibrium quality choices in the proportional mechanism. In contrast to the ranking mechanism, whether equilibrium quality choices converge to the optimal quality in the proportional mechanism *depends* on how quickly the number of potential contributors grows with the number of viewers. If $\frac{K(A)}{A} \to 0$ as $A \to \infty$, then equilibrium quality converges to one. But if not, equilibrium quality remains strictly less than one in the limit.

Theorem 4.3. *If* $\lim_{A\to\infty} \frac{K(A)}{A} = 0$, *then* $\lim_{A\to\infty} q_\tau(A) = 1$ *for all types τ and* $\beta_\tau(A) = 1$ *for all types τ for sufficiently large A. If* $\liminf_{A\to\infty} \frac{K(A)}{A} > 0$, *then* $\limsup_{A\to\infty} q_\tau(A) < 1$ *for all participating types τ and* $\liminf_{A\to\infty} \sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) > 0$.

We note that both regimes in this theorem, $\frac{K(A)}{A} \to 0$ and $\frac{K(A)}{A} \to r > 0$, are of interest in the context of user-generated content. In question-and-answer sites such as Yahoo! Answers or StackOverflow, the number of users $K(A)$ who can answer a question is often significantly smaller than the number of users who consume the answer, and $\frac{K(A)}{A} \to 0$ is likely. On the other hand, in settings like posts on discussion forums or comments on blogs where many consumers are also producers, the number of contributors may not be negligible compared to the number of viewers who consume the content, *i.e.*, $\frac{K(A)}{A}$ is not vanishingly small. The theorem says that the proportional mechanism elicits the optimal quality in the first kind of setting, but not in the second.

Theorem 4.3 also guarantees that the participation rate will remain bounded away from zero even if the number of potential contributors grows at the same rate as the number of viewers, in contrast to the rank-order mechanism. While we can guarantee that the participation rate remains bounded away from zero, it is not possible to further pin down the precise participation rate beyond this. If $K(A)$ is a sufficiently small fraction of A, then full participation will arise in equilibrium because the expected rewards to participating will exceed the costs. However, when $K(A) = A$, there will not generally be full participation since each agent could only expect to obtain a small fraction of the available attention by participating. Thus the precise participation rates will depend on how quickly $K(A)$ grows with A, even if we restrict attention to cases where $\liminf_{A\to\infty} \frac{K(A)}{A} > 0$.

The proportional mechanism can be implemented in a similar way to the rank-order mechanism. To achieve any needed difference in attention between two agents, the mechanism can increase this difference by showing one agent's contribution prominently more often than the other's, and decrease this difference by showing the agents' contributions in prominent positions roughly equally often.

5. Comparing mechanisms

In this section, we compare equilibrium qualities in the rank-order and proportional mechanisms. We already know from Theorems 3.2 and 4.3 that when $\lim_{A\to\infty} \frac{K(A)}{A} > 0$, the lowest quality in the support of the equilibrium distribution in the ranking mechanism converges to 1, but not in the proportional mechanism. For this case, therefore, the rank-order mechanism leads to higher quality contributions than the proportional mechanism for diverging attention. We now complete this comparison by investigating the case where $\lim_{A\to\infty} \frac{K(A)}{A} = 0$. We show that in this case, the ranking mechanism also elicits higher quality contributions than the proportional mechanism.

Theorem 5.1. *Suppose* $\lim_{A\to\infty} \frac{K(A)}{A} = 0$, $\lim_{A\to\infty} T(A) = \infty$, *and* α *in the rank-order mechanism satisfies* $\alpha_j(k) - \alpha_{j+1}(k) = \Theta(k^{-2})$ *for all* $j \le k-1$. *Let* $q_{\tau,r}(A)$ *denote the (possibly random) quality chosen by the contributors of type* τ *in some equilibrium of the ranking mechanism, and let* $q_{\tau,p}(A)$ *be an equilibrium quality for agents of type* τ *in the proportional mechanism. Then* $\lim_{A\to\infty} Pr(q_{\tau,r}(A) > q_{\tau,p}(A)) = 1$ *for all types* τ *that participate with non-vanishing probability in the rank-order mechanism, i.e. for all types* τ *such that* $\lim_{A\to\infty} \frac{\pi_\tau \beta_\tau(A)}{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)} \neq 0$.

To understand the intuition behind this result, suppose that agents chose quality in the rank-order mechanism according to a symmetric pure strategy, $q_r(A)$. Then if an agent makes a small change in quality from $q = q_r(A)$ to $q = q_r(A) + \epsilon$ for some $\epsilon > 0$, the agent goes from obtaining an average ranking in expectation to almost certainly being ranked near the very top. Thus such a change dramatically increases an agent's expected payoff. By contrast, in the proportional mechanism, increasing one's quality by $\epsilon > 0$ does relatively little to improve one's expected attention. Thus incentives to produce higher quality content are greater in the rank-order mechanism than in the proportional mechanism. Our proof extends this logic to mixed strategy equilibria.

It is also worth noting that this theorem illustrates that the lowest qualities used in the rank-order mechanism will be greater than all the qualities in the proportional mechanism. Thus even the agents who are most weakly incentivized to produce high-quality contributions in the rank-order mechanism produce higher quality content than all the agents in the proportional mechanism.

Our theorems so far have investigated diverging reward regimes, but have not considered regimes with a finite amount of attention. We illustrate below an example with non-diverging attention, where we can explicitly characterize the equilibrium conditions for both mechanisms. The incentives for agents to produce higher quality content in the rank-order mechanism than in the proportional mechanism still stand in this example, despite the finite amount of attention.

Example 5.1. Suppose there are $K = 2$ potential participants, each of whom has the same cost function $c(q)$ and values attention A at $v(A) = A$. Also suppose that there are $T = 2$ voters that will vote on the quality of the agents' contributions. Consider values of A that are large enough that both agents will participate with probability 1 in equilibrium (but not necessarily arbitrarily large). In this case, as we show in Appendix A, in the proportional mechanism, both agents contribute with a quality q that satisfies $\frac{c'(q)}{q^2/3-q+1} = A$ and in the rank-order mechanism in which all A units of attention are rewarded to the agent who finishes first, both agents contribute with a quality q that satisfies $\frac{c'(q)}{q^2-q+1} = A$. Since the functions $f_p(q) = \frac{c'(q)}{q^2/3-q+1}$ and $f_r(q) = \frac{c'(q)}{q^2-q+1}$ are both increasing in q for sufficiently convex $c(\cdot)$ and $f_p(q) > f_r(q)$ for all values of $q \in (0, 1)$, it then follows that the agents make higher quality contributions in equilibrium in the rank-order mechanism.

We now turn to questions related to the participation levels in the two mechanisms. While we have seen that the rank-order mechanism elicits higher quality contributions than the proportional mechanism, one might conjecture that these higher quality contributions only come at a cost of decreased participation. In our next theorem, we illustrate that these participation differences between the two mechanisms hold if and only if $\liminf_{A\to\infty} \frac{K(A)}{A} \neq 0$:

Theorem 5.2. *If* $\lim_{A\to\infty} \frac{K(A)}{A} = 0$*, then there exists a sequence of values of* $T(A)$*, with* $\lim_{A\to\infty} T(A) = \infty$*, such that full participation is achieved (i.e.* $\beta_\tau(A) = 1$ *for all* τ*) for sufficiently large* A *in both the rank-order and the proportional mechanism. If* $\liminf_{A\to\infty} \frac{K(A)}{A} \neq 0$*, then the expected number of participants in the rank-order mechanism becomes arbitrarily small compared to the expected number of participants in the proportional mechanism in the limit as* $A \to \infty$*.*

Theorems 5.1 and 5.2 indicate that there can be an interesting participation/quality tradeoff that results between the rank-order and proportional mechanisms—the rank-order mechanism always elicits higher quality contributions than the proportional mechanism, but this potentially comes at the cost of a smaller number of participants. Thus while both the rank-order mechanism and the proportional mechanism will elicit an arbitrarily large number of contributions in equilibrium, the expected number of contributors in the rank-order mechanism may diverge at a considerably slower rate than the expected number of contributors in the proportional mechanism.[8]

A natural next question to ask is whether there are tools that the mechanism designer can use to achieve more favorable points on the participation/quality tradeoff by varying the parameters of either mechanism. In particular, one might ask whether changes in the number of votes considered in ranking the agents would potentially have an effect on the equilibrium participation and quality choices. This unfortunately, is not the case in the regime with diverging numbers of votes.

Theorem 5.3. *Equilibrium participation and quality choices become independent of* T *in the limit as* $T \to \infty$ *for both the rank-order mechanism and the proportional mechanism.*

Thus when a large number of votes is considered in rewarding attention to the agents (as we have assumed for most of this paper), changes in the number of votes has little effect on equilibrium participation and quality choices. Thus while changing from a rank-order mechanism to a

[8] An implication of this result is that there may be more content diversity under the proportional mechanism than under the rank-order mechanism since the proportional mechanism elicits more contributions than the rank-order mechanism, and the larger number of contributions means there is a more diverse set of contributions to choose from. This is potentially important in a world with heterogeneous viewers if having a diverse set of contributions is important for being able to personalize which contributions are shown to different viewers.

proportional mechanism potentially has a significant effect on the participation/quality tradeoff that results, underlying changes in the number of votes considered in these mechanisms do not.

6. Conclusion

In this paper, we have analyzed the widely used rank-order mechanism for displaying user-contributed content in a model with strategic attention-driven contributors, and shown that the rank-order mechanism elicits contributions of optimal quality in the limit as the amount of attention diverges. By contrast, whether equilibrium quality in the proportional mechanism becomes optimal depends on how quickly the number of potential contributors grows with the number of viewers. Even when equilibrium quality in the proportional mechanism tends to the optimal quality, quality is almost always lower in the proportional mechanism than in the ranking mechanism. Thus, despite being more equitable, and sometimes eliciting more contributions, the proportional system creates inferior incentives for eliciting high quality contributions than the ranking mechanism.

There are a number of interesting directions for further work; we discuss three specific directions. On most sites, there are almost always some contributors who produce consistently low quality contributions, despite receiving little or no attention for it. This indicates that some subset of contributors have zero cost for producing low-quality content. An ideal mechanism in this setting would continue to elicit high quality contributions and high participation from the remaining contributors: an interesting question is how effective the ranking mechanism is when such contributors are present, and whether other mechanisms might be more effective. A second interesting direction regards questions related to malicious voters. The ranking mechanism is robust to voters who do not vote according to the model as long as they do this uniformly for all content. However, it is less clear how robust the results are in a model in which some malicious voters try to bring up specific contributions or put down others. Finally, we assume throughout our model that all voters are homogeneous in terms of their evaluation of content, *i.e.*, all voters upvote or downvote a given contribution with the same probability q_i. The problem of modeling and analyzing mechanisms for a setting in which readers have different 'tastes', as captured in a model where different reader types have different probabilities of upvoting the same contribution, with or without personalized content display, is an interesting direction for further work.

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Appendix A

Proof of Proposition 3.1. Note that as $T \to \infty$, the fraction of positive votes received by an agent who contributes with quality q_i, $\frac{m_i}{T}$, converges in probability to q_i and the fraction of positive votes received by an agent who contributes with quality q_j, $\frac{m_j}{T}$, converges in probability to q_j. Thus $\frac{m_i - m_j}{T}$ converges in probability to $q_i - q_j$, and thus if $q_i > q_j$, we have $\lim_{T\to\infty} Pr(m_i > m_j) = 1$. □

Proof of Theorem 3.1. First note that no player in this game would ever choose a quality $q > \max_\tau\{c_\tau^{-1}(v(A))\}$, as a player could always obtain a strictly greater expected payoff by not participating than by participating and choosing a quality $q > \max_\tau\{c_\tau^{-1}(v(A))\}$. Thus any mixed strategy equilibrium to the game in which players are restricted to choosing $q \in [0, \max_\tau\{c_\tau^{-1}(v(A))\}]$ is also a mixed strategy equilibrium of the original game.

Now note that this modified game in which players are restricted to choosing $q \in [0, \max_\tau\{c_\tau^{-1}(v(A))\}]$ is a symmetric game in which each player has a pure strategy space that is compact and Hausdorff. Also note that each player's expected payoff in this modified game is continuous in the actions of the players. It thus follows from Theorem 1 of [2] that there exists a symmetric mixed strategy equilibrium of this modified game. This in turn implies that there is a symmetric mixed strategy equilibrium of the original game. □

Proof of Theorem 3.2. Suppose by means of contradiction that there exists some $q^* < 1$ and some $\gamma > 0$ such that the probability a contributor chooses quality $q \le q^*$ is at least γ for an infinite number of A. If q_A denotes the minimum value of the set of all q in the supports of the distributions $F_{\tau,A}(q)$ for which $\beta_\tau(A) > 0$, then $q_A \le q^*$ holds for all such A. For small $\epsilon > 0$, let $p_A(\epsilon) = \frac{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) F_{\tau,A}(q_A+\epsilon)}{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)}$ denote the probability that a contributor chooses some quality $q \le q_A + \epsilon$ for a given A.

Our proof breaks down into three steps. We first show that if we restrict attention to a subsequence of A for which the probability a contributor chooses quality $q \le q^*$ is at least γ, it must be the case that $\lim_{A\to\infty} p_A(\epsilon) \le C\epsilon$ for some constant C and small $\epsilon > 0$, where the limit is taken along this subsequence. We then show that if $\lim_{A\to\infty} p_A(\epsilon) \le C\epsilon$, then it must be the case that $\lim_{A\to\infty} \frac{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) K(A)}{A} = 0$ along this subsequence. Finally, we show that if $\lim_{A\to\infty} p_A(\epsilon) \le C\epsilon$ and $\lim_{A\to\infty} \frac{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) K(A)}{A} = 0$ along this subsequence, then it cannot be the case that q_A is in the support of $F_{\tau,A}(q)$ for any τ.

Step 1: We first show that if we restrict attention to a subsequence of A for which the probability a contributor chooses quality $q \le q^*$ is at least γ (unconditional on the contributor's type), it must be the case that $\lim_{A\to\infty} p_A(\epsilon) \le C\epsilon$ for some constant C and small $\epsilon > 0$, where the limit is taken along this subsequence.

To see this, suppose by means of contradiction that there is no constant C such that $\lim_{A\to\infty} p_A(\epsilon) \le C\epsilon$ for small $\epsilon > 0$ along this subsequence. Note that if a contributor chooses quality $q = q_A$, then the probability she receives a higher ranking than a particular other contributor who chooses quality $q \le q_A + \epsilon$ is no greater than $\frac{1}{2}$. Thus if $\eta_A(\epsilon)$ denotes the expected fraction of contributors who choose quality $q > q_A + \epsilon$ and receive a lower ranking than this contributor for a given A, then the expected total fraction of contributors who receive a lower ranking than her is no greater than $\frac{p_A(\epsilon)}{2} + \eta_A(\epsilon)$.

Now note that if this contributor instead uses quality $q = q_A + 2\epsilon$ for some $\epsilon > 0$, then the probability she receives a higher ranking than a particular other contributor who chooses quality $q \le q_A + \epsilon$ goes to 1 in the limit as T goes to infinity. The expected fraction of contributors who choose quality $q > q_A + \epsilon$ and receive a lower ranking than a contributor who uses quality $q = q_A + 2\epsilon$ is at least as large as the expected fraction of contributors who choose quality $q > q_A + \epsilon$ and receive a lower ranking than a contributor who uses quality $q = q_A$. Thus if she instead uses quality $q = q_A + 2\epsilon$ for some $\epsilon > 0$, then the expected fraction of contributors who choose quality $q > q_A + \epsilon$ and receive a lower ranking than her is at least $\eta_A(\epsilon)$. From this it follows that the expected total fraction of contributors who receive a lower ranking than this contributor when she chooses quality $q = q_A + 2\epsilon$ is at least $p_A(\epsilon) + \eta_A(\epsilon)$ in the limit as T goes to infinity.

Thus, choosing quality $q = q_A + 2\epsilon$ instead of $q = q_A$ results in this contributor receiving a higher ranking than an expected fraction of at least $\frac{p_A(\epsilon)}{2}$ more contributors. Therefore, if there are k participating contributors, the expected number of contributors that she beats increases by at least $\frac{p_A(\epsilon)k}{2}$ as a result of this change. Moving up in the rankings by one spot increases a contributor's payoff by $\Theta(\frac{A}{k^2})$ since $\alpha_j(k) - \alpha_{j+1}(k) = \Theta(k^{-2})$ for all $j \le k-1$. Thus choosing quality $q = q_A + 2\epsilon$ instead of $q = q_A$ increases her expected benefits by $\Theta(\frac{p_A(\epsilon)k}{2}(\frac{A}{k^2})) = \Theta(\frac{p_A(\epsilon)A}{k})$. So if $b_A(\epsilon)$ denotes the change in expected benefits from choosing quality $q = q_A + 2\epsilon$ instead of choosing quality $q = q_A$, there is no constant C such that $\lim_{A\to\infty} b_A(\epsilon) \le C\epsilon$ for small $\epsilon > 0$.

Now let $c_A(\epsilon)$ denote the maximum added cost (taken over all possible types) that a contributor incurs by choosing quality $q = q_A + 2\epsilon$ instead of choosing quality $q = q_A$. Note that there exists some constant C such that $c_A(\epsilon) \le C\epsilon$ for small $\epsilon > 0$ since $q_A \le q^* < 1$ implies $c'_\tau(q_A) \le c'_\tau(q^*)$ for all τ, which is finite. Combining this with the result in the previous paragraph shows that there exists some large A and some small $\epsilon > 0$ such that a contributor obtains a strictly larger expected payoff from choosing the quality $q = q_A + 2\epsilon$ instead of choosing quality $q = q_A$. This contradicts the fact that q_A is in the support of $F_{\tau,A}(q)$ for some participating type τ and proves that there exists some constant C such that $\lim_{A\to\infty} p_A(\epsilon) \le C\epsilon$ for small $\epsilon > 0$ along this subsequence.

Step 2: Now we show that if there exists some constant $C > 0$ such that $\lim_{A\to\infty} p_A(\epsilon) \le C\epsilon$ for small $\epsilon > 0$ along this subsequence, then it must be the case that $\lim_{A\to\infty} \frac{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)K(A)}{A} = 0$ along this subsequence.

To see this, note that if there exists some constant $C > 0$ such that $\lim_{A\to\infty} p_A(\epsilon) \le C\epsilon$ for small $\epsilon > 0$ along this subsequence, then as T becomes large, the expected fraction of contributors who receive a lower ranking than a contributor who participates with quality $q = q_A$ goes to zero, for the following reason. Let $\epsilon(T)$ denote the largest value of $\epsilon > 0$ such that the probability of an agent with quality q_A receiving a higher ranking than one with quality $q_A + \epsilon(T)$ is at least $\frac{1}{T}$; then, $\lim_{T\to\infty} \epsilon(T) = 0$. Thus the expected fraction of contributors who receive a lower ranking than a contributor who participates with quality q_A is no greater than $p_A(\epsilon(T)) + \frac{1}{T}$ for any T. This tends to zero as T goes to infinity.

Thus if $g(A)$ denotes the expected fraction of contributors who receive a lower ranking than a contributor who participates with quality q_A, then $\lim_{A\to\infty} g(A) = 0$. Now if there are k contributors, then the expected number of contributors who receive a lower ranking than a contributor who participates with quality q_A is $g(A)k$. Combining this with the facts that $\alpha_j(k) - \alpha_{j+1}(k) = \Theta(k^{-2})$ for all $j \le k-1$ and $\alpha_k(k) = 0$ for sufficiently large k shows that a contributor who participates with quality q_A obtains an expected amount of attention $\Theta(g(A)k\frac{A}{k^2}) = \Theta(g(A)\frac{A}{k})$.

Now $\frac{k}{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)K(A)}$ converges in probability to 1 as $A \to \infty$, so $\Theta(g(A)\frac{A}{k}) = \Theta(g(A)\frac{A}{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)K(A)})$. Thus if $\lim_{A\to\infty} \frac{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)K(A)}{A} = 0$ does not hold along this subsequence, then $\lim_{A\to\infty} g(A) = 0$ implies it must be the case that a contributor who contributes with quality q_A receives an expected amount of attention that approaches zero as $A \to \infty$. But this means that an agent could obtain a strictly higher payoff by not contributing than by contributing with quality q_A for some large A, contradicting the fact that

q_A is in the support of $F_{\tau,A}(q)$ for some participating type τ. This contradiction shows that $\lim_{A\to\infty} \frac{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) K(A)}{A} = 0$ must hold along this subsequence.

Step 3: Finally we show that if $\lim_{A\to\infty} \frac{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) K(A)}{A} = 0$ holds along this subsequence, then it cannot be the case that q_A is in the support of $F_{\tau,A}(q)$ for any τ.

To see this, let $\gamma_A = \frac{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) F_{\tau,A}(q^*)}{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)}$ denote the probability that an arbitrary contributor chooses a quality $q \le q^*$ for a given A, and note that $\gamma_A \ge \gamma$ for all A in the subsequence. Also note that if a contributor uses quality $q = q^* + \epsilon$ instead of $q = q_A$, then this costs her no more than $c(q^* + \epsilon)$, where $c(q^* + \epsilon)$ denotes the maximum value of $c_\tau(q^* + \epsilon) - c_\tau(q_A)$ over all types τ.

Now if this contributor uses quality $q = q_A$, then the probability she receives a higher ranking than a particular other contributor who chooses quality $q \le q^*$ is no greater than $\frac{1}{2}$. Thus if δ_A denotes the expected fraction of contributors who choose quality $q > q^*$ and receive a lower ranking than this contributor using quality q_A, then the expected total fraction of contributors that receive a lower ranking than her is no greater than $\frac{\gamma_A}{2} + \delta_A$.

Now, if she instead uses quality $q = q^* + \epsilon$ for some $\epsilon > 0$, then the probability she receives a higher ranking than a particular other contributor who chooses quality $q \le q^*$ goes to 1 in the limit as T goes to infinity. The expected fraction of contributors who choose quality $q > q^*$ and receive a lower ranking than a contributor who uses quality $q = q^* + \epsilon$ is at least as large as the expected fraction of contributors who choose quality $q > q^*$ and receive a lower ranking than a contributor who uses quality $q = q_A$. Thus if the contributor instead uses quality $q = q^* + \epsilon$ for some $\epsilon > 0$, then the expected fraction of contributors who choose quality $q > q^*$ and receive a lower ranking than the contributor is at least δ_A. From this it follows that the expected fraction of contributors who receive a lower ranking than this contributor when she chooses quality $q = q^* + \epsilon$ is at least $\gamma_A + \delta_A$ in the limit as $T \to \infty$.

Thus choosing quality $q = q^* + \epsilon$ instead of $q = q_A$ results in receiving a higher ranking than an expected fraction of at least $\frac{\gamma_A}{2}$ additional contributors. Thus if there are k participating contributors, this contributor increases the expected number of agents she beats by at least $\frac{\gamma_A k}{2}$ as a result of this change. As before, moving up in the rankings by one spot increases the payoff by $\Theta(\frac{A}{k^2})$ since $\alpha_j(k) - \alpha_{j+1}(k) = \Theta(k^{-2})$ for all $j \le k-1$. Thus if a contributor chooses quality $q = q^* + \epsilon$ instead of choosing quality $q = q_A$, then she increases her expected benefits by $\Theta(\frac{\gamma_A k}{2}(\frac{A}{k^2})) = \Theta(\frac{A}{k}) = \Theta(\frac{A}{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) K(A)})$.

But for sufficiently large A in the subsequence, it follows that this increase in expected benefits is greater than $c(q^* + \epsilon)$ since $\lim_{A\to\infty} \frac{A}{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) K(A)} = \infty$ in the subsequence but $c(q^* + \epsilon)$ is independent of A. Thus from this it follows that a contributor obtains a strictly greater expected payoff from choosing quality $q = q^* + \epsilon$ instead of choosing quality $q = q_A$ for sufficiently large A in the subsequence. This contradicts the existence of some such q^* and proves the desired result. □

Proof of Theorem 3.3. Let $c(\frac{1}{2})$ denote the maximum value of $c_\tau(\frac{1}{2})$ over all types τ, and let $g(A) \equiv \frac{A}{K(A)}$. Note that the probability that an agent who contributes with quality $q = \frac{1}{2}$ receives only positive votes is $\frac{1}{2^{T(A)}}$. From this it follows that the expected benefit an agent obtains from participating with quality $q = \frac{1}{2}$ is always at least $\frac{g(A)}{2^{T(A)}}$ (even if all other participants also receive all $T(A)$ positive votes, each participant receives an equal amount of attention in expectation, which is no smaller than $\frac{A}{K(A)}$). Thus if $\frac{g(A)}{2^{T(A)}} \ge c(\frac{1}{2})$ for sufficiently large A, then all agents will strictly prefer participating with quality $\frac{1}{2}$ to not participating at all. From this it follows that if $T(A)$ satisfies $T(A) \le \log_2(\frac{g(A)}{c(\frac{1}{2})})$ for sufficiently large A, then all agents will strictly prefer participating with quality $\frac{1}{2}$ to not participating at all for sufficiently large A. Since $\frac{A}{K(A)}$ diverges as $A \to \infty$, it then follows that there exists some sequence $\{T(A)\}_{A=1}^{\infty}$ satisfying $\lim_{A\to\infty} T(A) = \infty$ such that $\beta_\tau(A) = 1$ must hold for all types τ for sufficiently large A. □

Proof of Theorem 3.4. To see that $\lim_{A\to\infty} \beta_\tau(A) = 0$ for all τ, suppose by means of contradiction that there exists some $\beta^* > 0$ such that there exists some type τ for which $\beta_\tau(A) > \beta^*$ for infinitely many values of A. Restrict attention to values of A for which $\beta_\tau(A) > \beta^*$ for this particular type τ. For such A, $\limsup_{A\to\infty} \frac{A}{\pi_\tau \beta_\tau(A) K(A)} = s$ for some $s < \infty$. Thus for sufficiently large A, the expected benefit to participating for an agent of type τ is no greater than $s + 1$. But since we know from Theorem 3.2 that the probability an agent chooses a quality $q \ge q^*$ for any $q^* < 1$ goes to 1 as $A \to \infty$, it follows that the expected cost to participating and following equilibrium strategies for an agent of type τ becomes arbitrarily large in the limit as $A \to \infty$. From this it follows that for sufficiently large A in this subsequence, an agent of type τ strictly prefers not to participate. This contradicts our assumption that agents of type τ participate with positive probability

Now suppose that there is an infinite sequence of values of A such that, for each A, there is a symmetric equilibrium to the ranking mechanism with $\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) K(A) < y$, where $y < \infty$. Since $\lim_{A\to\infty} \beta_\tau(A) = 0$ for all τ, we know that $\beta_\tau(A) < 1$ for sufficiently large A. Furthermore, we know that there is at least one type τ for which $\beta_\tau(A) > 0$ for all A, so from this it follows that there is at least one type τ for which $\beta_\tau(A) \in (0, 1)$ for sufficiently large A. Any agent of such a type τ must be indifferent between participating and not participating, so the expected costs from participating for this type must equal expected benefits.

For any given A, consider the set of types τ for which $\beta_\tau(A) \in (0, 1)$, and let $\tau(A)$ denote the value of τ for which the total expected amount of attention received by agents of type τ is largest in equilibrium (where if there are multiple types τ that receive the same expected amount of attention in equilibrium, then we let $\tau(A)$ be the first such type). Consider a type τ for which $\tau = \tau(A)$ for infinitely many values of A (note that there are a finite number of types) and restrict attention to values of A for which $\tau = \tau(A)$. Note that the total amount of attention received by agents of type τ is $\Theta(A)$, so from the condition on indifference between participating and not participating, it follows that $E[c(q)|q \sim F_{\tau,A}(q)] = \Theta(\frac{A}{\beta_\tau(A)K(A)}) = \Theta(A)$ for all A in this sequence. Thus if $q^*(A)$ denotes the largest quality in the support of $F_{\tau,A}(q)$, then we know that $c_\tau(q^*(A)) = \Omega(A)$ for all A in this sequence. Thus there exists some function $g(A) = \Theta(A)$ independent of $T(A)$ such that $c_\tau(q^*(A)) \ge g(A)$ and $q^*(A) \ge c_\tau^{-1}(g(A))$ for all A in this sequence. Thus if $\hat{q}(A) = c_\tau^{-1}(g(A))$, then $\lim_{A\to\infty} \hat{q}(A) = 1$ and $q^*(A) \ge \hat{q}(A)$ for all A in this sequence.

Now note that if a contributor chooses some quality $q = q^*(A) + \epsilon(A)$ instead of choosing quality $q = q^*(A)$ for some infinitesimal amount $\epsilon(A) = o(\frac{1}{T(A)})$ (which may either be positive or negative), then the contributor changes the probability that she receives an additional positive vote by an amount $\Theta(\epsilon(A)T(A))$. Thus the contributor changes the probability that she receives a higher ranking than a particular other contributor by $O(\epsilon(A)T(A))$, which means that the expected number of other contributors that this contributor beats changes by $O(\epsilon(A)T(A)\sum_{\tau=1}^{t} \pi_\tau \beta(A) K(A)) = O(\epsilon(A)T(A))$. When a contributor moves up in the rankings by one spot, she increases her payoff by an amount $\Theta(A)$ since there are only a finite number of contributors, the difference in attention between any two positions is $(\alpha_j(k) - \alpha_{j+1}(k))A$ for some j and k, and $\alpha_j(k) - \alpha_{j+1}(k) > 0$ for all finite j and k by assumption. Thus if a contributor chooses some quality $q = q^*(A) + \epsilon(A)$ instead of choosing quality $q = q^*(A)$ for some infinitesimal amount $\epsilon(A)$, then the contributor changes her expected benefits by $O(\epsilon(A)T(A)A)$.

Thus if $b_A(\epsilon(A))$ denotes the difference between the expected benefits from attention that a contributor obtains by choosing $q = q^*(A) + \epsilon(A)$ instead of choosing quality $q = q^*(A)$ for some infinitesimal amount $\epsilon(A)$, then $|b_A(\epsilon(A))| = O(|\epsilon(A)T(A)A|)$, meaning $b'_A(0) = O(T(A)A)$. But in order for a contributor of type τ to not be able to profitably deviate from choosing quality $q = q^*(A)$, it is necessary that $b'_A(0) = c'_\tau(q^*(A))$. Thus $c'_\tau(q^*(A)) = O(T(A)A)$.

Recall that $c_\tau(q^*(A)) = \Omega(A)$. Combining this with the result in the previous paragraph shows that $\frac{c'_\tau(q^*(A))}{c_\tau(q^*(A))} = O(T(A))$. Thus if $\bar{q}(A)$ denotes a minimizer of $\frac{c'_\tau(q)}{c_\tau(q)}$ subject to the constraint $q \in [\hat{q}(A), 1]$, then $\log \frac{c'_\tau(\bar{q}(A))}{c_\tau(\bar{q}(A))}$ is less than the value of $\frac{c'_\tau(q)}{c_\tau(q)}$ for every $q \in [\hat{q}(A), 1]$. Thus if $T(A) = \Theta(\log \frac{c'_\tau(\bar{q}(A))}{c_\tau(\bar{q}(A))})$, then $T(A)$ is less than the value of $\frac{c'_\tau(q)}{c_\tau(q)}$ for every $q \in [\hat{q}(A), 1]$ for sufficiently large A. From this it follows that if $\frac{c'_\tau(q^*(A))}{c_\tau(q^*(A))} = O(T(A))$, then $q^*(A)$ cannot be in $[\hat{q}(A), 1]$, and $q^*(A) < \hat{q}(A)$ for sufficiently large A, which would contradict the fact that $q^*(A) \geq \hat{q}(A)$ for all A in this sequence.

Now since $\lim_{q\to 1} c_\tau(q) = \infty$, it follows that $\lim_{q\to 1} \log c_\tau(q) = \infty$, $\lim_{q\to 1} \frac{d}{dq} \log c_\tau(q) = \infty$, and $\lim_{q\to 1} \frac{c'_\tau(q)}{c_\tau(q)} = \infty$. Thus $\lim_{A\to\infty} \bar{q}(A) = 1$ implies $\lim_{A\to\infty} \frac{c'_\tau(\bar{q}(A))}{c_\tau(\bar{q}(A))} = \infty$. Combining this with the results in the previous paragraphs shows that if $T(A) = \Theta(\log \frac{c'_\tau(\bar{q}(A))}{c_\tau(\bar{q}(A))})$, then $\{T(A)\}_{A=1}^{\infty}$ is a sequence satisfying $\lim_{A\to\infty} T(A) = \infty$ such that $\lim_{A\to\infty} \sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) K(A) = \infty$. □

Proof of Theorem 3.5. First note that the probability that at least l agents participate goes to one in the limit as $A \to \infty$. To see this, suppose by means of contradiction that there is some $\pi > 0$ such that there are an infinite number of values of A for which the probability that fewer than l agents participate is greater than π. Then for any such sufficiently large A, an agent obtains a strictly greater expected payoff by participating with quality 0 (and obtaining at least $\alpha_l A$ attention with probability greater than π) than by not participating at all. This contradicts the fact that agents do not participate with some strictly positive probability and proves that the probability that at least l agents participate goes to one in the limit as $A \to \infty$. From this and the fact that all agents of the same type participate with the same probability, it follows that the probability that at least $l + 1$ agents participate goes to one in the limit as $A \to \infty$ as well.

Now suppose by means of contradiction that there is some $\rho < 1$ such that the probability that at least l agents make a contribution of quality $q \geq q^*$ is less than ρ for an infinite number of A. Let $\underline{q}(A)$ denote the smallest q in the supports of the distributions $F_{\tau,A}(q)$. Since there is a positive probability that agents make contributions of quality $q < q^*$, it follows that $\underline{q}(A) < q^*$.

Now suppose an agent i deviates from participating with quality $q = \underline{q}(A)$ to participating with quality $q^* + \epsilon$ for some small $\epsilon > 0$. Note that this deviation costs an agent no more than $c_\tau(q^* + \epsilon)$ for some τ, which remains bounded and finite as $A \to \infty$. This deviation also increases the probability that agent i will be ranked ahead of some other agent j who makes a contribution of quality $q \leq q^*$ by at least $\frac{1}{2}$ in the limit as $A \to \infty$ (and thus $T(A) \to \infty$): Since agent i was initially choosing the lowest possible quality, agent i would initially never beat any other agent with probability greater than $\frac{1}{2}$. But when agent i chooses quality $q = q^* + \epsilon$, the agent now beats any other agent who makes a contribution of quality $q \leq q^*$ with probability arbitrarily close to 1 in the limit as $A \to \infty$. Thus this deviation increases the probability that agent i will be ranked ahead of some other agent j who makes a contribution of quality $q \le q^*$ by at least $\frac{1}{2}$ in the limit as $A \to \infty$.

Thus if there is a positive probability that fewer than l agents make a contribution of quality $q \ge q^*$, this deviation increases the probability that an agent will be ranked amongst the top l contributions by an amount that remains bounded away from zero in the limit as $A \to \infty$. Thus this deviation increases an agent's expected benefit by an unbounded amount in the limit as $A \to \infty$, while the cost from this deviation remains bounded above by $c_\tau(q^* + \epsilon)$, which is finite since $q^* < 1$. Therefore this is a profitable deviation, contradicting our assumption that there is some $\rho < 1$ such that the probability that at least l agents make a contribution of quality $q \ge q^*$ is less than ρ for an infinite number of values of A. From this it follows that for any $q^* < 1$, the probability that at least l agents make a contribution of quality $q \ge q^*$ goes to 1 in the limit as $A \to \infty$. □

Proof of Theorem 4.1. First note that no player in this game would ever choose a quality $q > \max_\tau\{c_\tau^{-1}(v(A))\}$, as a player could always obtain a strictly greater expected payoff by not participating than by participating and choosing a quality $q > \max_\tau\{c_\tau^{-1}(v(A))\}$. Thus any mixed strategy equilibrium to the game in which players are restricted to choosing $q \in [0, \max_\tau\{c_\tau^{-1}(v(A))\}]$ is also a mixed strategy equilibrium of the original game.

Now note that this modified game in which players are restricted to choosing $q \in [0, \max_\tau\{c_\tau^{-1}(v(A))\}]$ is a symmetric game in which each player has a pure strategy space that is compact and Hausdorff. Also note that each player's expected payoff in this modified game is continuous in the actions of the players. It thus follows from Theorem 1 of [2] that there exists a symmetric mixed strategy equilibrium of this modified game. This in turn implies that there is a symmetric mixed strategy equilibrium of the original game.

But in Theorem 4.2 we show that there cannot exist an equilibrium in which there is some type τ that chooses qualities drawn from a distribution $F_\tau(q)$, where $F_\tau(q)$ is not a point mass. From this and the previous equilibrium existence result, it follows that there must exist an equilibrium of the form given in the statement of the theorem. □

Proof of Theorem 4.2. Suppose by means of contradiction that there exists a symmetric mixed strategy equilibrium in which there is some type τ that chooses qualities drawn from a distribution $F_\tau(q)$, where $F_\tau(q)$ is not a point mass. Let q_L denote the infimum of the support of $F_\tau(q)$ and q_H denote the supremum of the support of $F_\tau(q)$. Since $F_\tau(q)$ is not a point mass, $q_L < q_H$.

If $\vec{\beta} \equiv (\beta_1, \ldots, \beta_t)$ and $b_\tau(q_i, q_{-i}, \vec{\beta})$ denotes the expected benefit to an agent of type τ from participating with quality q_i when all other agents have chosen qualities q_{-i} and participate with probabilities given by $\vec{\beta}$, then we know that $\frac{\partial b_\tau}{\partial q_i}(q_i, q_{-i}, \vec{\beta})$ is decreasing in q_i for the following reason: The marginal benefit from receiving an additional positive vote is smaller when m_i is larger for any fixed values of m_{-i} since the difference between $\frac{m_i+1}{m_i+1+\sum_{j\neq i} m_j}$ and $\frac{m_i}{m_i+\sum_{j\neq i} m_j}$ is decreasing in m_i. Now if $G(m_i|q_i)$ denotes the distribution of the values of m_i given that agent i is producing quality q_i, then $q_i' > q_i$ implies that $G(m_i|q_i')$ first order stochastically dominates $G(m_i|q_i)$. Thus if $q_i' > q_i$, then the expected benefit from receiving an additional positive vote is smaller when an agent is producing with quality q_i' than it is when an agent is producing with quality q_i. From this it follows that $\frac{\partial b_\tau}{\partial q_i}(q_i, q_{-i}, \vec{\beta})$ is decreasing in q_i.

At the same time, we know that $c_\tau'(q_i)$ is nondecreasing in q_i because c_τ is convex. From this it follows that if agents are following a mixed strategy equilibrium in which agents participate with probabilities given by $\vec{\beta}$ and agents of type τ choose qualities that are random draws from the distribution $F_\tau(q)$, then $E[\frac{\partial b}{\partial q_i}(q_L, q_{-i}, \vec{\beta})] - c'_\tau(q_L) > E[\frac{\partial b_\tau}{\partial q_i}(q_H, q_{-i}, \vec{\beta})] - c'_\tau(q_H)$. But this implies that either $E[\frac{\partial b_\tau}{\partial q_i}(q_L, q_{-i}, \vec{\beta})] - c'_\tau(q_L) > 0$ or $E[\frac{\partial b_\tau}{\partial q_i}(q_H, q_{-i}, \vec{\beta})] - c'_\tau(q_H) < 0$. In the first case, an agent can profitably deviate by choosing a quality slightly higher than q_L, and in the second case an agent can profitably deviate by choosing a quality slightly lower than q_H. This contradicts the possibility that there is a symmetric mixed strategy equilibrium in which there is some type τ that choose qualities drawn from a distribution $F_\tau(q)$, where $F_\tau(q)$ is not a point mass. □

Lemma 4.1. *Let $\beta_\tau(A)$ and $q_\tau(A)$ denote equilibrium participation probabilities and quality choices in the proportional mechanism for agents of type τ for a given A. Then $c'_\tau(q_\tau(A)) = \Theta(\frac{A}{\sum_{\tau=1}^{t} \pi_\tau q_\tau(A)\beta_\tau(A)K(A)})$ for any type τ for which $\beta_\tau(A) > 0$ for sufficiently large A.*

Proof.

$$E\left[V(m_i, m_{-i}) \mid (\vec{\beta}, q_i, q_{-i})\right] = E\left[v\left(\frac{m_i}{m_i + \sum_{j\neq i} m_j} A\right) \middle| (\vec{\beta}, q_i, q_{-i})\right].$$

Note that for a given value of k, as T goes to infinity, $\frac{m_i}{m_i+\sum_{j\neq i} m_j} = \frac{m_i/T}{m_i/T+\sum_{j\neq i} m_j/T}$ converges in probability to $\frac{q_i}{q_i+\sum_{j\neq i} q_j}$. Now if $q_j = q_\tau$ for all $j \neq i$, where τ denotes the type corresponding to agent j, then $\frac{q_i}{q_i+\sum_{j\neq i} q_j} = \frac{q_i}{q_i+\sum_{\tau=1}^{t} k_\tau q_\tau}$, where k_τ denotes the number of participating agents of type τ. Thus as A and $K(A)$ go to infinity, $\frac{q_i}{q_i+\sum_{\tau=1}^{t} k_\tau q_\tau}$ converges in probability to $\frac{q_i}{q_i+\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)K(A)q_\tau}$. Thus $\frac{d}{dq_i} E[V(m_i, m_{-i})|(\vec{\beta}, q_i, \vec{q_\tau})] = \Theta(\frac{d}{dq_i} \frac{q_i A}{q_i+\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)K(A)q_\tau})$ for large A, $K(A)$, and $T(A)$. But $\frac{d}{dq_i} \frac{q_i A}{q_i+\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)K(A)q_\tau} = \Theta(\frac{A}{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)K(A)q_\tau})$. Thus in equilibrium it must be the case that $c'_\tau(q_\tau(A)) = \Theta(\frac{A}{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)K(A)q_\tau(A)})$. □

Proof of Theorem 4.3. Note that if $\lim_{A\to\infty} \frac{K(A)}{A} = 0$, then the fact that $c'_\tau(q_\tau(A)) = \Theta(\frac{A}{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)K(A)q_\tau(A)}) = \Omega(\frac{A}{K(A)})$ for all participating types τ implies $\lim_{A\to\infty} c'_\tau(q_\tau(A)) = \infty$ and $\lim_{A\to\infty} q_\tau(A) = 1$ for all participating types τ. Also note that an agent can always obtain a diverging amount of attention in the limit as $A \to \infty$ by contributing with a quality $q = \frac{1}{2}$ because such a contribution would result in at least $\frac{A}{2K(A)}$ units of attention in expectation in the limit even if all other agents participate and contribute with quality $q = 1$. However, such a contribution only incurs a finite cost, so all agents strictly prefer to participate than to not participate in the limit as $A \to \infty$. From this it follows that $\beta_\tau(A) = 1$ for all types τ for sufficiently large A.

Now suppose that $\liminf_{A\to\infty} \frac{K(A)}{A} > 0$. In this case, if $\beta_\tau(A) = 1$ for some type τ for sufficiently large A, then the fact that $c'_\tau(q_\tau(A)) = \Theta(\frac{A}{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)K(A)q_\tau(A)})$ implies that $c'_\tau(q_\tau(A)) = \Theta(1)$ and $\limsup_{A\to\infty} q_\tau(A) < 1$ for this type τ. And if $\beta_\tau(A) < 1$ for some participating type τ for some infinite subsequence of A, then $c_\tau(q_\tau(A)) = \Theta(\frac{q_\tau(A)A}{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)K(A)q_\tau(A)})$ since an agent of type τ receives $\Theta(\frac{q_\tau(A)A}{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)K(A)q_\tau(A)})$ units of attention in expectation and must be indifferent between entry and exit if $\beta_\tau(A) < 1$. Thus for such an agent of type τ, it must be the case that $\frac{c'_\tau(q_\tau(A))}{c_\tau(q_\tau(A))} = \Theta(\frac{1}{q_\tau(A)})$, meaning $\frac{c'_\tau(q_\tau(A))}{c_\tau(q_\tau(A))} = \Theta(1)$ if $q_\tau(A) \to 1$ for some infinite subsequence of A. But since $\lim_{q\to 1} c_\tau(q) = \infty$, $\lim_{q\to 1} \log c_\tau(q) = \infty$ as well, $\lim_{q\to 1} \frac{d}{dq} \log c_\tau(q) = \infty$, and $\lim_{q\to 1} \frac{c'_\tau(q)}{c_\tau(q)} = \infty$. Thus the fact that $\lim_{A\to\infty} q_\tau(A) = 1$ implies that $\lim_{A\to\infty} \frac{c'_\tau(q_\tau(A))}{c_\tau(q_\tau(A))} = \infty$, meaning $\frac{c'_\tau(q_\tau(A))}{c_\tau(q_\tau(A))} = \Theta(1)$ cannot hold. From this it follows that $q_\tau(A) \to 1$ cannot hold for any infinite subsequence of A, so it must be the case that $\limsup_{A\to\infty} q_\tau(A) < 1$ for any such type τ. Thus $\limsup_{A\to\infty} q_\tau(A) < 1$ holds for all participating types τ if $\liminf_{A\to\infty} \frac{K(A)}{A} > 0$.

Finally, to prove that $\liminf_{A\to\infty} \sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) > 0$, suppose by means of contradiction that there exists some subsequence of values of A for which $\lim_{A\to\infty} \sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) = 0$. Then if an agent contributes with a quality $q = \frac{1}{2}$, the agent would obtain a diverging amount of attention because such a contribution would result in at least $\frac{A}{2\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) K(A)}$ units of attention in expectation in the limit even if all other participating agents contribute with quality $q = 1$. However, such a contribution only incurs a finite cost, so all non-participating agents strictly prefer to participate than to not participate in the limit as $A \to \infty$. This contradicts our assumption that there exists some subsequence of values of A for which $\lim_{A\to\infty} \sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A) = 0$ and proves the result. □

Lemma 5.1. *The minimum value of* $\sum_{i=1}^{n} x_i^2$ *subject to the constraints* $x_i \geq 0$ *and* $\sum_{i=1}^{n} x_i = 1$ *is* $\frac{1}{n}$.

Proof. Note that subject to the constraints $x_i \geq 0$ and $\sum_{i=1}^{n} x_i = 1$, the expression $\sum_{i=1}^{n} x_i^2$ is minimized when $x_i = \frac{1}{n}$ for all i. To see this, note that if $x_i = \frac{1}{n} + \delta_i$ for some $\{\delta_i\}_{i=1}^{n}$ satisfying $\sum_{i=1}^{n} \delta_i = 0$, then $\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} (\frac{1}{n} + \delta_i)^2 = \sum_{i=1}^{n} (\frac{1}{n^2} + \frac{2}{n}\delta_i + \delta_i^2) = \frac{1}{n} + \sum_{i=1}^{n} \delta_i^2$. But $\frac{1}{n} + \sum_{i=1}^{n} \delta_i^2$ is minimized when $\delta_i = 0$ for all i. From this it follows that the expression $\sum_{i=1}^{n} x_i^2$ is minimized when $x_i = \frac{1}{n}$ for all i, and the minimum value of $\sum_{i=1}^{n} x_i^2$ subject to the constraints $x_i \geq 0$ and $\sum_{i=1}^{n} x_i = 1$ is $\frac{1}{n}$. □

Proof of Theorem 5.1. Suppose by means of contradiction that there exists some type τ that participates with non-vanishing probability for which $Pr(q_{\tau,r}(A) \leq q_{\tau,p}(A)) \geq \gamma > 0$ for an infinite number of A. Restrict attention to values of A satisfying $Pr(q_{\tau,r}(A) \leq q_{\tau,p}(A)) \geq \gamma$, and define $q^*(A)$ to be the largest quality such that $Pr(q_{\tau,r}(A) < q^*(A)) \leq \frac{\gamma}{2}$ for any such A. From this it follows that $Pr(q^*(A) \leq q_{\tau,r}(A) \leq q_{\tau,p}(A)) \geq \frac{\gamma}{2}$ for all A in this subsequence. We also know from Theorem 3.2 that $\lim_{A\to\infty} q^*(A) = 1$.

Now suppose that a contributor of type τ deviates from $F_{\tau,A}(q)$ by making the following change: if she draws a quality $q \in [q^*(A), q_{\tau,p}(A)]$, then she instead chooses a quality $q + \epsilon$ for some infinitesimal amount $\epsilon > 0$; if she draws a quality $q \notin [q^*(A), q_{\tau,p}(A)]$, then she makes no change to her quality. We seek to show that this is a profitable deviation for a contributor for sufficiently large A in the subsequence. To do this, we first show that the additional expected cost from this deviation is bounded above by the marginal costs of producing higher quality content in the proportional mechanism, and then show that the expected benefit in the ranking mechanism from this deviation exceeds these marginal costs.

First note that this change in quality costs a contributor of type τ an additional amount no greater than $E[c_\tau(q+\epsilon) - c_\tau(q) | q \in [q^*(A), q_{\tau,p}(A)]]$ which, in the limit as $\epsilon \to 0$, converges to $\epsilon E[c'_\tau(q) | q \in [q^*(A), q_{\tau,p}(A)]] \leq \epsilon c'_\tau(q_{\tau,p}(A)) = \Theta(\epsilon \frac{A}{K(A)})$ (since $\beta_\tau(A) = 1$ for all types τ

for large A in the proportional mechanism when $\lim_{A\to\infty} \frac{K(A)}{A} = 0$). Next we calculate the expected benefits from this increase in quality.

Suppose a contributor i of type τ chooses a quality $q \in [q^*(A), q_{\tau,p}(A)]$. (This happens with probability at least $\gamma/2 > 0$ by assumption.) Then there exists an infinite sequence $\hat{q}(A)$ such that $\lim_{A\to\infty} \hat{q}(A) = 1$ and $Pr(\frac{m_i}{T} \geq \hat{q}(A)) \geq \frac{1}{2}$ for all A. Thus if two contributors of type τ both choose qualities in $[q^*(A), q_{\tau,p}(A)]$, then the probability that they both receive at least $\hat{q}(A)T$ positive votes is at least $\frac{1}{4}$. Moreover, conditional on both receiving at least $\hat{q}(A)T$ positive votes, the probability a contributor receives any particular number of votes is the same for both contributors.

From Lemma 5.1, we know that if two contributors of type τ both receive at least $\hat{q}(A)T$ positive votes and they both receive any particular number of votes with the same probability, then the probability that both contributors receive an equal number of votes is at least $\frac{1}{(1-\hat{q}(A))T}$. Thus if one contributor receives an additional vote when both receive at least $\hat{q}(A)T$ positive votes, then this additional vote increases her probability of being ranked ahead of the other contributor by at least $\frac{1}{2(1-\hat{q}(A))T}$ (since the additional vote increases the agent's probability of being ranked ahead of the other agent by a factor of $\frac{1}{2}$).

Now the probability both contributor i and some other contributor of type τ choose qualities $q \in [q^*(A), q_{\tau,p}(A)]$ is at least $(\frac{\gamma}{2})^2$. And we have seen that the probability that both contributors receive at least $\hat{q}(A)T$ positive votes if they choose qualities $q \in [q^*(A), q_{\tau,p}(A)]$ is at least $\frac{1}{4}$. Combining this with the results in the previous paragraph shows that an additional vote for contributor i increases the probability of her being ranked ahead of a particular other contributor of type τ by at least $(\frac{\gamma}{2})^2 \frac{1}{4} \frac{1}{2(1-\hat{q}(A))T(A)} = \frac{\gamma^2}{32(1-\hat{q}(A))T(A)}$. Thus an additional vote for contributor i also increases her probability of being ranked ahead of some particular other contributor by at least $\frac{\pi_\tau \beta_\tau(A)}{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)} \frac{\gamma^2}{32(1-\hat{q}(A))T(A)}$.

Now increasing quality by $\epsilon(A) = o(\frac{1}{T(A)})$ leads to an additional positive vote with probability $\Theta(\epsilon(A)T(A))$. Therefore, increasing quality by $\epsilon(A)$ increases the probability of being ranked ahead of a particular other contributor by $\Omega(\epsilon(A)T(A) \frac{\pi_\tau \beta_\tau(A)}{\sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)} \frac{\gamma^2}{32(1-\hat{q}(A))T(A)}) = \Omega(\frac{\epsilon(A)}{1-\hat{q}(A)})$.

This implies that a contributor's change in expected ranking from increasing quality by $\epsilon(A)$ when $q \in [q^*(A), q_{\tau,p}(A)]$ is $\Omega(\frac{\epsilon(A)k}{1-\hat{q}(A)})$. And moving up in the rankings by one spot increases one's payoff by $\Theta(\frac{A}{k^2})$ since $\alpha_j(k) - \alpha_{j+1}(k) = \Theta(k^{-2})$ for all $j \leq k-1$. Thus if an agent increases her quality by $\epsilon(A)$ when $q \in [q^*(A), q_{\tau,p}(A)]$, then her expected benefit increases by an amount $\Omega(\frac{\epsilon(A)k}{1-\hat{q}(A)} \frac{A}{k^2}) = \Omega(\frac{\epsilon(A)A}{(1-\hat{q}(A)) \sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)K(A)})$.

Thus, increasing quality by $\epsilon(A)$ when $q \in [q^*(A), q_{\tau,p}(A)]$ leads to an additional benefit of $\Omega(\frac{\epsilon(A)A}{(1-\hat{q}(A)) \sum_{\tau=1}^{t} \pi_\tau \beta_\tau(A)K(A)})$, at an additional cost of $O(\frac{\epsilon(A)A}{K(A)})$. Since $\lim_{A\to\infty} \hat{q}(A) = 1$, it follows that this is a profitable deviation for sufficiently large A. This contradicts the assumption that there is a sequence of equilibria for which $Pr(q_{\tau,r}(A) \leq q_{\tau,p}(A)) \geq \gamma > 0$ for an infinite number of A and proves the desired result. □

Proof of Example 5.1. Note that in the proportional mechanism, an agent will obtain a total of A units of attention if the agent obtains at least one positive vote and the other agent does not obtain any positive votes. An agent will obtain (i) a total of $A/2$ units of attention if both agents receive the same number of positive votes, (ii) a total of $2A/3$ units of attention if she obtains two positive votes and the other agent obtains one positive vote, (iii) $A/3$ units of attention if she obtains one positive vote and the other agent obtains two positive votes, and finally (iv) no attention at all if she does not obtain any positive votes and the other agent obtains at least one positive vote.

Now if agents i and j contribute with qualities q_i and q_j, then the probability that both agents obtain the same number of votes is $q_i^2 q_j^2 + 4q_i(1-q_i)q_j(1-q_j) + (1-q_i)^2(1-q_j)^2$. The probability that agent i obtains at least one positive vote and the other agent does not obtain any positive votes is $q_i^2(1-q_j)^2 + 2q_i(1-q_i)(1-q_j)^2$; that agent i obtains two positive votes while agent j obtains one positive vote is $2q_i^2 q_j(1-q_j)$; and that agent i obtains one positive vote while agent j obtains two positive votes is $2q_i(1-q_i)q_j^2$. From this it follows that agent i's expected utility from contributing with quality q_i given that agent j contributes with quality q_j is

$$\begin{aligned} u_i(q_i; q_j) &= \big[q_i^2 q_j^2 + 4q_i(1-q_i)q_j(1-q_j) + (1-q_i)^2(1-q_j)^2\big]\frac{A}{2} + \big[q_i^2(1-q_j)^2 \\ &\quad + 2q_i(1-q_i)(1-q_j)^2\big]A \\ &\quad + 2q_i^2 q_j(1-q_j)\frac{2A}{3} + 2q_i(1-q_i)q_j^2\frac{A}{3} - c(q_i) \end{aligned}$$

Differentiating this expression with respect to q_i gives

$$\begin{aligned} u_i'(q_i; q_j) &= \big[2q_i q_j^2 + 4(1-q_i)q_j(1-q_j) - 4q_i q_j(1-q_j) - 2(1-q_i)(1-q_j)^2\big]\frac{A}{2} \\ &\quad + \big[2q_i(1-q_j)^2 + 2(1-q_i)(1-q_j)^2 - 2q_i(1-q_j)^2\big]A \\ &\quad + 4q_i q_j(1-q_j)\frac{2A}{3} + \big[2(1-q_i)q_j^2 - 2q_i q_j^2\big]\frac{A}{3} - c'(q_i) \end{aligned}$$

In a symmetric pure-strategy equilibrium in which both agents participate with the same quality q (if one exists), it must be the case that the above derivative is equal to zero when evaluated at $q_i = q_j = q$. Thus if there is a pure strategy equilibrium in which both agents contribute with quality q, it must be the case that

$$\begin{aligned} &\big[2q^3 + 4q(1-q)^2 - 4q^2(1-q) - 2(1-q)^3\big]\frac{A}{2} + \big[2q(1-q)^2 + 2(1-q)^3 \\ &\quad - 2q(1-q)^2\big]A + 4q^2(1-q)\frac{2A}{3} + \big[2(1-q)q^2 - 2q^3\big]\frac{A}{3} = c'(q) \end{aligned}$$

By simplifying the left-hand-side of this equation, it then follows that q must satisfy

$$\frac{1}{3}\big[q^3 + 4q^2(1-q) + 6q(1-q)^2 + 3(1-q)^3\big]A = c'(q).$$

Further simplifying the left-hand-side of this equation then gives $A[\frac{q^2}{3} - q + 1] = c'(q)$, which in turn implies that q must satisfy $\frac{c'(q)}{q^2/3-q+1} = A$.

Next we turn to the rank-order mechanism in which all A units of attention are awarded to the agent who finishes first. Note that in the rank-order mechanism, if both agents receive the same number of positive votes, then both agents obtain $\frac{A}{2}$ units of attention in expectation. If one agent receives a strictly greater number of positive votes than the other agent, then the agent with the larger number of positive votes obtains all A units of attention, and the agent with the strictly lower number of positive votes obtain 0 units of attention.

Now if agent i contributes with quality q_i and agent j contributes with quality q_j, then the probability that both agents receive the same number of positive votes is $q_i^2 q_j^2 + 4q_i(1-q_i)q_j(1-q_j) + (1-q_i)^2(1-q_j)^2$ and that agent i obtains a strictly larger number of positive votes than agent j is $q_i^2(1-q_j)^2 + 2q_i(1-q_i)(1-q_j)^2 + 2q_i^2 q_j(1-q_j)$. From this it follows that if agent i contributes with quality q_i and agent j contributes with quality q_j, then agent i obtains an expected utility of

$$u_i(q_i;q_j) = \left[q_i^2 q_j^2 + 4q_i(1-q_i)q_j(1-q_j) + (1-q_i)^2(1-q_j)^2\right]\frac{A}{2} + \left[q_i^2(1-q_j)^2 + 2q_i(1-q_i)(1-q_j)^2 + 2q_i^2 q_j(1-q_j)\right]A - c(q_i).$$

Differentiating this expression with respect to q_i gives

$$u_i'(q_i;q_j) = \left[2q_i q_j^2 + 4(1-q_i)q_j(1-q_j) - 4q_i q_j(1-q_j) - 2(1-q_i)(1-q_j)^2\right]\frac{A}{2} + \left[2q_i(1-q_j)^2 + 2(1-q_i)(1-q_j)^2 - 2q_i(1-q_j)^2 + 4q_i q_j(1-q_j)\right]A - c'(q_i).$$

As before, in a symmetric pure-strategy equilibrium in which both agents participate with the same quality q (if one exists), it must be the case that the above derivative is equal to zero when evaluated at $q_i = q_j = q$. Thus if there is a pure strategy equilibrium in which both agents contribute with quality q, it must be the case that

$$\left[q^3 + 2q(1-q)^2 - 2q^2(1-q) - (1-q)^3\right]A + \left[2(1-q)^3 + 4q^2(1-q)\right]A = c'(q).$$

Simplifying the left-hand side of this expression then gives $[q^3 + 2q(1-q)^2 + 2q^2(1-q) + (1-q)^3]A = c'(q)$, which in turn simplifies to $[q^2 - q + 1]A = c'(q)$. From this it follows that in equilibrium, the agents contribute with quality q satisfying $\frac{c'(q)}{q^2-q+1} = A$. □

Proof of Theorem 5.2. If $\lim_{A\to\infty}\frac{K(A)}{A} = 0$, then we know from Theorem 3.3 that there exists some sequence of values of $T(A)$ with $\lim_{A\to\infty} T(A) = \infty$ such that $\beta_\tau(A) = 1$ for all types τ for sufficiently large A in the rank-order mechanism. We also know from Theorem 4.3 that $\beta_\tau(A) = 1$ for sufficiently large A in the proportional mechanism. From this it follows that if $\lim_{A\to\infty}\frac{K(A)}{A} = 0$, then there exists some sequence of values of $T(A)$ with $\lim_{A\to\infty} T(A) = \infty$ such that $\beta_\tau(A) = 1$ for all types τ for sufficiently large A in both the rank-order and proportional mechanisms.

If $\lim_{A\to\infty}\frac{K(A)}{A} \neq 0$, then we know from Theorem 3.4 that $\lim_{A\to\infty}\beta_\tau(A) = 0$ for all types τ in the rank-order mechanism. We also know from Theorem 4.3 that if $q_{\tau,p}(A)$ denotes the quality that an agent of type τ chooses in the proportional mechanism in equilibrium for a given A, then $\limsup_{A\to\infty} q_{\tau,p}(A) < 1$, so $\limsup_{A\to\infty} c_\tau'(q_{\tau,p}(A)) < \infty$, meaning $c_\tau'(q_{\tau,p}(A)) = \Theta(1)$. We further know from Lemma 4.1 that $c_\tau'(q_{\tau,p}(A)) = \Theta(\frac{A}{\sum_{\tau=1}^t \pi_\tau q_{\tau,p}(A)\beta_\tau(A)K(A)})$ for any type τ for which $\beta_\tau(A) > 0$ for sufficiently large A. From this it follows that $\Theta(\frac{A}{\sum_{\tau=1}^t \pi_\tau q_{\tau,p}(A)\beta_\tau(A)K(A)}) = \Theta(1)$.

But this last result means that the expected number of participants in the game, $\sum_{\tau=1}^t \pi_\tau\beta_\tau(A)K(A)$, will satisfy $\Theta(\sum_{\tau=1}^t \pi_\tau\beta_\tau(A)K(A)) = \Theta(A)$ for the proportional mechanism. And we have seen that $\lim_{A\to\infty}\beta_\tau(A) = 0$ for all types τ in the rank-order mechanism, meaning the expected number of participants in the rank-order mechanism is $o(A)$. From this it follows that the expected number of participants in the rank-order mechanism becomes arbitrarily small compared to the expected number of participants in the proportional mechanism in the limit as $A \to \infty$. □

Proof of Theorem 5.3. To prove this it suffices to show that for any fixed strategies of the other players, a player's expected payoff from choosing a given action becomes independent of T for large T, in both the proportional and the rank-order mechanism—if all expected payoffs are independent of T, best responses and equilibria must be independent of T as well. First note that this holds whenever a player chooses not to participate because a player's payoff from not participating is always zero regardless of the value of T in both mechanisms.

Now note that this holds when a player i participates and chooses quality q_i in the proportional mechanism. The cost of choosing quality q_i is $c_\tau(q_i)$ for a participant of type τ regardless of the value of T. The amount of attention this participant receives when the other participating agents participate with qualities q_{-i} is $\frac{m_i A}{m_i+\sum_{j\neq i} m_j}$ where m_j denotes the number of positive votes received by agent j. For any fixed strategies of the other agents, this in turn converges in probability to $\frac{q_i A}{q_i+\sum_{j\neq i} q_j}$, where the sum is over all agents $j \neq i$ who participate, in the limit as $T \to \infty$. Thus for any fixed strategies of the other players, a player's expected payoff from participating with quality q_i becomes independent of T for large T in the proportional mechanism.

Finally note that this holds when a player i participates and chooses quality q_i in the rank-order mechanism. As before, the cost of choosing quality q_i is $c_\tau(q_i)$ for a participant of type τ regardless of the value of T. The number of agents who receive a higher rank than player i converges in probability to the number of agents who participate and choose a higher quality than q_i, and the number of agents who receive a lower rank than player i converges in probability to the number of agents who participate and choose a lower quality than q_i. Since the number of other agents who choose to participate and the number of these who receive a higher rank than agent i uniquely determine the expected amount of attention that agent i will receive, it then follows that for any fixed strategies of the other players, a player's expected payoff from participating with quality q_i becomes independent of T for large T in the rank-order mechanism as well.

These results indicate that for both the rank-order mechanism and the proportional mechanism, a player's expected payoff from any given action becomes independent of the value of T for large T for fixed strategies of the other players. The result then follows. □

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