The Markov chain on a network simply uses edge-weights as transition rates:

$$x \to y \text{ rate } w_{xy}$$

and has uniform stationary distribution; indeed this is the general form of a reversible chain with uniform stationary distribution.

A "compactification" result conjectured by me and proved in a weak form by Henry Towsner (Limits of sequences of Markov chains, *Electron. J. Probab.* 2015).

Theorem

An arbitrary sequence of networks with $n \to \infty$ has a subsequence in which (after time-scaling) the Markov chain either

- *has the L^2 cutoff property*
- *or converges (in a certain subtle sense) to a limit Markov process of the form described below.*

What is the form of the limit process?

Important note: this here is purely Measure Theoretic – no topology. So we can take state space as $([0,1], \mathcal{B}, \mathrm{Leb})$. Consider measurable functions $p^\infty(x,y,t)$ for $x, y \in [0,1]$ and $t > 0$ such that

- $p^\infty(x,y,t) \equiv p^\infty(y,x,t)$.
- $y \to p^\infty(x,y,t)$ is a probability density function.
- $p^\infty(x,z,t+s) = \int p^\infty(x,y,t) p^\infty(y,z,s) dy$. (Chapman-Kolmogorov)
- some $t \downarrow 0$ pinning.

This specifies the finite-dimensional distributions of a symmetric Markov process on $[0,1]$ started at x.

The proof uses the 1980s Hoover-Aldous-Kallenberg work on structure of random arrays with an exchangeability property

$$(Z_{ij}) =_d (Z_{\pi(i)\pi(j)}) \text{ for all permutations } \pi.$$

For the chain on a finite network define

$$p^n(x,y,t) = n\ P(X_t = y | X_0 = x).$$

Take V_i IID uniform on the n states and define

$$Z^n_{ij} \text{ is the random function } t \to p^n(V_i, V_j, t) \tag{7}$$

Towsner's theorem: for arbitrary sequence of networks there is a subsequence with either the L^2 cut-off property or with $(Z^n_{ij}) \stackrel{d}{\to} (Z^\infty_{ij})$, the limit defined as at (7) with IID uniform[0,1] (V_i).

Not easy to see what this means . . . . . .

but the point is that the sampled transition densities (Z^n_{ij}) in the finite case identify the Markov chain "up to relabeling of states", that is up to a bijection $\phi : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

So the limit (Z^∞_{ij}) identifies a process on states $[0, 1]$ "up to measure-preserving transformation of $[0, 1]$".

As mentioned before, this is all measure-theoretic, and (in the spirit of the famous quote *History doesn't repeat itself but it often rhymes*) Towsner repeats Hoover in giving a "logic" proof.

It would be nice to have a "standard" proof.

Conjecture There is some natural way to define a topology, for instance

$$d(x_1, x_2) := \sqrt{\int \int (p^\infty(x_1, y, t) - p^\infty(x_2, y, t))^2 e^{-t} dydt}$$

which makes $[0, 1]$ into a complete separable metric space and makes the Markov process have the Feller property.

This in turn could used to define a distance function of the vertices of approximating networks.