Childrens’ game *Up The River*. Each player has three boats; throw a non-standard die, choose one boat to advance. At end of round, every boat is moved backwards one step; boats at last step are lost.
If we chose a random boat each turn, then the numbers on the die make each of 3 boats do roughly a mean-zero random walk.

Suggests (to me) the following math problem. Let $K$ particles perform independent Brownian motion on $[0, \infty)$, started at 1, killed on reaching 0. Then suppose we have drift rate $+1$, to be distributed amongst surviving particles according to some strategy we choose.

Write $N = N^K = \text{number of particles which survive forever}$. Ask

- what strategy maximizes $EN^K$?
- what is maximized value of $EN^K$?
Cute exercise for first course in BM or SDEs: show

\[
\max_{\text{strategy}} \mathbb{E} N^{(K)} \in [c_1 K^{1/2}, c_2 K^{1/2}] \text{ as } K \to \infty.
\]

**Lower bound.** Consider the following strategy. Give a particle zero drift until it hits \( K^{1/2} \): then give it drift \( 1/K^{1/2} \).

**Upper bound.** Arbitrary strategy; want to upper bound mean number of particles surviving until time \( K \). Write \( X_i(t) \) for position of particle \( i \) and consider

\[
Y(t) := \sum_i h_{K-t}(X_i(t))
\]

for suitable \( h_\tau(x) \). After some thought choose

\[
h_\tau(x) = P(\inf_{0 \leq t \leq \tau} B_t > 0|B_0 = x)
\]

because if no drift then \( Y(t) \) is the martingale \( E(\text{number particles survive to } K|X_i(t)), 0 < t < K. \)
\[ Y(t) := \sum_i h_{K-t}(X_i(t)) \]

Now if \( A_i \) is drift rate assigned to particle \( i \), can calculate

\[ dY_t = dM_t + \sum_i A_i(t) g_{K-t}(X_i(t)) \]

where \( M_t \) is a martingale and \( g_s(x) \leq cs^{-1/2} \).

Integrating over \( 0 \leq t \leq K \) gives

\[ EY_K \leq EY_0 + cK^{1/2}. \]

But \( EY_0 = cK^{1/2} \) and

\[ Y_K = \text{number of particles survive until } K. \]
For rest of talk we analyze the strategy

assign drift 1 to the lowest particle.

We believe
- this is optimal strategy
- with this strategy $E_N^{(K)} \sim 5\pi^{-1/2}K^{1/2}$.

Will give back-of-envelope calculations for the latter.

[Joint work with Saul Jacka?]
**Brownian scaling.** For BM $B(t)$ on $(-\infty, \infty)$, we know we can “rescale time by $K$ and rescale space by $K^{1/2}$”, that is

- replace $B(t)$ by $K^{-1/2}B(tK)$

and we get back BM.

Consider our setting but without any drift. That is, start $K$ independent Brownian particles at 1, and kill a particle when it hits 0. Assign “mass” $1/K^{1/2}$ to each particle. Then under Brownian scaling, there is a $K \to \infty$ limit “mass density” $\tilde{f}(t,x)$ at time $t$. There are several ways to think about/calculate $\tilde{f}$; in particular it solves the heat equation

$$\frac{df}{dt} = \frac{1}{2} \frac{d^2 f}{dx^2}$$

with boundary condition

$$f(t,0) = 0.$$

Now consider our strategy **assign drift 1 to the lowest particle.** Presumably, under the same limit procedure we get a (different) limit mass density $f$, satisfying the heat equation but with some different boundary condition.
Picture suggests: position $L(t)$ of lowest particle is (after rescaling) approx. deterministic $\ell(t)$ of form

$$\ell(t) = \begin{cases} 0, & t \leq t_0 \\ > 0, & t > t_0. \end{cases}$$

Remarkably, this is enough to calculate

- $t_0 = 2$
- $EN(K) \sim 5\pi^{-1/2}K^{1/2}$. 
Let’s change the initial conditions: instead of $K$ particles at 1, suppose initially we have a Poisson point process of particles on $(0, \infty)$, with some rate $\mu > 0$. Allow particles to move freely on $(-\infty, \infty)$ without drift.

**Question.** At time $t$, if I want to move particles to restore original Poisson distribution, how much “work” (= sum of distances moved) is needed?

**Answer** is easy using reflection principle and fact that Poisson scatter is stationary for reflecting process: $\mu t/2$.

Now assign drift 1 to lowest particle. We see heuristically

$\mu = 2$ is **critical** value: at that value, adding the drift mimics a reflecting boundary at 0.
For general \( \mu \) the position \( L(t) \) of lowest particle will be to first order a deterministic \( \ell(t) \) and the spatial density of particles near the boundary will always be \( \approx 2 \).

Simple instance of **self-organized criticality**. Same argument shows heuristically: for \( \mu < 2 \)

\[
\ell(t) = a_\mu t^{1/2}
\]

\[
a_\mu = (1 - \frac{\mu}{2})\sqrt{2/\mu}.
\]

So absorbing boundary at 0 doesn’t matter in subcritical setting.