Childrens' game Up The River. Each player has three boats; throw a non-standard die, choose one boat to advance. At end of round, every boat is moved backwards one step; boats at last step are lost.

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If we chose a random boat each turn, then the numbers on the die make each of 3 boats do roughly a mean-zero random walk.

Suggests (to me) the following math problem. Let K particles perform independent Brownian motion on $[0, \infty)$, started at 1, killed on reaching 0. Then suppose we have drift rate $+1$, to be distributed amongst surviving particles according to some strategy we choose.

Write $N = N^{(K)} =$ number of particles which survive forever. Ask

- what strategy maximizes $EN^{(K)}$?
- what is maximized value of $EN^{(K)}$?

Cute exercise for first course in BM or SDEs: show

$$\max_{strategy} EN^{(K)} \in [c_1 K^{1/2}, c_2 K^{1/2}] \text{ as } K \to \infty.$$

Lower bound. Consider the following strategy. Give a particle zero drift until it hits $K^{1/2}$: then give it drift $1/K^{1/2}$.

Upper bound. Arbitrary strategy; want to upper bound mean number of particles surviving until time K. Write $X_i(t)$ for position of particle i and consider

$$Y(t) := \sum_i h_{K-t}(X_i(t))$$

for suitable $h_\tau(x)$. After some thought choose

$$h_\tau(x) = P(\inf_{0 \le t \le \tau} B_t > 0 | B_0 = x)$$

because if no drift then $Y(t)$ is the martingale

$E($ number particles survive to $K | X_i(t)), 0 < t < K.$

$$Y(t) := \sum_i h_{K-t}(X_i(t))$$

Now if A_i is drift rate assigned to particle i, can calculate

$$dY_t = dM_t + \sum_i A_i(t) g_{K-t}(X_i(t))$$

where M_t is a martingale and $g_s(x) \leq cs^{-1/2}$. Integrating over $0 \leq t \leq K$ gives

$$EY_K \leq EY_0 + cK^{1/2}.$$

But $EY_0 = cK^{1/2}$ and

$Y_K =$ number of particles survive until K.

For rest of talk we analyze the strategy

assign drift 1 **to the lowest particle**.

We believe

- this is optimal strategy
- with this strategy $EN^{(K)} \sim 5\pi^{-1/2}K^{1/2}$.

Will give back-of-envelope calculations for the latter.

[Joint work with Saul Jacka?]

Brownian scaling. For BM $B(t)$ on $(-\infty, \infty)$, we know we can "rescale time by K and rescale space by $K^{1/2}$", that is
• replace $B(t)$ by $K^{-1/2}B(tK)$
and we get back BM.

Consider our setting but without any drift. That is, start K independent Brownian particles at 1, and kill a particle when it hits 0. Assign "mass" $1/K^{1/2}$ to each particle. Then under Brownian scaling, there is a $K \to \infty$ limit "mass density" $\tilde{f}(t, x)$ at time t. There are several ways to think about/calculate $\tilde{f}$; in particular it solves the **heat equation**

$$df/dt = \tfrac{1}{2} d^2 f/dx^2$$

with boundary condition

$$f(t, 0) = 0.$$

Now consider our strategy **assign drift** 1 **to the lowest particle**. Presumably, under the same limit procedure we get a (different) limit mass density f, satisfying the heat equation but with some different boundary condition.

Picture suggests: position $L(t)$ of lowest particle is (after rescaling) approx. determinisic $\ell(t)$ of form

$$\begin{aligned} \ell(t) &= 0, \ t \leq t_0 \\ &> 0, \ t > t_0. \end{aligned}$$

Remarkably, this is enough to calculate

- $t_0 = 2$
- $EN^{(K)} \sim 5\pi^{-1/2} K^{1/2}$.

Let's change the initial conditions: instead of K particles at 1, suppose initially we have a Poisson point process of particles on $(0, \infty)$, with some rate $\mu > 0$. Allow particles to move freely on $(-\infty, \infty)$ without drift.

Question. At time t, if I want to move particles to restore original Poisson distribution, how much "work" (= sum of distances moved) is needed?

Answer is easy using reflection principle and fact that Poisson scatter is stationary for <u>reflecting</u> process: $\mu t/2$.

Now assign drift 1 to lowest particle. We see heuristically

$\mu = 2$ is **critical** value: at that value, adding the drift mimics a reflecting boundary at 0.

For general μ the position $L(t)$ of lowest particle will be to first order a deterministic $\ell(t)$ and the spatial density of particles near the boundary will always be ≈ 2.

Simple instance of **self-organized criticality**. Same argument shows heuristically: for $\mu < 2$

$$\ell(t) = a_\mu t^{1/2}$$

$$a_\mu = (1 - \tfrac{\mu}{2})\sqrt{2/\mu}.$$

So absorbing boundary at 0 doesn't matter in subcritical setting.