



An $O(n^2)$ Bound for the Relaxation Time of a Markov Chain on Cladograms

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ABSTRACT: A cladogram is an unrooted tree with labeled leaves and unlabeled internal branchpoints of degree 3. Aldous has studied a Markov chain on the set of n -leaf cladograms in which each transition consists of removing a random leaf and its incident edge from the tree and then reattaching the leaf to a random edge of the remaining tree. Using coupling methods, Aldous showed that the relaxation time (i.e., the inverse of the spectral gap) for this chain is $O(n^3)$. Here, we use a method based on distinguished paths to prove an $O(n^2)$ bound for the relaxation time, establishing a conjecture of Aldous. © 2001 John Wiley & Sons, Inc. *Random Struct. Alg.*, 20, 59–70, 2001

Key Words: mixing times; cladograms; Markov chains; distinguished paths; Poincaré inequalities; continuum random tree

1. INTRODUCTION

An n -leaf cladogram is an unrooted tree with n leaves and internal branchpoints of degree 3. The leaves are labeled, usually by the integers $1, \dots, n$, but the internal branchpoints are unlabeled. We denote the set of n -leaf cladograms by T_n . Each n -leaf cladogram contains $2n - 3$ edges. The cardinality of T_n , which we denote by

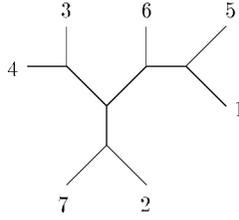


Fig. 1. A 7-leaf cladogram.

c_n , is given by

$$c_n = (2n - 5)(2n - 7) \cdots (3)(1) = \frac{(2n!)}{(2n - 1)(2n - 3)2^n n!}. \quad (1)$$

Figure 1 shows an example of a 7-leaf cladogram.

As mentioned in the introduction to [2], n -leaf cladograms are used in biology to represent the evolutionary relationship among n species. Because Markov chain Monte Carlo methods have been used to reconstruct cladograms from DNA data, it is of interest to study Markov chains on T_n . Aldous [2] introduced a simple Markov chain on cladograms. He proved that the relaxation time (i.e., the inverse of the spectral gap) is $O(n^3)$ and conjectured that it is $O(n^2)$. The purpose of this article is to prove Aldous's conjecture. Here we recall for reference Aldous's definitions and some basic properties of the chain.

We first define the following two operations on cladograms:

- (1) To *remove a leaf i* means to remove the edge from the leaf i to its branchpoint b , and then remove the branchpoint b to collapse the two edges into one (see Figure 2).
- (2) To *add a leaf i to the edge g* means to create a new branchpoint b in the interior of g and then add a new edge connecting b to the leaf i (see Figure 3).

We now define a Markov chain $(X_t)_{t=0}^\infty$ with state space T_n . To define this chain, we construct X_{t+1} from X_t by removing a leaf chosen uniformly at random from X_t and then adding this leaf to an edge chosen uniformly at random from the remaining tree. For trees x and y in T_n , we write $y \sim x$ if $y \neq x$ and y can be obtained from x by removing one leaf of x and then adding it to an edge of the remaining tree. Because there are n leaves that can be removed and $2n - 5$ edges to which the leaf can be reattached, the transition probabilities for this chain are given by

$$P(X_{t+1} = y | X_t = x) = \begin{cases} 1/n(2n - 5) & \text{if } y \sim x, \\ 1/(2n - 5) & \text{if } y = x. \end{cases} \quad (2)$$



Fig. 2. Removing the leaf i .



Fig. 3. Adding the leaf i to the edge g .

It follows that this chain has a symmetric transition matrix and therefore is reversible with a uniform stationary distribution.

The chain $(X_t)_{t=0}^\infty$ is irreducible and aperiodic. Therefore, the distribution of X_t converges to the uniform stationary distribution as $t \rightarrow \infty$. For designing Markov chain Monte Carlo algorithms, it is useful to know the time it takes for the chain to converge to the stationary distribution. There are different ways of formalizing this idea of “mixing time.” See [11] for a detailed comparison of several mixing-time parameters. In this article, we will consider the *relaxation time* τ_n for the Markov chain on T_n described above. The relaxation time is defined to be the inverse of the spectral gap. That is, $\tau_n = 1/(1 - \lambda_{n,2})$, where $\lambda_{n,2}$ is the second-largest eigenvalue of the transition matrix for the chain.

A general discussion of the relaxation time for reversible Markov chains and methods for bounding it is given in chapter 4 of [4]. In [2], Aldous used a coupling argument to show that for the Markov chain on n -leaf cladograms defined above, we have $\tau_n = O(n^3)$. In lectures given in a graduate course at U.C. Berkeley in the fall of 1999, Aldous outlined an alternative approach to bounding τ_n using a “distinguished paths” method. This approach did not yield a bound of $O(n^2)$, but we show in this article that a variation of the distinguished paths method does give an $O(n^2)$ bound for τ_n . We state this result below.

Theorem 1. *Let τ_n be the relaxation time of the Markov chain $(X_t)_{t=0}^\infty$ on n -leaf cladograms defined above. Then $\tau_n = O(n^2)$.*

Theorem 1.1 of [2] shows that $\tau_n = \Omega(n^2)$, so it follows from Theorem 1 that $\tau_n = \Theta(n^2)$.

In Section 2, we introduce the modification of the “distinguished paths” method that we will use to prove Theorem 1. We prove Theorem 1 in Section 3 and give some brief concluding remarks in Section 4.

2. A MODIFIED DISTINGUISHED PATHS METHOD

Let $(Z_t)_{t=0}^\infty$ be an irreducible, aperiodic, reversible Markov chain with finite state space V and stationary distribution π . Define the transition probabilities by $p_{xy} = P(Z_{t+1} = y | Z_t = x)$. Let G be a graph with vertex set V and edge set E such that G has a directed edge from x to y if and only if $p_{xy} > 0$. Let $Q(x, y) = \pi_x p_{xy}$, and if e is an edge from x to y , let $Q(e) = Q(x, y)$. Let τ denote the relaxation time of the chain $(Z_t)_{t=0}^\infty$.

A well known method for upper-bounding τ is the method of distinguished paths. The idea is to choose paths connecting each pair of vertices in the graph G . If it

is possible to avoid bottlenecks, in which many paths go through the same edges of the graph, then one can obtain a good bound on the relaxation time. Jerrum and Sinclair [9] used distinguished paths to prove rapid mixing for a Markov chain that can be used to approximate the permanent of a matrix. Diaconis and Stroock [5] used distinguished paths to prove “Poincaré inequalities” and showed how to use these inequalities to get bounds on the relaxation time in various examples. Sinclair [15] and Kahale [10] obtained related inequalities. Fulman and Wilmer [8] compared the bounds from Poincaré inequalities to bounds derived from Cheeger’s inequality.

In this section, we present a modification of the Poincaré inequalities which is useful for studying the Markov chain described in the introduction. We first recall the following theorem, which is proved in chapter 4 of [4] and can be viewed as a probabilistic version of inequalities in [5] and [15].

Theorem 2. *Suppose, for all x and y in V , that γ_{xy} is a path in G , possibly random, from x to y such that no edge is traversed more than once. Then,*

$$\tau \leq \max_{e \in E} \frac{1}{Q(e)} \sum_{x \in V} \sum_{y \in V} \pi_x \pi_y E[|\gamma_{xy}| 1_{\{e \in \gamma_{xy}\}}],$$

where $|\gamma_{xy}|$ is the number of edges in the path γ_{xy} .

To analyze the Markov chain defined in the introduction, we will need the following corollary of Theorem 2, which can be useful if one can only easily define a short path γ_{xy} when y is in some subset B of V .

Corollary 3. *Let B be a subset of V . Suppose, for all $x \in V$ and $y \in B$, that γ_{xy} is a path in G , possibly random, from x to y which has at most L edges. Then,*

$$\tau \leq \frac{4L}{\pi(B)} \max_{e \in E} \frac{1}{Q(e)} \sum_{x \in V} \sum_{y \in B} \pi_x \pi_y P(e \in \gamma_{xy}). \quad (3)$$

Proof. For all x and z in V , we define a random path $\tilde{\gamma}_{xz}$ from x to z as follows. First, we choose a random vertex Y in B such that $P(Y = y) = \pi_y / \pi(B)$. Next, we concatenate the paths γ_{xY} and γ'_{zY} , where γ'_{zY} is the path from z to Y traversed in the opposite direction. Finally, we erase all loops, so that no edge of $\tilde{\gamma}_{xz}$ is traversed more than once. Note that the length of $\tilde{\gamma}_{xz}$ is at most $2L$. By Theorem 2,

$$\begin{aligned} \tau &\leq \max_{e \in E} \frac{1}{Q(e)} \sum_{x \in V} \sum_{z \in V} \pi_x \pi_z E[|\tilde{\gamma}_{xz}| 1_{\{e \in \tilde{\gamma}_{xz}\}}] \\ &\leq 2L \max_{e \in E} \frac{1}{Q(e)} \sum_{x \in V} \sum_{z \in V} \pi_x \pi_z P(e \in \tilde{\gamma}_{xz}). \end{aligned} \quad (4)$$

Denote by e^r the directed edge connecting the same two vertices as e but in the reverse direction. Then, we have

$$\begin{aligned} \tau &\leq 2L \max_{e \in E} \frac{1}{Q(e)} \sum_{x \in V} \sum_{z \in V} \pi_x \pi_z (P(e \in \gamma_{xY}) + P(e^r \in \gamma_{zY})) \\ &= 2L \max_{e \in E} \frac{1}{Q(e)} \left\{ \sum_{x \in V} \pi_x P(e \in \gamma_{xY}) \left(\sum_{z \in V} \pi_z \right) + \sum_{z \in V} \pi_z P(e^r \in \gamma_{zY}) \left(\sum_{x \in V} \pi_x \right) \right\} \\ &\leq 2L \max_{e \in E} \frac{1}{Q(e)} \sum_{x \in V} \pi_x P(e \in \gamma_{xY}) + 2L \max_{e \in E} \frac{1}{Q(e)} \sum_{z \in V} \pi_z P(e^r \in \gamma_{zY}). \end{aligned}$$

For all $e \in E$, we have $Q(e) = Q(e^r)$ because the chain is reversible. Therefore,

$$\begin{aligned} \tau &\leq 4L \max_{e \in E} \frac{1}{Q(e)} \sum_{x \in V} \pi_x P(e \in \gamma_{xY}) \\ &= 4L \max_{e \in E} \frac{1}{Q(e)} \sum_{x \in V} \pi_x \left(\sum_{y \in B} \frac{\pi_y}{\pi(B)} P(e \in \gamma_{xy}) \right) \\ &= \frac{4L}{\pi(B)} \max_{e \in E} \frac{1}{Q(e)} \sum_{x \in V} \sum_{y \in B} \pi_x \pi_y (P(e \in \gamma_{xy})), \end{aligned}$$

as claimed. ■

3. PROOF OF THE $O(n^2)$ BOUND

In this section, we use Corollary 3 to prove Theorem 1. Our strategy is as follows. We will construct a path between each pair of cladograms. To get a path from x to y , we will remove and reattach the leaves one at a time in such a way that at each stage, all the leaves that have already been moved are in the same relative position as they are in the tree y . We will show that the diameter (i.e., maximum distance between two leaves) of most cladograms is $O(n^{1/2})$, and we will define the set B to consist of all cladograms whose diameter is at most $A_1 n^{1/2}$, where A_1 is a constant. We will then show that the right-hand side of (3) is $O(n^2)$, using the fact that the paths end at a tree whose diameter is not too large to show that no “edge” between two cladograms appears in too many paths.

Given a subset U of $\{1, \dots, n\}$ and a cladogram $x \in T_n$, define the U -tree derived from x to be the tree obtained by removing from x all leaves whose labels are not in U . Note that we obtain the same tree regardless of the order in which the leaves are removed.

Lemma 4. *If U is a k -element subset of $\{1, \dots, n\}$ and x is a uniform random n -leaf cladogram, then the U -tree derived from x is a uniform random k -leaf cladogram.*

Proof. The natural inductive proof of formula (1) for the cardinality of T_n establishes that a uniform random cladogram can be built by inductively adding leaves to uniform random edges. In fact, as can be easily checked, this construction works for

any ordering of the leaves. More precisely, suppose σ is a permutation of $\{1, \dots, n\}$. Let x_3 be the unique 3-leaf cladogram with leaves labeled $\sigma(1)$, $\sigma(2)$, and $\sigma(3)$. For $j = 4, \dots, n$, define x_j by adding a leaf labeled $\sigma(j)$ to an edge chosen uniformly at random from the edges of x_{j-1} . Then each x_j is a uniform j -leaf cladogram with leaves labeled $\sigma(1), \dots, \sigma(j)$.

Now, choose σ such that $U = \{\sigma(1), \dots, \sigma(k)\}$. Then x_k is the U -tree derived from x_n . Because x_k is a uniform random k -leaf cladogram and x_n is a uniform random n -leaf cladogram, we see that the U -tree derived from a uniform random n -leaf cladogram is a uniform random k -leaf cladogram, as claimed. ■

If $x \in T_n$, then any two leaves in x are connected by a unique path that does not intersect itself. Let $d(x)$ denote the *diameter* of x , which is the maximum number of edges in the path between two leaves of x . Lemma 5 below states that the diameter of a typical n -leaf cladogram is $O(n^{1/2})$.

Lemma 5. *There exists a constant $A_1 < \infty$, not depending on n , such that if π denotes the uniform distribution on T_n and $B = \{x \in T_n: d(x) \leq A_1 n^{1/2}\}$, then $\pi(B) \geq 1/2$.*

Proof. Let x_n be a uniform random n -leaf cladogram. Following [2], we define a tree y_n by assigning random lengths to the $2n - 3$ edges of x_n such that the joint density of the edge lengths is given by

$$g(l_1, \dots, l_{2n-3}) = se^{-s^2/2}, \quad \text{where } s = \sum_{i=1}^{2n-3} l_i.$$

Note that the edge lengths are exchangeable and are independent of the shape of the tree. In the proof of Lemma 21 of [1], Aldous constructs a tree whose shape and edge lengths have the same distribution as y_n from the times $C_1 < C_2 < \dots$ of a nonhomogeneous Poisson process on $(0, \infty)$ with rate $r(t) = t$. It follows from this construction that the sum of the lengths of the edges of y_n has the same distribution as C_{n-1} . This fact and the exchangeability of the edge lengths imply that there exists $A_2 > 0$ such that the expected length of any edge of y_n is at least $A_2 n^{-1/2}$ for all n .

Let v and w be leaves of x_n such that the number of edges on the path from v to w equals the diameter $d(x_n)$. Then the expected lengths, conditional on x_n , of the path in y_n from v to w is at least $A_2 n^{-1/2} d(x_n)$. Therefore, the expected length of the longest path between two leaves of y_n is at least $A_2 n^{-1/2} E[d(x_n)]$.

Next, let $(\mathcal{T}_\infty, \mu)$ be the Brownian continuum random tree, as defined in Section 4.3 of [1]. We may assume that \mathcal{T}_∞ is constructed from the continuous function $f(t) = 2B_t$ as described in Theorem 13 of [1], where $(B_t)_{0 \leq t \leq 1}$ is a normalized Brownian excursion. Let \mathcal{T}_n be the reduced subtree of \mathcal{T}_∞ having $n - 1$ leaves, which are chosen at random from \mathcal{T}_∞ according to the probability measure μ . Regard the root of \mathcal{T}_n as a leaf, so \mathcal{T}_n becomes an unrooted n -leaf tree. By Theorem 3 and Corollary 22 of [1], \mathcal{T}_n has the same distribution as y_n . By Theorem 13 of [1], the length of any path between two vertices of \mathcal{T}_n is at most $2 \max_{t \in [0, 1]} f(t) = 4M_1$, where $M_1 = \max_{t \in [0, 1]} B_t$. Therefore, $E[d(x_n)] \leq 4A_2^{-1} E[M_1] n^{1/2}$. Because $E[M_1] < \infty$, as shown, for example, in [13],

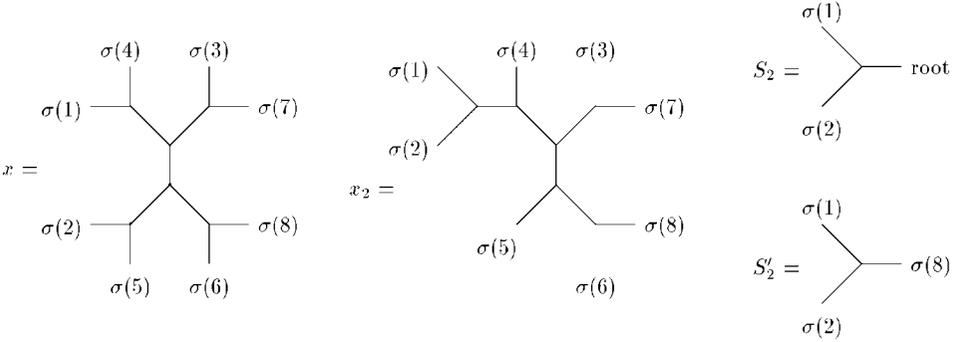


Fig. 4. An example of x , x_2 , S_2 , and S'_2 .

Markov’s inequality implies that the condition of Lemma 5 holds with $A_1 = 8A_2^{-1}E[M_1]$. ■

We now work towards constructing our distinguished paths. We use the notation from Section 2, so $V = T_n$ and G is the graph with vertex set V such that there is an edge in G from x to y if and only if $y \sim x$. The stationary distribution π is uniform on T_n .

For all $x, y \in V$, we will construct a random path γ_{xy} from x to y . First, let σ be a uniform random permutation of $\{1, \dots, n\}$. Let $x_1 = x$. Define x_2 to be the cladogram that we obtain by removing the leaf labeled $\sigma(2)$ from x_1 and then adding it to the edge incident to $\sigma(1)$. Let S_2 denote the rooted subtree of x_2 whose root is a branchpoint of x_2 and whose only leaves are $\sigma(1)$ and $\sigma(2)$. Let S'_2 be the unrooted tree obtained by regarding the root of S_2 as a leaf and labeling it $\sigma(n)$. Note that S'_2 is the $\{\sigma(1), \sigma(2), \sigma(n)\}$ -tree derived from y because there is only one 3-leaf cladogram. See Figure 4 for an example.

Next, suppose for some $k \in \{2, 3, \dots, n - 2\}$, we have defined cladograms x_2, \dots, x_k , and x_k has a rooted subtree S_k with leaves $\sigma(1), \dots, \sigma(k)$ such that if we define S'_k by regarding the root of S_k as a leaf and labeling it $\sigma(n)$, then S'_k is the $\{\sigma(1), \dots, \sigma(k), \sigma(n)\}$ -tree derived from y . Then, we can define x_{k+1} by removing the leaf $\sigma(k + 1)$ from x_k and adding it to the unique edge of S_k such that, after the leaf is added, the new subtree S_{k+1} has the property that when the root is labeled $\sigma(n)$ and regarded as a leaf, the new unrooted tree S'_{k+1} is the $\{\sigma(1), \dots, \sigma(k + 1), \sigma(n)\}$ -tree defined from y . See Figure 5, which shows the trees x_3 and x_4 defined when constructing the path from x to y , where x is the same tree as in Figure 4. Note that in going from x_3 to x_4 , the leaf $\sigma(4)$ is removed and then added to the edge incident to $\sigma(2)$. Also, note that the $\{\sigma(1), \sigma(2), \sigma(3), \sigma(8)\}$ -tree derived from x_3 is the same as the $\{\sigma(1), \sigma(2), \sigma(3), \sigma(8)\}$ -tree derived from y , and the $\{\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(8)\}$ -tree derived from x_4 is the same as the $\{\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(8)\}$ -tree derived from y . When x_1, \dots, x_{n-1} are defined by this process, the tree S_{n-1} is just x_{n-1} with $\sigma(n)$ regarded as the root, so $x_{n-1} = S'_{n-1} = y$. Thus, we have inductively defined random cladograms x_1, \dots, x_{n-1} such that $x_1 = x, x_{n-1} = y$, and, for $1 \leq k \leq n - 2$, either $x_k = x_{k+1}$ or $x_{k+1} \sim x_k$. We now define γ_{xy} to be the path consisting of all edges e that connect x_k to x_{k+1} for some k . If e is the edge from x_k to x_{k+1} , then we say $e \in \gamma_{xy}$ at step k .

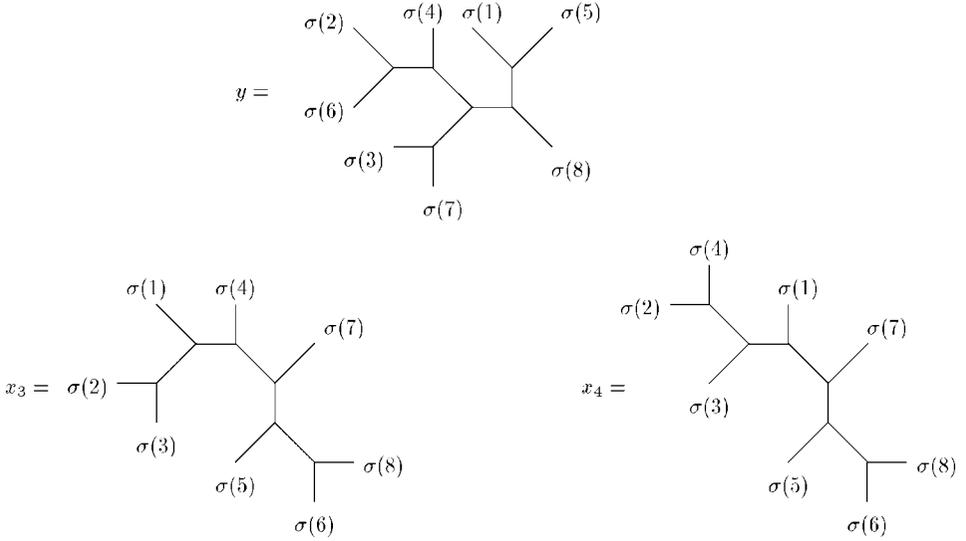


Fig. 5. The trees x_3 and x_4 .

Note that $|\gamma_{xy}| \leq n - 2$ for all $x, y \in V$. Therefore, Corollary 3 implies that

$$\tau_n \leq \frac{4(n-2)}{\pi(B)} \max_{e \in E} \frac{1}{Q(e)} \sum_{x \in V} \sum_{y \in B} \pi_x \pi_y P(e \in \gamma_{xy}),$$

where $B = \{x \in T_n : d(x) \leq A_1 n^{1/2}\}$ as defined in Lemma 5. By Lemma 5, $\pi(B) \geq 1/2$. Because π is uniform on V , we have $\pi_x = c_n^{-1}$ for all $x \in V$. By (2), we have $p_{xy} = 1/n(2n - 5)$ whenever $y \sim x$. Thus, for all edges e of G , we have $Q(e) = \pi_x p_{xy} = 1/c_n n(2n - 5)$. It follows that

$$\begin{aligned} \tau_n &\leq 8(n-2) \max_{e \in E} c_n n(2n-5) \sum_{x \in V} \sum_{y \in B} \pi_x \pi_y P(e \in \gamma_{xy}) \\ &\leq 16n^3 c_n \max_{e \in E} \sum_{x \in V} \sum_{y \in B} \pi_x \pi_y P(e \in \gamma_{xy}). \end{aligned}$$

For all $e \in E$, define $K(e) = \{k : P(e \in \gamma_{xy} \text{ at step } k) > 0 \text{ for some } x \in V, y \in B\}$. Now, fix an edge e . Let v and w be the cladograms such that e is the edge from v to w , and let i and g be the leaf and edge respectively in v such that w is obtained from v by removing the leaf i and then adding it to the edge g . Let p be the unique

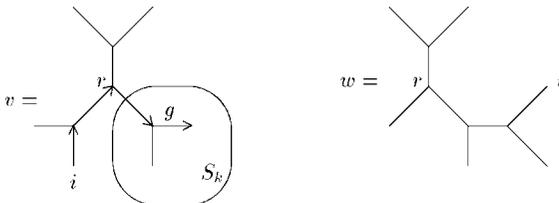


Fig. 6. Arrows mark the path p ; the subtree S_k is circled.

path in v that begins by traversing the edge incident to i , ends by traversing the edge g , and does not intersect itself. Suppose $e \in \gamma_{xy}$ at step k . Then $v = x_k$ and $w = x_{k+1}$. Also, v has a rooted subtree S_k , as defined above in the description of the construction of γ_{xy} , which contains k leaves and the edge g . Note that the root r of S_k must be one of the branchpoints on the path p (see Fig. 6).

If $y \in B$, then y has diameter at most $A_1 n^{1/2}$. Because S'_k is the $\{\sigma(1), \dots, \sigma(k), \sigma(n)\}$ -tree derived from y and S'_k contains g , the portion of the path p that starts at r and ends after traversing g has length at most $A_1 n^{1/2}$. Therefore, there are at most $A_1 n^{1/2}$ branchpoints on p that could be the root r of S_k . There are three subtrees of v rooted at each branchpoint, but only one of them contains the edge g . Therefore, there are at most $A_1 n^{1/2}$ subtrees of v which could be the subtree S_k . Because S_k has k leaves, it follows that there are at most $A_1 n^{1/2}$ possible values for k , so $K(e)$ contains at most $A_1 n^{1/2}$ integers. Thus,

$$\begin{aligned} \tau_n &\leq 16n^3 c_n \max_{e \in E} \sum_{x \in V} \sum_{y \in B} \pi_x \pi_y \left(\sum_{k \in K(e)} P(e \in \gamma_{xy} \text{ at step } k) \right) \\ &\leq 16A_1 n^{7/2} c_n \max_{e \in E} \max_{k \in K(e)} \sum_{x \in V} \sum_{y \in V} \pi_x \pi_y P(e \in \gamma_{xy} \text{ at step } k). \end{aligned} \tag{5}$$

Note that

$$\sum_{x \in V} \sum_{y \in V} \pi_x \pi_y P(e \in \gamma_{xy} \text{ at step } k) \tag{6}$$

is the probability that $e \in \gamma_{xy}$ at step k when x and y are chosen uniformly at random from T_n . We now assume x and y are independent uniform random n -leaf cladograms and bound the expression in (6) for fixed $e \in E$ and $k \in K(e)$. Define v , w , i , p , and g as in the previous paragraph. We have seen that, because $k \in K(e)$, the tree v must have a rooted subtree S_k with k leaves which contains the edge g and has its root on the path p . Also, for any fixed k , v has only one such subtree. Now let σ be a uniform random permutation of $\{1, \dots, n\}$, and construct a random path γ_{xy} from x , y , and σ as described earlier. We have $e \in \gamma_{xy}$ at step k if and only if $x_k = v$ and $x_{k+1} = w$, where x_k and x_{k+1} are as defined in the construction of γ_{xy} . We claim that if $x_k = v$ and $x_{k+1} = w$, then the following three events must occur:

- (a) The leaves of S_k are $\sigma(1), \dots, \sigma(k)$, and $\sigma(k+1) = i$.
- (b) The $\{\sigma(1), \sigma(k+1), \dots, \sigma(n)\}$ -tree derived from x is the same as the $\{\sigma(1), \sigma(k+1), \dots, \sigma(n)\}$ -tree derived from v .
- (c) The $\{\sigma(1), \dots, \sigma(k+1), \sigma(n)\}$ -tree derived from y is the same as the $\{\sigma(1), \dots, \sigma(k+1), \sigma(n)\}$ -tree derived from w .

Event (a) must occur because, to obtain x_{k+1} from x_k , the leaf $\sigma(k+1)$ is removed and then added to an edge in a subtree of x_k whose k leaves are $\sigma(1), \dots, \sigma(k)$. Event (b) must occur because none of the leaves $\sigma(1), \sigma(k+1), \dots, \sigma(n)$ is moved in the first $k-1$ steps in the construction of γ_{xy} , so the $\{\sigma(1), \sigma(k+1), \dots, \sigma(n)\}$ -tree derived from x is the $\{\sigma(1), \sigma(k+1), \dots, \sigma(n)\}$ -tree derived from x_k . Also, event (c) must occur because the leaves $\sigma(1), \dots, \sigma(k+1), \sigma(n)$ are not moved again after the k th step in the construction, so the $\{\sigma(1), \dots, \sigma(k+1), \sigma(n)\}$ -tree

derived from y is the same as the $\{\sigma(1), \dots, \sigma(k+1), \sigma(n)\}$ -tree derived from x_{k+1} .

Because σ is a uniform random permutation of $\{1, \dots, n\}$, we have that $\sigma(1), \dots, \sigma(k)$ are the leaves of S_k with probability $1/\binom{n}{k}$, and conditional on the event that $\sigma(1), \dots, \sigma(k)$ are the leaves of S_k , we have $\sigma(k+1) = i$ with probability $1/(n-k)$. By Lemma 4, the $\{\sigma(1), \sigma(k+1), \dots, \sigma(n)\}$ -tree derived from x is a uniform random element of T_{n-k+1} . Therefore, the probability of event (b) is $1/c_{n-k+1}$. Likewise, the $\{\sigma(1), \dots, \sigma(k+1), \sigma(n)\}$ -tree derived from y is a uniform random element of T_{k+2} , so the probability of event (c) is $1/c_{k+2}$. Because σ, x , and y are independent, so are the events (a), (b), and (c). Therefore, if $k \in K(e)$, then

$$\begin{aligned} & \sum_{x \in V} \sum_{y \in V} \pi_x \pi_y P(e \in \gamma_{xy} \text{ at step } k) \\ & \leq \frac{1}{\binom{n}{k} (n-k) c_{k+2} c_{n-k+1}} \\ & = \frac{1}{\binom{n}{k} (n-k) (2k-1) (2k-3) (2(n-k)-3) c_k c_{n-k}} \end{aligned} \quad (7)$$

because $c_{k+2} = (2k-1)(2k-3)c_k$ and $c_{n-k+1} = (2(n-k)-3)c_{n-k}$ by (1). Combining (5) and (7), we get

$$\tau_n \leq 16A_1 n^{7/2} \max_{e \in E} \max_{k \in K(e)} \frac{c_n}{\binom{n}{k} (n-k) (2k-1) (2k-3) (2(n-k)-3) c_k c_{n-k}}. \quad (8)$$

By Stirling's formula, we have

$$\frac{(2n)!}{n!} \approx \frac{(2n)^{2n+1/2} e^{-2n}}{n^{n+1/2} e^{-n}} \approx 2^{2n} n^n e^{-n},$$

where \approx means that the ratio of two sides is bounded away from zero and infinity as n varies. Therefore, using (1), we have

$$c_n = \frac{(2n)!}{(2n-1)(2n-3)2^n n!} \approx \frac{2^{2n} n^n e^{-n}}{(2n-1)(2n-3)2^n} \approx 2^n n^{n-2} e^{-n}.$$

Also, we have

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} \approx \frac{n^{n+1/2} e^{-n}}{(k^{k+1/2} e^{-k})((n-k)^{n-k+1/2} e^{-(n-k)})} \\ &= \frac{n^{n+1/2}}{k^{k+1/2} (n-k)^{n-k+1/2}}, \end{aligned}$$

where here \approx means that the ratio of the two sides is bounded away from zero and infinity as n and k vary. Thus, there exists a constant $A_3 > 0$ such that for all $n \in \mathbb{N}$

and $k \in \{1, \dots, n - 1\}$, we have

$$\begin{aligned} & \frac{c_n}{\binom{n}{k}(n-k)(2k-1)(2k-3)(2(n-k)-3)c_k c_{n-k}} \\ & \leq \frac{A_3}{k^2(n-k)^2} \frac{k^{k+1/2}(n-k)^{n-k+1/2}}{n^{n+1/2}} \frac{2^n n^{n-2} e^{-n}}{(2^k k^{k-2} e^{-k})(2^{n-k}(n-k)^{n-k-2} e^{-(n-k)})} \\ & = \frac{A_3}{k^2(n-k)^2} \left(\frac{k(n-k)}{n} \right)^{5/2} = A_3 n^{-5/2} (k(n-k))^{1/2} \leq A_3 n^{-3/2} \end{aligned} \tag{9}$$

for all e and all $k \in K(e)$. Thus, from (8) and (9), we have $\tau_n \leq 16A_1 A_3 n^2$. Hence, $\tau_n = O(n^2)$, as claimed.

4. CONCLUDING REMARKS

Although it would be easy to sample directly from the set of n -leaf cladograms (see the proof of Lemma 4), Markov chains on other combinatorial sets have been studied for the purpose of obtaining fast algorithms for generating approximately uniform samples from these sets. For example, Randall and Tetali [14] studied Markov chains on lozenge tilings, domino tilings, triangulations of convex polygons, and binary trees. A chain on $m \times n$ contingency tables was studied in [7], and chains on independent sets of a graph were studied in [6, 12]. However, the upper bounds obtained for the mixing times of these chains are not known to be sharp. The chain in this article provides an example of a chain on a combinatorial set for which the relaxation time is known up to a constant factor.

Regarding the techniques used in this article, the advantage that we get from using Corollary 3, rather than one of the standard Poincaré inequalities, is that we only need to bound the bottlenecking for the paths that end in the set B . Consequently, we do not need to consider the paths ending in trees which are unusually long and thin. Although the author is not currently aware of other useful applications of Corollary 3, it seems likely that there would be other Markov chains for which one could obtain a better bound by avoiding having to consider paths which end in a “bad” part of the state space.

The Markov chain in this article appears to be the easiest Markov chain on cladograms to analyze. A natural next step might be to consider other chains on n -leaf cladograms. For example, Aldous [3] describes a chain in which each transition consists of picking a random edge (not necessarily adjacent to a leaf), removing the entire branch of the chain below that edge, and then reattaching the branch to one of the edges adjacent to where it was before. Aldous poses the problem of proving that this chain has a relaxation time of $O(n^{3/2})$. He also mentions that this chain is believed to be related to a chain on triangulations of the regular n -gon.

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