Patience Sorting, Longest Increasing Subsequences and a Continuous Space Analog of the Simple Asymmetric Exclusion Process

David Aldous *
Department of Statistics
University of California
Berkeley CA 94720

Persi Diaconis
Department of Mathematics
Harvard University
Cambridge MA 02138

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Abstract

There is a broad circle of ideas relating to the length L_n of the longest increasing subsequence of a random n-permutation. After reviewing known results and methods, we develop two new themes. The simplest algorithm for computing the length of the longest increasing subsequence can be viewed as a card gard, *patience sorting*, and one theme is to give the first asymptotic probabilistic analysis of this game. The second theme is that a continuous limit process, which we call *Hammersley's process*, exists as a continuous-space interacting particle process. This turns out to be analogous to the (discrete-space) simple asymmetric exclusion process, and also to be an elaboration of Hammersley's Poisson process representation. The celebrated result $EL_n \sim 2n^{1/2}$ is intimately tied to the hydrodynamical limit theorem for Hammersley's process.

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1 Introduction

Section 1 reviews known material, and our new results are outlined in section 1.7.

1.1 Patience Sorting

A concrete and elementary starting point is provided by a solitaire card game which we call *patience sorting*. To describe the game, consider a deck of cards labeled $1, 2, 3, \ldots, n$. The deck is shuffled, cards are turned up one at a time and dealt into piles on the table. A low card can be placed on a higher card (e.g. a 2 may be placed on a 7). When the turned up card is higher than the cards showing, it is put into a new pile to the right of the others. The object is to finish with as few piles as possible.

Suppose a shuffled deck of 10 cards is in the order

$$7 \quad 2 \quad 8 \quad 1 \quad 3 \quad 4 \quad 10 \quad 6 \quad 9 \quad 5$$

The top card, 7, is dealt face up on the table. The next card, 2, can be put on top of the 7. But the 8 has to form a new pile, giving

2
7 8

The 1 can be placed on either the 2 or the 8. If it were placed on the 8, and if any of $\{3, 4, 5, 6, 7\}$ were to come next, the next card would have to form a new pile, whereas if the 1 were placed on the 2, the 8 is still open for these cards. The *greedy* strategy is to always place a card on the leftmost possible pile. Adopting the greedy strategy (which we do unless otherwise stated) leads to

1	**1**	**1**	**1**	**1**	**1**	**1** **5**
2	2 **3**	2 **3**	2 **3**	2 **3** **6**	2 **3** **6**	2 **3** 6
7 **8**	7 8	7 8 **4**	7 8 **4** **10**	7 8 **4** 10	7 8 **4** 10 **9**	7 8 **4** 10 **9**

At each step, the top card in each pile is in boldface. It is easy to see inductively that, under the greedy strategy, the top cards are increasing from left to right. It is also easy to check the following *compatibility property* of the greedy algorithm. The final allocation of cards labeled $1, \ldots, j$ to piles is the same as if we deleted the higher-numbered cards from the deck before starting the game. So in our example, taking $j = 6$, starting with cards in the order

$$2 \quad 1 \quad 3 \quad 4 \quad 6 \quad 5$$

leads to

1			5
2	3	4	6

Motivation for patience sorting is provided by its connection with the *longest increasing subsequence*. The permutation we used above

$$7 \quad 2 \quad 8 \quad \underline{1} \quad \underline{3} \quad \underline{4} \quad 10 \quad \underline{6} \quad \underline{9} \quad 5$$

has an increasing subsequence

$$1\ 3\ 4\ 6\ 9 \tag{1}$$

of length 5, and that is the longest possible for the permutation. We saw that patience sorting played with this permutation ended with 5 piles. This is no coincidence. Define $l(\pi)$ the be the length of the longest increasing subsequence of a permutation π.

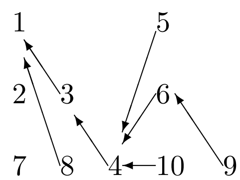
Lemma 1 *With deck π, patience sorting played with the greedy strategy ends with exactly $l(\pi)$ piles, and played with any legal strategy ends with at least $l(\pi)$ piles.*

Proof. If cards $a_1 < a_2 < \ldots < a_l$ appear in increasing order, then under any legal strategy each a_i must be placed in some pile to the right of the pile containing a_{i-1}, because the card number on top of that pile can only decrease. Thus the final number of piles is at least l, and hence at least $l(\pi)$. Conversely, using the greedy strategy, when a card c is placed in a pile other than the first pile, put a pointer from that card to the currently top card $c' < c$ in the pile to the left. At the end of the game, let a_l be the card on top of the rightmost pile l. The sequence

$$a_1 \leftarrow a_2 \leftarrow \ldots \leftarrow a_{l-1} \leftarrow a_l$$

obtained by following the pointers is an increasing subsequence whose length is the number of piles.

In the example, we recover the subsequence (1).



Note that the top cards are in general not an increasing subsequence: in the example, 5 came after 9 in the deck.

1.2 Remarks on Lemma 1

These remarks are not used in the sequel.

(a) *History.* The name *patience sorting* comes from Mallows [**?**], who in [**?**] credits A.S.C. Ross for its invention with the following motivation. At the end of the game, card 1 is always on top of the leftmost pile. Removing card 1, card 2 is now at the top of some pile; removing card 2, card 3 is now at the top of some pile, and so we have a natural algorithm for manually sorting cards. Mallows' analysis [**?**] was done in 1960 but not published until much later (see section 1.5 for more remarks). Independently, patience sorting was discovered by computer scientist Bob Floyd in 1964 and developed briefly in letters between Floyd and Knuth, but their work apparently has never been published. And Hammersley [**?**] p. 362 independently recognized its use an an algorithm for computing $l(\pi)$.

(b) *Algorithms.* Implicit in the proof of Lemma 1 is an algorithm for computing $l(\pi)$ and exhibiting a maximal-length increasing subsequence. Fredman [**?**] has shown this algorithm can be implemented using $n \log n - n \log\log n + O(n)$ comparisons in the worst case, and that no algorithm has better worst-case behavior.

One can easily extend to an algorithm for exhibiting *every* maximal-length increasing subsequence. When a card c is placed in a pile other than the first pile, put a pointer from that card to *every* card $c' < c$ in the pile to the left. At the end of the game, let a_l be any card in the rightmost pile l. A sequence

$$a_1 \leftarrow a_2 \leftarrow \ldots \leftarrow a_{l-1} \leftarrow a_l$$

obtained by following the pointers making arbitrary choices is an increasing subsequence with length $l(\pi)$, and every such subsequence arises from some choice of a_l and pointers. In our example, the only new pointer added is from 3 to 2, and we get only one other maximal-length increasing subsequence 2 3 4 6 9.

Persi ? I just made up the above. I assume its true and known! Does the Schensted correspondence give all maximal subsequences?

The Schensted correspondence described in section 1.5 was partly motivated as an algorithm for computing $l(\pi)$, though patience sorting leads to a slightly simpler algorithm.

(c) *Duality.* The piles in patience sorting provide a decomposition of our permutation into a disjoint union of *decreasing* subsequences: the record values in the first pile, the second record values in the second pile, and so on. In our example we get

$$(7, 2, 1)\ (8, 3)\ (4)\ (10, 6, 5)\ (9)$$

This is closely related to theorems of Dilworth [**?**]. One such theorem says that in any partially ordered set, the length of the longest chain equals the size of the minimal partition into antichains. A permutation π determines a partial order on $\{1, \ldots, n\}$ by $i \ll j$ iff $i \leq j$ and $\pi(i) \leq \pi(j)$. Here a maximal chain is a longest increasing subsequence and an antichain is a decreasing subsequence. Lovasz and Plummer [**?**] Theorem 1.4.12 give Dilworth's result and relate it to a host of classical combinatorial topics (the matching theorem, max-flow min-cut, theorems of Konig and Frobenius). Greene [**?**] has given a far-reaching extension of Dilworth's result, relating the largest cardinality of the union of k chains to the minimum decomposition of suitably weighted antichains. When specialized to permutations, Greene's result implies that the largest union of k increasing subsequences in π equals the number of cells in the first k rows of the partition associated to π in the Schensted correspondence (section 1.5). We have not found a sharp connection between patience sorting and these results.

(d) *Ulam's metric.* For a permutation π, the minimum number of insertion-deletion steps needed to go from the identity permutation to π is $n - l(\pi)$. This connection motivates use of

$$d(\pi, \sigma) \equiv n - l(\sigma\pi^{-1})$$

as a metric on permutations [**?**]. Diaconis [**?**] Chapter 6B and Critchlow [**?**] discuss statistical uses of various metrics on permutations.

Persi ? I omitted your "generating function factorization" remark. Should I put it back?

1.3 Distributional results for L_n

Playing patience sorting with a randomly-shuffled n-card deck, we finish with L_n, say, piles. By Lemma 1 L_n is the length of the longest increasing subsequence of a random permutation π_n. With this latter interpretation, distributional aspects of L_n have been studied from several different viewpoints.

(a) *Subadditive ergodic theory.* There is a representation (reviewed in section 1.4) of L_n in terms of random points in the plane. With this representation, one can apply the subadditive ergodic theorem to prove

$$n^{-1/2} L_n \stackrel{p}{\to} c$$

for some unspecified c.

(b) *Young tableaux .* The Schensted correspondence , reviewed in section 1.5, relates longest increasing subsequences to Young tableaux . Independently, Vershik and Kerov [**?**] and Logan and Shepp [**?**] indicated how to use the Schensted correspondence and analysis of random Young tableaux to show $c = 2$. These arguments use classical hard analysis far removed from probability theory – e.g. Logan and Shepp solve an infinite-dimensional optimization problem in a cleverly chosen function space. Part of our purpose is to give an alternate proof that $c = 2$, so we shall not appeal to that result.

Persi ? Any work on the distribution of the number of maximal-length increasing subsequences? Does this follow from our stuff?

Persi ? Average-case number of comparisons for Floyd's algorithm for computing L_n?

(c) *Concentration inequalities.* There has been recent work using martingale concentration inequalities to bound the spread of L_n. The latest result, due to Talagrand (personal communication) and improving on results of Frieze [**?**] and Bollobas xxx, is

Theorem 2 *Let m_n be the median of L_n. Then for $u > 0$ and all $n \geq 1$,*

$$P(L_n \geq m_n + u) \leq 2 \exp\left(-\frac{u^2}{4(m_n + u)}\right)$$

$$P(L_n \leq m_n + u) \leq 2 \exp\left(-\frac{u^2}{4m_n}\right)$$

(d) *Open Problems.* For theoretical reasons we would expect one of the following alternatives to be true. Either

$$\frac{L_n - EL_n}{\sqrt{\text{var } L_n}} \stackrel{d}{\to} \text{Normal}(0,1) \text{ and var } L_n = n^{1/2} s(n) \qquad (2)$$

where $s(\cdot)$ is slowly varying; or

$$\frac{L_n - EL_n}{\sqrt{\text{var } L_n}} \stackrel{d}{\to} \xi \text{ and var } L_n = o(n^{1/2} s(n)) \qquad (3)$$

for all slowly varying $s(\cdot)$, where ξ is not Normal. Simulation evidence is ambiguous: with $n = 1,000,000$ our simulations give $EL_n \approx 1982$, var $L_n \approx 110$ and a good fit to Normal.

As for large deviations, a subadditivity argument can be used to show the existence of the limit

$$n^{-1/2} \log P(L_n > c' n^{1/2}) \to \alpha(c') \in (-\infty, 0], \quad c' > c.$$

Then Theorem 2 implies $\alpha(c') < 0$ for $c' > c$. We do not know any plausible conjecture for the actual value of $\alpha(c')$. It is interesting that the large deviation behavior has a different order of magnitude in the left tail.

Proposition 3 $n^{-1/2} P(L_n < c' n^{1/2}) \to -\infty, \ c' < c.$

This is proved in section 5 by soft arguments. We do not know what the correct order of magnitude is.

Persi ? Does the Young tableaux method say anything about large deviations?

1.4 Hammersley's representation and subadditivity

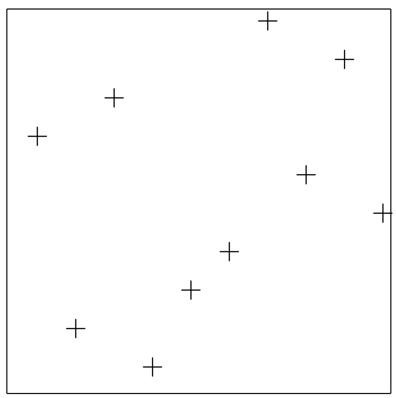
The ideas here are due to Hammersley [**?**], and were one of the motivations for the development of subadditive ergodic theory. They are now a textbook application of that theory (see [**?**] sec 6.7).

Consider n points (x_i, y_i) in the rectangle $[0, z] \times [0, t]$ with all coordinates distinct. The set of points specifies a permutation π by:

> the point with i'th smallest x-coordinate has the $\pi(i)$'th smallest y-coordinate.

The length $l(\pi)$ of the longest increasing subsequence is the maximal number of points on an up-right path from $(0,0)$ to (z,t), i.e. the maximal length l of a sequence (i_j) such that

$$x_{i_1} < x_{i_2} < \ldots < x_{i_l}, \quad y_{i_1} < y_{i_2} < \ldots < y_{i_l}.$$



Now take a Poisson process $\mathcal{N}$ of rate 1 in $[0,\infty)^2$ and let $\mathbf{L}^{\nearrow}_{z,t}$ be the maximal number of points on an up-right path from $(0,0)$ to (z,t). The number of points in the rectangle $[0,z] \times [0,t]$, say $M(z,t)$, has Poisson(zt) distribution, and the associated random permutation of $M(z,t)$ is uniform. Thus

$$\mathbf{L}^{\nearrow}_{z,t} \stackrel{d}{=} L_{M(z,t)}. \tag{4}$$

Define

$$g(t) = E\mathbf{L}^{\nearrow}_{t,t}. \tag{5}$$

By considering paths from $(0,0)$ to $(t+s,t+s)$ via (t,t) we see that g is *superadditive*:

$$g(t+s) \geq g(t) + g(s); \;\; s,t \geq 0. \tag{6}$$

This implies that, defining $c = \limsup g(t)/t$, we have

$$g(z)/z \to c \tag{7}$$

$$g(z) \leq cz; \;\; z \geq 0. \tag{8}$$

Moreover ([**?**] sec. 6.7) the subadditive ergodic theorem can be applied to $\mathbf{L}^{\nearrow}_{t,t}$ to show

$$t^{-1}\mathbf{L}^{\nearrow}_{t,t} \to c \text{ a.s.} \tag{9}$$

and simple estimates show $1.59 < c < 2.49$.

Now (4) says that $\mathbf{L}^{\nearrow}_{t,t}$ is almost the same as L_{t^2}, and elementary dePoissonization arguments show that (7,9) imply

$$n^{-1/2} E L_n \to c, \quad n^{-1/2} L_n \stackrel{p}{\to} c.$$

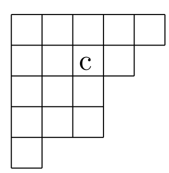
xxx details later

Hammersley [**?**] discusses three heuristic methods of determining c. Our work in section 3 is essentially formalizing his "third method".

1.5 Young tableaux and the Schensted correspondence

For the benefit of any probabilists unfamiliar with the topic, we state some standard facts taken from the undergraduate textbook Stanton and White [**?**], sec. 3.5-3.7. More sophisticated treatments, emphasizing representation theory for the symmetric group, can be found in Sagan [**?**], MacDonald [**?**] and James and Kerber [**?**].

A *partition* $\lambda = (\lambda_1, \ldots, \lambda_j)$ of an integer $n \geq 1$ is a sequence with $\lambda_1 \geq \lambda_2 \geq \ldots \geq 1$ and $\sum_i \lambda_i = n$. We may identify a partition λ with its associated *Ferrers diagram* of n cells with λ_i cells in row i, drawn as shown below. The *hook length* h_c of a cell c in a Ferrers diagram is the number of cells in the set consisting of the cells to the right of c in its row, the cells below c in its column, and cell c itself.



$\lambda = (5, 4, 3, 3, 1)$

$h_c = 4$

A (standard) *Young tableau* of shape λ is a Ferrers diagram with the cells occupied by the numbers $1, 2, \ldots, n$ in such a way that the numbers increase along each row and down each column. Examples are in the picture below. The number of Young tableaux of a given shape λ is denoted by d_λ and is given by

Lemma 4 (The hook formula)

$$d_\lambda = \frac{n!}{\prod_c h_c}.$$

The *Schensted* (or Robinson-Schensted-Knuth) *correspondence* is a bijection between permutations of n cards and ordered pairs of Young tableaux of size n with the same (unspecified) shape λ. The existence of such a bijection implies the formula

$$n! = \sum_{\lambda \vdash n} d_\lambda^2$$

where $\lambda \vdash n$ is an abbreviation for "λ is a partition of n".

The two tableaux are called the P-tableau and the Q-tableau. As with patience sorting, the correspondence can be constructed inductively by specifying where to place the next card.

> Place the next card in the top row of the P-tableau, in the position of the smallest higher-numbered card if any exists (thereby "bumping" that card), or append to the right end of the top row if no such higher-numbered card exists. If a card is bumped, consider the P-tableau with top row deleted, and recursively use the same rule to insert the bumped card into the remaining tableau. This eventually yields a new P-tableau whose shape is the previous tableau with one extra cell. Make the new Q-tableau be the previous Q-tableau with this extra cell and with the number m in that cell, where the added card is the m'th card dealt.

The figure illustrates the construction on our example

$$7 \;\; 2 \;\; 8 \;\; 1 \;\; 3 \;\; 4 \;\; 10 \;\; 6 \;\; 9 \;\; 5$$

	2	2 8	1 8	1 3	1 3 4	1 3 4 10	1 3 4 6	1 3 4 6 9	1 3 4 5 9
P	7	7	2	2 8	2 8	2 8	2 8 10	2 8 10	2 6 10
			7	7	7	7	7	7	7 8
	1	1 3	1 3	1 3	1 3 6	1 3 6 7	1 3 6 7	1 3 6 7 9	1 3 6 7 9
Q	2	2	2	2 5	2 5	2 5	2 5 8	2 5 8	2 5 8
			4	4	4	4	4	4	410

The top row in the P-tableau is the same as the top cards in patience sorting. So the length of the top row in the Young tableaux associated with a permutation π is the length $l(\pi)$ of the longest increasing subsequence of π. So we can write

$$P(L_n = l) = \frac{1}{n!} \sum_{\substack{\lambda \vdash n \\ \lambda_1 = l}} d_\lambda^2. \qquad (10)$$

The essence of the proofs [**?**, **?**] that $c = 2$ is to let $\Lambda^{(n)} = (\Lambda_i^{(n)}; i \geq 1)$ be a random partition of n with distribution

$$p(\lambda) \equiv d_\lambda^2/n!$$

rescale to define

$$Z_n(t) = n^{-1/2}\Lambda^{(n)}_{\lfloor n^{1/2}t \rfloor}; \quad t \geq 0$$

and then to show that Z_n converges to a certain deterministic function $z(t)$ with $z(0) = 2$.

Returning to patience sorting, Mallows [**?**] records how he discovered that the number of piles relates to Young tableaux . D.E. Barton [**?**] pointed out the connection with the Schensted correspondence . Using these, Mallows proved that the chance that patience sorting yields l piles and also yields r piles when played with the deck in reversed order equals the sum (10) further restricted to tableaux λ with r rows.

1.6 Nonasymptotic upper bound for longest increasing subsequence

The known proofs that $c \geq 2$ all use some notion of "continuous limit object", and there is no known *explicit* function $f(n)$ such that

$$EL_n \geq f(n) \text{ and } f(n) \sim 2n^{1/2}.$$

It seems to have been folklore for some time that one could use the Schensted correspondence to give purely discrete proofs that $c \leq 2$; one such proof has been given by Pilpel [**?**]. By the elementary Lemma 7 below, $c \leq 2$ implies $EL_n \leq 2n^{1/2} + O(1)$. But in fact one can use the Schensted correspondence to get the natural nonasymptotic bound $EL_n \leq 2n^{1/2}$. The simple proof we give is due to Kerov and Vershik – we learned it from unpublished work of Curtis Green and from lectures by Larry Shepp – but apparently has not been published.

Theorem 5 *$EL_n \leq 2n^{1/2}$ for all $n \geq 1$.*

The precise asymptotic behavior of EL_n is unclear, but as a corollary of our continuous-world results we shall see in section 3.7 that $2n^{1/2} - EL_n \to \infty$.

xxx conjecture the difference grows as order $\log n$.

Proof. The proof uses only the Q-tableau of the Schensted correspondence . Writing Q_n for the Q-tableau derived from a uniform random permutation π_n, the correspondence implies

$$P(Q_n = q) = \frac{d_{\Lambda(q)}}{n!} \tag{11}$$

where $\Lambda(q)$ denotes the shape of q. First note that deleting n from a tableau q on n cells gives a legal tableau $r_{n-1}(q)$, say. And this is the tableau obtained when the bottom card is deleted from the deck. (So we are using the natural process of uniform random permutations associated with card-dealing – see section 6.1.) Thus we have the natural consistency condition

$$P(r_{n-1}(Q_n) = q_0) = P(Q_{n-1} = q_0). \tag{12}$$

Let A_n denote the event that n is in the top row (and hence at the right end of the top row) of Q_n. For a tableau q_0 of size $n-1$ let q_0^+ denote the tableau of size n obtained by adding n to the right end of the top row. Since q_0^+ is the unique q with $r_{n-1}(q) = q_0$ and with n in the top row,

$$\begin{aligned} P(A_n | r_{n-1}(Q_n) = q_0) &= P(Q_n = q_0^+ | r_{n-1}(Q_n) = q_0) \\ &= \frac{P(Q_n = q_0^+)}{P(r_{n-1}(Q_n) = q_0)} \\ &= \frac{P(Q_n = q_0^+)}{P(Q_{n-1} = q_0)} \text{ by (12)} \\ &= \frac{1}{n} \frac{d_{\Lambda(q_0^+)}}{d_{\Lambda(q_0)}} \text{ by (11)} \end{aligned}$$

Note the right side depends only on the shape of q_0. Writing $\Lambda_{n-1}(Q_n)$ for the shape of $r_{n-1}(Q_n)$, and λ^+ for the shape obtained from λ by adding an extra cell at the right end of the top row,

$$\begin{aligned} P(A_n) &= \sum_{\lambda \vdash n-1} P(A_n | \Lambda_{n-1}(Q_n) = \lambda) P(\Lambda_{n-1}(Q_n) = \lambda) \\ &= \sum_{\lambda \vdash n-1} \frac{1}{n} \frac{d_{\lambda^+}}{d_\lambda} P(\Lambda_{n-1}(Q_n) = \lambda) \\ &\leq \left(\sum_{\lambda \vdash n-1} \frac{1}{n^2} \left(\frac{d_{\lambda^+}}{d_\lambda} \right)^2 P(\Lambda_{n-1}(Q_n) = \lambda) \right)^{1/2} \end{aligned} \tag{13}$$

But the Schensted correspondence says

$$P(\Lambda(Q_n) = \lambda) = d_\lambda^2/n!,\ \ \lambda \vdash n \tag{14}$$

and, appealing also to (12),

$$P(\Lambda_{n-1}(Q_n) = \lambda) = \frac{d_\lambda^2}{(n-1)!},\ \ \lambda \vdash n-1 \tag{15}$$

Substituting (15) into (13) gives the first inequality below.

$$\begin{aligned} P(A_n) &\leq n^{-1/2} \sum_{\lambda \vdash n-1} \frac{d_{\lambda^+}^2}{n!} \\ &\leq n^{-1/2} \sum_{\lambda \vdash n} \frac{d_\lambda^2}{n!} \\ &= n^{-1/2} \text{ using (14)} \end{aligned}$$

Interpreting L_n as the length of the top row of Q_n, the consistency condition (12) implies

$$P(A_n) = EL_n - EL_{n-1}$$

and so

$$EL_n \leq \sum_{m=1}^{n} m^{-1/2} \leq 2n^{1/2}.$$

1.7 What's new in this paper?

We have two main goals. The first is to say what we can about patience sorting in the random setting. This occupies section 2.

xxx state results; lean on Schensted correspondence and on continuous world

Our second goal is more diffuse. There is a well-established topic in theoretical probability called "interacting particle systems" [**?**] which studies various different models of continuous-time evolution of discrete-space configurations of particles. In section 3 we introduce and study a continuous-space interacting particle process ("Hammersley's process") which is analogous to a certain well-studied discrete-space process, the simple asymmetric exclusion process. This is interesting in its own right, because it seems widely believed (by default, at least) that the familiar interacting particle processes do not have natural continuous-space analogs. Hammersley's process is most naturally motivated in terms of a limit of patience sorting, but in retrospect it can also be regarded as an elaboration of Hammersley's representation. The main result, Theorem 30, is the hydrodynamical limit for Hammersley's process on R^+. Theorem 30 implies, and we regard as "the correct explanation of", the fact $c = 2$. We would like to have a self-contained probabilistic proof of Theorem 30 (and hence of $c = 2$), but have not yet found such a proof.

xxx but do have proof of $c \geq 2$.

xxx say Hammersley understood this outline

xxx other stuff

1.8 Elementary facts about EL_n

For comparison with later results and for technical use, we record some simple extensions of the subadditive result $g(t)/t \to c$ at (7). Clearly EL_n is increasing: the following simple estimate shows it cannot increase locally too fast.

Lemma 6 *For $m < n$,*

$$EL_m \geq \frac{m}{n} EL_n.$$

Proof. Given a random permutation π_n of n cards, construct a random permutation π_m of m cards by deleting $n - m$ cards chosen at random. Let L_n and L_m be the longest increasing subsequences. It is enough to show

$$E(L_m | L_n = l) \geq \frac{m}{n} l.$$

But this is clear because, given an increasing subsequence of l cards in π_n, the mean number of these cards which remain after the random deletions equals $\frac{m}{n} l$.

Note this argument uses a random permutation process which is different from the usual one – see section 6.1.

Write M_t for a Poisson(t) r.v., independent of random permutations.

Lemma 7 *$|EL_{M_n} - EL_n|$ is bounded as $n \to \infty$. So in particular, $EL_n \sim cn^{1/2}$ and $EL_n \leq cn^{1/2} + O(1)$.*

Proof.

$$\begin{aligned}
|EL_{M_n} - EL_n| &\leq \sum_j |EL_j - EL_n| P(M_n = j) \\
&\leq \sum_{j \geq n} \frac{j-n}{j} EL_j P(M_n = j) + \sum_{j<n} \frac{n-j}{n} EL_n P(M_n = j) \text{ by Lemma 6} \\
&\leq \sum_j \frac{|j-n|}{n} EL_n P(M_n = j) \text{ by Lemma 6 again} \\
&= n^{-1} EL_n E|M_n - n| \\
&\leq n^{-1/2} EL_n \text{ because } \mathrm{var} M_n = n.
\end{aligned}$$

By (7) $EL_{M_n} \leq cn^{1/2}$ and so

$$EL_n \leq (1 - n^{-1/2})^{-1} cn^{1/2} = cn^{1/2} + O(1)$$

and the Lemma follows.

Remark. Consider the chance p_n that the card in position n goes into a new pile in patience sorting (equivalently, goes into the first row in the Schensted correspondence). Clearly $p_n = EL_n - EL_{n-1}$.

Conjecture 8 $p_n = EL_n - EL_{n-1}$ *is decreasing.*

This seems intuitively obvious. We will occasionally point out consequences of Conjecture 8, for instance

$$p_n \sim \frac{1}{2} cn^{-1/2} \tag{16}$$

$$EL_{M_n} - EL_n \to 0 \tag{17}$$

Persi ? can you prove any of this?

2 Pile sizes in patience sorting

2.1 The first pile

The cards which enter the first pile are exactly the *record* cards, i.e. those whose number is smaller than any previous card. Thus in our example

$$\underline{7}\ \ \underline{2}\ \ 8\ \ \underline{1}\ \ 3\ \ 4\ \ 10\ \ 6\ \ 9\ \ 5$$

the records are $7, 2, 1$. Records form a classical topic in probability theory. Writing A_i for the event "the i'th card is a record", it is elementary that $P(A_i) = 1/i$ and that the events $(A_1, \ldots, A_n)$ are independent. Thus $S_n(1)$, the total number of records in a random permutation of n cards (and the number of cards in pile 1 in patience sorting) satisfies

$$ES_n(1) = \sum_{i=1}^{n} \frac{1}{i} \sim \log n \tag{18}$$

$$\text{var } S_n(1) = \sum_{i=1}^{n} \frac{1}{i}\left(1 - \frac{1}{i}\right) \sim \log n$$

And a textbook application (e.g. [**?**] p. 102) of the central limit theorem for independent non-identically-distributed random variables shows

$$\frac{S_n(1) - \log n}{\sqrt{\log n}} \stackrel{d}{\to} \text{Normal}(0, 1). \tag{19}$$

Persi ? I took out your elementary discussion about records. Do you want it put back?

We shall study the analogous questions for $S_n(i)$, the number of cards in pile i. In a certain sense the cards in pile i are "i'th records", but the sense is different from that used in the well-known and remarkable Ignatov's theorem [**?**].

Proposition 9

$$ES_n(i) = \sum_{k=1}^{n} P(L_{k-1} = i - 1) \sum_{j=k}^{n} \frac{1}{j}.$$

Interpreting L_0 as 0, we recover formula (18) for $ES_n(1)$.

Proof. The key fact is that

$$P(\text{card labeled } j \text{ in } j\text{-card deck put in pile } i)$$

$$= \sum_{k=1}^{j} \frac{1}{j} P(L_{k-1} = i-1). \tag{20}$$

Because for each $1 \leq k \leq j$ the card has chance $1/j$ to be in position k. Given it is in position k, it is put into pile i iff the $k-1$ previous cards formed exactly $i-1$ piles, which has chance $P(L_{k-1} = i-1)$ because the $k-1$ previous cards are in uniform random order.

Now the compatibility property of section 1.1 shows that (20) remains true for the card labeled j in a n-card deck, for any $n \geq j$. Summing over j gives the result.

2.2 Asymptotics for the leftmost piles

Corollary 10 *For fixed $i \geq 1$,*

$$ES_n(i) = c(i) \log n \ + O(1) \ \text{as } n \to \infty \tag{21}$$

where

$$c(i) \equiv \sum_{k=i}^{\infty} P(L_{k-1} = i-1) < \infty.$$

And $c(1) = 1, \quad c(2) = e-1,$

$$c(3) = \sum_{k=2}^{\infty} \frac{1}{k!} \left(\binom{2k}{k} \frac{1}{k+1} - 1 \right) \approx 2.3724$$

$$c(4) = \sum_{k=3}^{\infty} \frac{1}{k!} \left(2 \sum_{l=0}^{k} \binom{2l}{l} \binom{k}{l}^2 \frac{3l^2 + 2l + 1 - k - 2kl}{(l+1)^2(l+2)(k-l+1)} - \binom{2k}{k} \frac{1}{k+1} \right) \approx xxx.$$

Persi ? Numerical value for $c(3)$ used Mathematica and is different from yours. I copied your formula for $c(4)$ – please check it's right, then I'll do numerical computation. Also, where does formula for $c(4)$ come from?

Proof. Writing $\sum_{j=k}^{n} \frac{1}{j} = \log n - \log(k-1) + O(1)$ in the formula of Proposition 9, the only issue in deriving (21) is to prove

$$\sum_{k} P(L_k = i-1) \log k < \infty. \tag{22}$$

For an elementary proof, considering whether disjoint packets of i cards are in increasing order gives the bound

$$P(L_k = i-1) \leq \left(1 - \frac{1}{i!}\right)^{\lfloor k/i \rfloor}$$

which is enough to establish (22). One could also appeal to the sharp asymptotics, stated as (23) below.

Clearly $P(L_n = 1) = 1/n!$, giving

$$c(2) = \sum_{k=2}^{\infty} \frac{1}{(k-1)!} = e - 1.$$

And the expression for $c(3)$ follows from the formula

$$k!P(L_k = 2) = \binom{2k}{k} \frac{1}{k+1} - 1$$

which goes back to MacMahon [**?**] p. 130-131.

Persi ? story for $c(4)$?

Remarks. In fact sharp asymptotics are known:

$$P(L_k = i) \sim \frac{b(i) i^{2k}}{k^{i(i+1)/4} k!} \text{ as } k \to \infty \tag{23}$$

$$b(i) = \frac{2^{1/2}}{4\pi} \frac{i^{i/2}(i-1)}{i^{1/2}} \frac{2^{i^2/2}}{i!} \sum_{j=1}^{i} j!.$$

See Cohen and Negev [**?**, **?**] for the simplest proof. Goulden [**?**] gives formulas for $P(L_n = i)$ for large i.

Persi ? I omitted some refs, but could be put back.

Persi ? Your comment on "effective computability" doesn't seem quite right: you need an explicit bound in place of the asymptotics (23).

The expression for $c(i)$ in Corollary 10 has the following interpretation. Play patience sorting with an infinite number of cards (c.f. section 3.3), and at each step note the number of piles. Then $c(i)$ is the mean number of steps after which there are exactly $i-1$ piles. So trivially $c(i) \geq 1$. The fact $c = 2$ easily implies that $\sum_{i=1}^{m} c(i) \sim m^2/4$. Thus if we could prove

$$c(i) \text{ is increasing in } i \tag{24}$$

then it would immediately follow that

$$c(i) \sim i/2 \text{ as } i \to \infty.$$

Persi ? Can you prove (24) ?

In section 4 we will see a different expression (44) for $c(i)$ and will use some "continuous" machinery to prove a central limit theorem.

Theorem 11 *For fixed $i \geq 1$, there exists $0 < \sigma(i) < \infty$ such that* $\mathrm{var} S_n(i) \sim \sigma^2(i) \log n$ *and*

$$\frac{S_n(i) - c(i)\log n}{\sqrt{\log n}} \xrightarrow{d} \mathrm{Normal}(0, \sigma^2(i))$$

We already know (19) this holds for $i = 1$ with $\sigma(1) = 1$. Below we give a "discrete" computation for the number $\sigma(2)$ such that var $S_n(2) \sim \sigma^2(2) \log n$,

xxx we check this with a "continuous" computation in section 4.2.

But we do not have computable expressions for $\sigma^2(i)$ for $i \geq 3$.

Theorem 12 var $S_n(2) = \sigma^2(2) + O(1)$ *where* $\sigma^2(2) = (3 - e)(2e - 1) \approx 1.24986$.

This will be proved via a sequence of lemmas. To set up notation, write $S_n(2) = \sum_{i=2}^n Y_i$ where Y_i is the indicator of the event "cards labelled i ends up in pile 2 in patience sorting". We know from (20)

xxx explain better?

$$\mu_i \equiv EY_i = \frac{1}{i} \sum_{k=1}^{i-1} 1/k!$$

Writing

$$\mu_{ij} \equiv EY_iY_j = P(\text{cards labeled } i \text{ and } j \text{ end in pile 2})$$

the general formula for variance of a counting r.v. is

$$\mathrm{var}\ S_n(2) = \sum_{i=2}^{n} \mu_i(1 - \mu_i) + 2 \sum_{2 \leq i < j \leq n} (\mu_{ij} - \mu_i \mu_j). \tag{25}$$

Lemma 13

$$\mu_{ij} = \frac{1}{j(j-1)} \sum_{2 \leq a < b \leq n} \sum_{d \geq 1} \frac{\binom{i-1}{d} \binom{j-2-(i-1)}{b-d-2}}{\binom{j-2}{b-2} (b-2)!}$$

$$\sum_{l \geq 0} \binom{b-d-2}{a-l-1} \binom{b-d-2}{a-l-1} (b+l-(a+d)-1)!$$

Persi ? I've copied your stuff verbatim – you need to proofread!

Proof. In carrying out this computation cards with labels bigger than j do not enter so it may be assumed we are working with a j-card deck. Condition on cards j and i being in positions $a < b$ (which occurs with probability $\frac{1}{j(j-1)}$.

$$\begin{array}{ccccc} & j & & i & \\ ---\; & - & ----- & - & -- \\ & a & & b & \end{array}$$

The conditional probability these two cards end in pile 2 is computed in two stages. First, only the top b cards matter and two of these are fixed. The remaining $b-2$ can be chosen in $\binom{j-2}{b-2}$ ways. Of these, there must be at least d "low cards" with labels from $\{1, 2, \ldots, i-1\}$ and $b-2-d$ high cards, chance

$$\frac{\binom{i-1}{d}\binom{j-2-(i-1)}{b-d-2}}{\binom{j-2}{b-2}}.$$

Condition on such a set being chosen for the first $b-2$ open places. The second stage is to compute the number of ways of arranging such a set. Obviously they must fall as

$$\begin{array}{ccccc} \searrow & j & \searrow & i & \\ ---\; & - & ----- & - & -- \\ & a & & b & \end{array}$$

Thus, the $a-1$ cards to the left of position a must be decreasing. The d low cards to the left of position d must be decreasing. The arrangements are grouped by l, the number of low cards to the left of position a. The chance of a successful arrangement is

$$\frac{1}{(b-2)!}\sum_{l \geq 0}\binom{b-2-d}{a-1-l}\binom{b-a-1}{d-l}(b+l-(a+d)-1)!$$

The first binomial coefficient counts choices of high card values to the left of position a. The second coefficient counts places for low card values to the right of position a (all low cards to the left are forced to lie in order in the l places immediately to the left of position a). The final factorial counts arrangements of the high card values in the remaining positions to the right of position a. These choices specify the arrangement.

Remark. The formula of the Lemma has been checked in various ways. For example, with a 4-card deck the chance that the cards labelled $(2,3)$ or $(2,4)$ or $(3,4)$ end up in pile 2 is $1/6$ by direct enumeration and this checks with the Lemma. As a further check it is straightforward to show directly

$$\mu_{2j} = \frac{1}{j(j-1)(j-2)} \sum_{a=2}^{j-1} \left(\frac{j-a}{(a-2)!} + \frac{\binom{j-a}{2}}{(a-1)!} \right)$$

$$\mu_{j-1,j} = \frac{1}{j(j-1)} \sum_{b=0}^{j-3} \frac{1}{b!}.$$

This also follows from the Lemma. Note that for j large

$$\mu_{2j} = \frac{1}{2j} + O(1/j^2), \quad \mu_{j-1,j} = \frac{e}{j^2} + O(1/j^3).$$

Thus, asymptotically low and high cards are independent (with respect to falling into pile 2) but high cards are dependent. This is made rigorous in the next Lemma.

Lemma 14

$$\mu_{ij} - \mu_i \mu_j = \frac{d(i)}{j^2} + O\left(\frac{1}{ij^2}\right) \text{ as } j \to \infty$$

where $d(i) \to e - (e-1)^2$ *as* $i \to \infty$.

Proof. Consider the formula for μ_{ij} given in Lemma 13. With i fixed, the inner sum in l goes between $0 \le l < d \le i-1$. The lead term comes from $l = 0$. Using just this, as $j \to \infty$

$$\mu_{ij} \sim \frac{1}{j^2} \sum_{2 \le a < b \le j} \frac{\binom{i-1}{d}\binom{j-i-1}{d}}{\binom{j-2}{b-2}(b-2)!} \binom{b-d-2}{a-1} \binom{b-a-1}{d} (b-(a+d)-1)!$$

Expanding the ratio of the binomial coefficients containing j,

$$\mu_{ij} \sim \frac{1}{j^{i+1}} \sum_{2 \le a < b \le j} \binom{i-1}{d} \frac{(j-b)!}{(j-b-(i-1-d))!} \frac{\binom{b-a-1}{d}}{(a-1)!}$$

Now any sum involving a fixed power of a is of lower order summed over $(a-1)!$. Thus replace $\begin{pmatrix} b-a-1 \\ d \end{pmatrix}$ by $b^d/d!$. Similarly replace $\frac{(j-b)!}{(j-b-(i-1-d))!}$ by $(j-b)^{i-1-d}$. This gives

$$\mu_{ij} \sim \sum_{2 \le a < b \le j} \sum_{d=1}^{i-1} \begin{pmatrix} i-1 \\ d \end{pmatrix} \frac{(j-b)^{i-1-d} b^d}{(a-1)!d!}$$

Consider the sum in b (bringing the sum in d outside). With $c = i - 1 - d$, this is

$$\sum_{b=1}^{j} (j-b)^c b^d = j^{c+d+1} \frac{1}{j} \sum_{b=1}^{j} \left(1 - \frac{b}{j}\right)^c \left(\frac{b}{j}\right)^d \sim j^{c+d+1} \int_0^1 (1-x)^c x^d = \frac{j^{c+d+1} c! d!}{(c+d+1)!}.$$

Making this substitution

$$\mu_{ij} \sim \frac{1}{j} \sum_{2 \le a \le j} \frac{1}{(a-1)!} \sum_{d=1}^{i-1} \begin{pmatrix} i-1 \\ d \end{pmatrix} \frac{d!(i-1-d)!}{i!d!} = \frac{1}{ij} \left(\sum_{a=2}^{j} \frac{1}{(a-1)!} \right) \sum_{d=1}^{j} \frac{1}{d!} \sim \frac{(e-1)^2}{ij}$$

This shows that $\mu_{ij} \sim \mu_i \mu_j$ as $j \to \infty$ for fixed i. A more careful analysis shows $\mu_{ij} - \mu_i \mu_j = \frac{c_i}{j^2} + O(1/(ij^2))$ for a certain bounded sequence c_i. To study c_i for large i, reconsider the formula for μ_{ij}. An argument simpler than the preceding argument shows

$$\mu_{j-i,j} \sim \frac{e}{j^2} \text{ as } j \to \infty \text{ for fixed } i$$

and therefore

$$\mu_{j-i,j} - \mu_{j-i}\mu_j \sim \frac{e - (e-1)^2}{j^2}$$

Proof of Theorem. Since $\mu_i = \frac{e-1}{i} = O(1/i!)$, the first term of (25) is

$$\sum_{2 \le i \le n} \mu_i (1 - \mu_i) \sim (e-1) \log n + O(1).$$

Using Lemma 13, the second term is

$$2 \sum_{2 \le i < j \le n} (\mu_{ij} - \mu_i \mu_j) = 2 \sum_{2 \le i < j \le n} \left(\frac{c_i}{j^2} + O\left(\frac{1}{ij^2}\right)\right)$$

$$= 2 \sum_{2 \le i \le n} \left(\frac{c_i}{i} + O\left(\frac{1}{i^2}\right)\right) = 2(e - (e-1)^2) \log n + O(1).$$

2.3 Asymptotics for the central piles

The pile sizes have a deterministic shape in the limit. In section xxx we will prove the "integrated" version of this fact.

Theorem 15 *For fixed* $0 < \alpha < 2$*,*

$$n^{-1} \sum_{i \leq \lfloor \alpha n^{1/2} \rfloor} S_n(i) \stackrel{p}{\to} \frac{\alpha^2(1 - 2\log\frac{2}{\alpha})}{4}.$$

What is undoubtedly true is the corresponding local version.

Conjecture 16 *As* $n \to \infty$ *and* $i \sim \theta n^{1/2}$ *for* $0 < \theta < 2$*,*

$$n^{-1/2} S_n(i) \to_{L^1} \theta \log \frac{2}{\theta}.$$

2.4 Asymptotics for the rightmost piles

Write $\bar{S}_n(1)$ for the number of cards in the rightmost pile, i.e. pile L_n.

Theorem 17 *Assuming* $\bar{S}_n(1) \stackrel{d}{\to} \bar{S}(1)$ *for some limit distribution, the limit can be described as follows.*

$$\bar{S}(1) = R(1 + \mathcal{P}(\xi_1 \xi_2))$$

where ξ_1 *and* ξ_2 *are independent with exponential(1) distribution,* $\mathcal{P}(\lambda)$ *denotes a Poisson r.v., and* $R(m) = S_m(1)$ *is the number of records in a random permutation of* m *cards.*

In fact we show in section xxx that the Poissonized version converges. Our probabilistic description of $\bar{S}(1)$ does not translate to a simple explicit formula, but numerical calculation gives

xxx do numerical calculation

xxx In fact there is a limit process $(\bar{S}(i); i \geq 1)$ describing the numbers of cards in the i'th-from-rightmost piles, for arbitrary fixed i.

3 Hammersley's process

A major purpose of this paper is to set out the connection between the longest increasing subsequence and certain continuous-space continuous-time interacting particle processes, which we'll call *Hammersley's processes*. We start in section 3.1 by defining two versions of Hammersley's process and stating our results, comparing them with the definitions and known results for the well-studied *exclusion process*. Our analysis of Hammersley's process implies (Theorem 20) $c \geq 2$ (without appealing to prior results about c). Then appealing to the fact $c \leq 2$ we can prove further results about both Hammersley's process and L_n.

xxx say what we prove

xxx Hammersley [**?**] sec. 12 understood that a proof of Theorem 30 would imply $c = 2$.

3.1 The exclusion process and Hammersley's process

The exclusion process is discussed in detail in Chapter 8 of Liggett [**?**], which we will reference often. We use the phrase "exclusion process" to mean the simple completely asymmetric exclusion process on Z^1, i.e. $p(x, x+1) \equiv 1$ in the notation of [**?**]. For readers unfamiliar with this topic, we now give an informal discussion.

Regard each (positive and negative) integer as a "site", which is either occupied by a single "particle" or unoccupied. Thus at any time t we see a random "configuration" of occupied and unoccupied sites. The time-dynamics follow the simple rule:

> For each pair $(x, x+1)$ of neighboring sites, if x is occupied and $x+1$ is unoccupied at time t, then with chance dt the particle at x jumps to $x+1$ by time $t + dt$.

We will see below a simple explicit construction using Poisson processes of points. Write μ and ν for distributions on configurations. Call μ (time)-*invariant* if, for the exclusion process with initial distribution μ, the distribution at each time $t > 0$ remains μ. Call ν *translation-invariant* if it is invariant under the shift map on the integers. Write ν_α for the distribution where each site is occupied independently with probability α. It is easy to show

Lemma 18 *A distribution is invariant and translation-invariant for the exclusion process iff it is a mixture of the* (ν_α).

This is a special case of results in Liggett [**?**]. Briefly, given a translation-invariant ν which is spatially ergodic with occupancy probability α, there is a natural coupling between the processes with initial distributions ν and ν_α, and the chance that the coupled processes agree at a fixed site tends to 1 as $t \to \infty$, so that if ν is time-invariant then $\nu = \nu_\alpha$. The general case follows by conditioning on the (spatial) invariant σ-field.

We now define *Hammersley's process on R*. A configuration is a set of particles at distinct positions on the (space) line $(-\infty, \infty)$, more precisely a set such that each finite interval has only finitely many particles. The time-dynamics follow the rule:

> For each interval $[x, x+dx]$ at time t, with probability $dx\ dt$ the nearest particle to the right of x is moved to x by time $t + dt$.

There is of course an issue in proving rigorously that such a process exists, but (for suitable initial configurations) we can give a constructive proof (Lemma 25). Our first observation is that the analog of Lemma 18 holds, with the same proof. Write ν_λ for the Poisson point process of rate λ on R.

Lemma 19 *A distribution is invariant and translation-invariant for Hammersley's process on R iff it is a mixture of the* (ν_λ).

Our main interest is in Hammersley's process on R^+, where particles are on the half-line $[0, \infty)$, so that in the time-dynamics rule above we consider only $x > 0$. Write $\hat{L}(x,t)$ for the number of particles at time t in the interval $[0, x]$. Because particles move only leftwards, for fixed x the number $\hat{L}(x,t)$ increases with t. As we shall explain in section 3.3, it makes sense to consider this process being started at time 0 with zero particles. Our main result is

Theorem 20 *(a) For Hammersley's process on* R^+ *started with zero particles,*

$$\frac{\hat{L}(x,t)}{\sqrt{tx}} \overset{p}{\to} c \text{ as } tx \to \infty.$$

(b) $c \geq 2$.

Here c is the constant defined by subadditivity in section 1.4, and we don't assume any prior knowledge about c except that $c < \infty$. Theorem 20 is analogous to a known result about the exclusion process. Consider starting the exclusion process with sites $x \leq 0$ occupied and sites $x \geq 1$ unoccupied. Call this the *nonequilibrium* exclusion process. Let $U(x,t)$ be the number of unoccupied sites amongst $(-\infty, x]$ at time t.

Theorem 21 *For the nonequilibrium exclusion process, for fixed* β

$$t^{-1}U(\lfloor \beta t \rfloor, t) \ \stackrel{p}{\to}\ r(\beta)$$

where $r(\beta) = 0,\ \beta < -1;\ r(\beta) = \beta,\ \beta > 1,$

$$r(\beta) = \frac{1}{4}(1+\beta)^2,\ -1 \leq \beta \leq 1.$$

Theorem 21 is due to Rost [**?**], with a slightly modified proof given in [**?**] sec. 8.5. (Their results are stated in terms of occupied sites to the right of x, so we have made a trivial reformulation). The proof of Theorem 21 starts by using the subadditive ergodic theorem to show that some limit $r(\beta)$ exists, then uses separate arguments to upper and lower bound $r(\beta)$. Our proof of Theorem 20 uses the same strategy, except that we have not succeeded in giving a self-contained proof that $c \leq 2$, though this is of course true. There is a "local" version of Theorem 21, asserting that the spatial configuration near site $\lfloor \beta t \rfloor$ at time t converges to the Bernouilli$(1-r(\beta))$ process ([**?**] Thm 8.5.12). The analogous local version of Theorem 20 is Corollary 31, which asserts that the spatial process around position x at time t approximates the Poisson process of rate $\lambda(x,t) = \sqrt{t/x}$.

3.2 Further remarks on the analogy

(a) *Hydrodynamical limits.* Results like Theorems 21 and 20 are often called *hydrodynamic limits*, and often have simple heuristic explanations by writing down and solving partial differential equations. Here's the heuristic explanation for Theorem 20. *Suppose* the spatial process around position x at time t approximates a Poisson process of some rate $\lambda(x,t)$. Clearly

$$\frac{d}{dt} E\hat{L}(x,t) = ED_{x,t}$$

where $D_{x,t}$ is the distance from x to the nearest particle to the left of x. For a Poisson process $ED_{x,t}$ would be 1/ (rate), so

$$ED_{x,t} \approx \frac{1}{\lambda(x,t)} \approx \frac{1}{\frac{d}{dx} E\hat{L}(x,t)}.$$

In other words, $w(x,t) = E\hat{L}(x,t)$ satisfies approximately the PDE

$$\frac{dw}{dt} = \frac{1}{\frac{dw}{dx}};\quad w(0,x) = w(t,0) = 0 \tag{26}$$

whose solution is $w(x,t) = 2\sqrt{tx}$. Note that "2" is not an arbitrary constant: no other constant will serve. See De Masi et al [?], Papanicolaou [?] or De Masi [?] for surveys of hydrodynamic limit theory. It is often technically difficult to make such arguments rigorous, and this is an active research field.

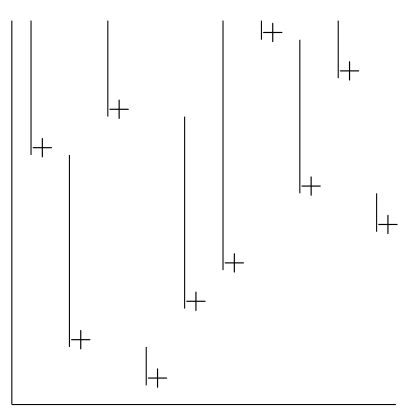
(b) *Large deviation bounds.* The large deviation behavior of L_n (mentioned as an open problem in section 1.3) is presumably analogous to the large deviation behavior of the asymmetric exclusion process in the context of Theorem 21. But the latter is also an open problem, according to Kipnis et al [?] who study this question for symmetric exclusion processes.

(c) *Maximal interpretation of Theorem 21.* Rost [?] p. 43 and Liggett [?] p. 412 have noted a reformulation of the asymmetric exclusion process as a certain geometric growth process. Gandolfi (xxx ref) pointed out a further reformulation, (27) below. Let $(\xi_{ij}; 1 \le i,j < \infty)$ be an array of independent exponential(1) r.v.'s. Let Γ_n be the set of paths $\gamma = \{(1,1) = z_1, z_2, \ldots, z_{2n-1} = (n,n)\}$ which are "up-right", i.e. such that each $z_{i+1} - z_i = (0,1)$ or $(1,0)$. Write $S_\gamma = \sum_{z \in \gamma} \xi_z$. Then Theorem 21 implies

$$n^{-1} \sup_{\gamma \in \Gamma_n} S_\gamma \stackrel{p}{\to} 4. \tag{27}$$

Underlying our work is the analogous connection between Hammersley's process and the length of the longest increasing subsequence of a random permutation.

xxx pictures of the 2 non-eq processes



3.3 Basic properties of Hammersley's process

Recall the definition in section 1.4 of $\mathbf{L}^{\nearrow}(x,t)$ as the maximal number of points of a Poisson process $\mathcal{N}$ on an up-right path from $(0,0)$ to (x,t). For fixed t the map $x \to \mathbf{L}^{\nearrow}(x,t)$ is the counting process associated with a point process on R^+, so we can regard $\mathbf{L}^{\nearrow}$ as a spatial point process evolving with time.

Lemma 22 $\mathbf{L}^{\nearrow}$ *is a version of Hammersley's process on* R^+ *started with zero particles.*

xxx proof

xxx intuitive derivation from patience sorting dumped into section 7 – put condensed version here?

Let us record two lemmas which are immediate from the "up-right path" definition of $\mathbf{L}^{\nearrow}$, but not obvious from the process definition (and which have no obvious analog for the exclusion process). Fix $0 < \kappa < \infty$. The space-time Poisson process $\mathcal{N}$ is invariant under compressing time by κ and dilating space by κ, and it follows that $\mathbf{L}^{\nearrow}$ has the corresponding scaling property.

Lemma 23 $(\mathbf{L}^{\nearrow}(x,t); x,t \geq 0) \stackrel{d}{=} (\mathbf{L}^{\nearrow}(\kappa x, t/\kappa); x,t \geq 0)$.

In particular, the distribution of $\mathbf{L}^{\nearrow}(x,t)$ depends only on the product tx.

The space-time Poisson process $\mathcal{N}$ is also invariant under interchanging space and time, so $\mathbf{L}^{\nearrow}$ inherits that property too.

Lemma 24 *Write* $\hat{L}(x,t) = \mathbf{L}^{\nearrow}(t,x)$. *Then*

$$(\hat{L}(x,t); x,t \geq 0) \stackrel{d}{=} (\mathbf{L}^{\nearrow}(x,t); x,t \geq 0).$$

Recall (5) the definition $g(t) = E\mathbf{L}^{\nearrow}_{t,t}$. Using the scaling property (Lemma 23)

$$E\mathbf{L}^{\nearrow}(x,t) = g(\sqrt{tx}) \qquad (28)$$

So Theorem 20 (a) is an immediate consequence of the subadditive ergodic result (9) and the scaling property.

We now turn to Hammersley's process on R.

Lemma 25 *Hammersley's process on* R *exists, for any initial configuration satisfying*

$$z^{-1/2}(\textit{number of points in } [-z,0]) \to \infty \textit{ as } z \to \infty.$$

xxx proof

The next lemma gives a space-time interchange property for Hammersley's process on R (c.f. Lemma 24).

Lemma 26 *Consider Hammersley's process on R, with the invariant distribution μ_λ, run for time $-\infty < t < \infty$. Then interchanging space and time gives Hammersley's process with the invariant distribution $\mu_{1/\lambda}$.*

xxx give proof.

3.4 A coupling construction

Consider two point process on R^+, such that the set of points in the first process is a subset of the set of points in the second. Define $\hat{L}^1(x,t)$ and $\hat{L}^2(x,t)$ to be Hammersley's process on R^+, started at time 0 with the given point processes as initial distributions, and run using the same space-time Poisson process $\mathcal{N}$. It is easy to see

(a) At each time $t > 0$, the set of particles of $\hat{L}^1$ is a subset of the set of particles of $\hat{L}^2$.

(b) The unmatched particles of $\hat{L}^2$ (i.e. those which do not coincide with a particle of $\hat{L}^1$) can move only to the right.

xxx draw picture

So (a) implies

$$\hat{L}^1(x,t) - \hat{L}^1(v,t) \le \hat{L}^2(x,t) - \hat{L}^2(v,t);\ t \ge 0; 0 \le v \le x. \qquad (29)$$

and (b) implies

$$\hat{L}^2(x,t) - \hat{L}^1(x,t) \le \hat{L}^2(x,s) - \hat{L}^2(x,s);\ x \ge 0; 0 \le s \le t. \qquad (30)$$

xxx probably don't need (30) ?

Now fix $t_0 > 0$, and apply the construction above where $\hat{L}^1$ is our usual version $\hat{L}$ of Hammersley's process on R^+ started with no particles, and $\hat{L}^2$ starts with the distribution of $\hat{L}$ at time t_0. So $\hat{L}^2(x,t) \stackrel{d}{=} \hat{L}(x,t_0+t)$. Taking expectations in (29),

$$E\hat{L}(x,t) - E\hat{L}(v,t) \le E\hat{L}(x,t_0+t) - E\hat{L}(v,t_0+t);\ t \ge 0; 0 \le v \le x. \quad (31)$$

We shall use this coupling in two different ways. First we will strengthen the convergence in (7) to give a more "local" result, which doesn't depend on knowing $c = 2$. (This is somewhat analogous to the coupling proof of the renewal theorem). In the next section we use it to give a probabilistic proof that $c \ge 2$.

Proposition 27 *(a) $g'(z) \to c$ as $z \to \infty$.*

(b) There exists a constant $B < \infty$ such that, for any $\xi > 0$, as $m, n \to \infty$ with $m^{1/2} - n^{1/2} \to \xi$,

$$\limsup |EL_m - EL_n - c\xi| \leq B.$$

Proof. Granted (a), part (b) follows by dePoissonization (Lemma 7), because $EL_{M_m} - EL_{M_n} \to c\xi$.

xxx If we have $o(1)$ in that Lemma then we can take $B = 0$ in (b) here.

Inequality (31) implies

$$g(\sqrt{(x+x_0)t}) - g(\sqrt{xt}) \leq g(\sqrt{(x+x_0)(t+t_0)}) - g(\sqrt{x(t+t_0)}); \; x, x_0, t, t_0 > 0.$$

Differentiating, we see

$$\frac{d}{dx}\frac{d}{dt}\, g(\sqrt{xt}) \geq 0$$

which, after a brief calculation, implies

$$g'(z) + zg''(z) \geq 0; \;\; z > 0 \tag{32}$$

Since the solution of $f(z) + zf'(z) = 0$ is $f(z) = A/z$, it follows that, for any fixed z_0,

$$g'(z) \geq \frac{g'(z_0)z_0}{z}; \;\; z > z_0. \tag{33}$$

We can now combine this with the superadditivity property to prove (a). First consider an interval $[z_1, z_1 + z_2]$. By (6) $g(z_1 + z_2) - g(z_1) \geq g(z_2)$, so there exists $z_0 \in [z_1, z_1 + z_2]$ such that $g'(z_0) \geq g(z_2)/z_2$. Then by (33),

$$g'(z_1 + z_2) \geq \frac{g(z_2)z_1}{z_2(z_1 + z_2)}.$$

Putting $z_2 = \sqrt{z}, z_1 = z - z_2$, gives the lower bound $\liminf_z g'(z) \geq c$. For the opposite bound, integrate (33) to get

$$g(z) - g(z_0) \geq z_0 g'(z_0) \log(z/z_0); z > z_0.$$

Use the fact (8) $g(z) \leq cz$, set $z = (1+\delta)z_0$ and rearrange to get

$$g'(z_0) \leq \frac{c(1+\delta) - \frac{g(z_0)}{z_0}}{\log(1+\delta)}$$

So

$$\limsup_z g'(z) \leq \frac{c\delta}{\log(1+\delta)}$$

and letting $\delta \downarrow 0$ establishes (a).

3.5 Proof of $c \geq 2$

For each $t > 0$, consider Hammersley's process on R^+ at time t, and shift all the particles left by distance t. That is, consider the process

$$\mathbf{L}(x,t) = \hat{L}(t+x,t)$$

as a point process on R (for fixed t), and write μ_t for its distribution.

xxx talk about topology.

Tightness of (μ_t) as $t \to \infty$ follows from

Corollary 28 *For fixed $-\infty < u < \infty$,*

$$E\mathbf{L}(u,t) - E\mathbf{L}(0,t) \to \frac{1}{2}cu \text{ as } t \to \infty.$$

Proof.

$$\begin{aligned} E\mathbf{L}(u,t) - E\mathbf{L}(0,t) &= E\mathbf{L}^{\nearrow}(t+u,t) - E\mathbf{L}^{\nearrow}(t,t) \\ &= g(\sqrt{t(t+u)}) - g(t) \end{aligned}$$

and the result follows from Proposition 27 (a), since $\sqrt{t(t+u)} - t \to \frac{1}{2}u$.

Lemma 29 *If $t_j \to \infty$ and $\mu_{t_j} \to \mu$ for some limit μ, then μ is translation-invariant and is an invariant distribution for Hammersley's process on R.*

Let's show how this implies $c \geq 2$. By Lemma 19, a limit μ must be a mixed Poisson process, i.e. a Poisson process of some random rate Λ. Next, the space-time interchange property (Lemma 24) for Hammersley's process on R^+, and the fact that $\mathbf{L}$ is defined relative to the diagonal $\{(t,t) : t \geq 0\}$ in space-time, implies that when we run Hammersley's process on R with distribution μ we get a process which is invariant under interchanging space and time. But by Lemma 26 this implies $\Lambda \stackrel{d}{=} \Lambda^{-1}$. Since Λ and Λ^{-1} are negatively correlated,

$$1 = E\Lambda\Lambda^{-1} \leq (E\Lambda)(E\Lambda^{-1}) = (E\Lambda)^2$$

and so $E\Lambda \geq 1$. By Fatou's Lemma, for (t_j) as in Lemma 29 and for any $u > 0$,

$$\begin{aligned} 2uE\Lambda &\leq \liminf_j E(\mathbf{L}(2u,t_j) - \mathbf{L}(0,t_j)) \\ &= cu \text{ by Corollary 28.} \end{aligned}$$

So

$$c \geq 2E\Lambda \tag{34}$$

and since $E\Lambda \geq 1$ we have shown $c \geq 2$.

Proof of Lemma 29. Fix t_0 and consider the coupling $\hat{L}^1, \hat{L}^2$ specified above (31). Fix B. Let U be the random set of (x,t) such that the particle processes $\hat{L}^1(\cdot, t)$ and $\hat{L}^2(\cdot, t)$ are not identical on $[x-B, x+B]$. And for $\delta > 0$ let $D = D(t_0, B, \delta)$ be the deterministic set of (x,t) such that

$$P(\ \hat{L}^1(\cdot, t) \text{ and } \hat{L}^2(\cdot, t) \text{ are not identical on } [x-B, x+B]\ ) > \delta.$$

Consider the curve

$$C_\kappa = \{(x,t) : xt = \kappa^2, \kappa/2 \leq x \leq 2\kappa\}.$$

Let Leb_κ be the measure on C_κ induced from Lebesgue measure on x. Then

$$\mathrm{Leb}_\kappa(U) \leq 2B\ \hat{L}^2(2\kappa + B, 0)$$

because an unmatched particle in the coupling whose trajectory crosses C_κ at (x', t') can only contribute to U for values $x \in (x' - B, x' + B)$, and only unmatched particles starting in $[0, 2\kappa + B]$ can contribute. Next,

$$E\mathrm{Leb}_\kappa(U) \leq 2BE\hat{L}^2(2\kappa + B, 0) = 2BE\hat{L}(2\kappa + B, t_0) \leq 2B\bar{c}\sqrt{t_0(2\kappa + B)}.$$

And

$$\mathrm{Leb}_\kappa(D) \leq \delta^{-1} E\mathrm{Leb}_\kappa(U).$$

As $\kappa \to \infty$ the right side is $O(\kappa^{1/2})$ whereas $\mathrm{Leb}_\kappa(C_\kappa) = 2.5\kappa$, and so $C_\kappa \setminus D$ is non-empty for all large κ. The diagonal principle now establishes the following. There exists a trajectory $\{(x_\kappa, t_\kappa) : 0 < \kappa < \infty\}$ with $(x_\kappa, t_\kappa) \in C_\kappa$ such that, for all rational (t_0, B, δ),

$$(x_\kappa, t_\kappa) \in D(t_0, B, \delta) \text{ for all } \kappa > \kappa_0(t_0, B, \delta) \tag{35}$$

where $\kappa_0(t_0, B, \delta) < \infty$. Write $\nu_{x,t}$ for the distribution of the point process $(\hat{L}(x+u, t); -\infty < u < \infty)$ and write d for a metrization of weak convergence of point processes. For fixed t_0, the fact that (35) holds for all rational (B, δ) implies

$$d(\nu_{x_\kappa, t_\kappa}, \nu_{x_\kappa, t_\kappa + t_0}) \to 0.$$

This extends from rational t_0 to real t_0, and then we can use the scaling property (Lemma 23) to show

$$d(\nu_{\kappa,\kappa}, \nu_{\kappa, \kappa + t_0}) \to 0.$$

Since $\mu_t = \nu_{t,t}$, we have established the "invariant for Hammersley's process" property, and the translation-invariance follows after another use of the scaling property.

xxx above proof slides over some details?

xxx I should think carefully about whether the arguments for the exclusion process can be used to show $c \leq 2$.

3.6 The hydrodynamical limit

We would like to be able to give a self-contained proof of Theorem 30 below. But there are two difficulties. First, we do not know how to prove the technical fact that for fixed u, $(\mathbf{L}(u,t) - \mathbf{L}(0,t))$ is uniformly integrable as $t \to \infty$. Second, we do not know how to show that a subsequential weak limit μ in Lemma 29 must be a pure Poisson process rather than a mixture of Poisson processes. If we could prove these assertions probabilistically, then Theorem 30 below and the fact $c = 2$ would easily follow. In fact we shall argue the other way round, and invoke the fact $c \leq 2$ to prove the Theorem.

Theorem 30 *$\mu_t \to \nu_1$ as $t \to \infty$.*

Proof. Granted $c \leq 2$, inequality (34) implies $E\Lambda \leq 1$, in the notation of the previous section. But $\Lambda \stackrel{d}{=} \Lambda^{-1}$, so $P(\Lambda = 1) = 1$. That is, a subsequential weak limit μ in Lemma 29 must be ν_1, so the Theorem follows by tightness.

Scaling gives the natural hydrodynamic limit.

Corollary 31 *Let $\mu_{x,t}$ be the distribution of the point process $(\mathbf{L}(x + \lambda_{x,t}u, t); -\infty < u < \infty)$ where $\lambda_{x,t} = \sqrt{t/x}$. Then*

$$\mu_{x,t} \to \nu_1 \text{ as } tx \to \infty.$$

xxx remaining parts of section 3 depend on knowing $c = 2$.

3.7 More about EL_n

Theorem 32 *$2n^{1/2} - EL_n \to \infty$.*

By Lemma 7 it suffices to prove

$$2t - E\hat{L}(t,t) \to \infty. \tag{36}$$

Fix $B < \infty$. Interpreting $\mathbf{L}^{\nearrow}(x,t)$ in terms of the space-time Poisson process $\mathcal{N}$ as the maximal number of points on an up-right path from $(0,0)$ to (x,t), we have

$$\mathbf{L}^{\nearrow}(2t,2t) = \sup_{-t<s<t} (\mathbf{L}^{\nearrow}(t,t+s) + \mathbf{L}^*(t,t-s)) \tag{37}$$

where $\mathbf{L}^*(t,x)$ is the maximal number of points on an up-right path from $(t,2t-x)$ to $(2t,2t)$. By symmetry and independence of the Poisson process,

$$(\mathbf{L}^*(t,u); 0 \le u \le 2t) \ \stackrel{d}{=}\ (\mathbf{L}^{\nearrow}(t,u); 0 \le u \le 2t)$$

and the two processes are independent. By Theorem 30

$$(\mathbf{L}^{\nearrow}(t,t+s) - \mathbf{L}^{\nearrow}(t,t); -B \le s \le B) \ \stackrel{d}{\to}\ (N(s); -B \le s \le B)$$

where the limit is the Poisson(1) counting process. Thus

$$\liminf_t E \sup_{-B\le s\le B} (\mathbf{L}^{\nearrow}(t,t+s) - \mathbf{L}^{\nearrow}(t,t)) + (\mathbf{L}^*(t,t-s) - \mathbf{L}^*(t,t)) \ge r(B)$$

where

$$r(B) \equiv E \sup_{-B\le s\le B} (N(s) + N^*(-s))$$

where N^* is an independent Poisson(1) counting process. Appealing to (37),

$$\liminf_t (E\mathbf{L}^{\nearrow}(2t,2t) - 2E\mathbf{L}^{\nearrow}(t,t)) \ge r(B).$$

But $r(B) \to \infty$ as $B \to \infty$ and so $E\mathbf{L}^{\nearrow}(2t,2t) - 2E\mathbf{L}^{\nearrow}(t,t) \to \infty$, which easily implies (36).

3.8 xxx

xxx prove Theorem 15

xxx the conjecture reduces to a tagged-particle result

xxx anomalous size of difference between adjacent piles?

xxx old heuristics for conjecture follow.

In terms of the Poisson process approximation to $\hat{L}(x,t)$, the rate of growth of a pile with top card x is

$$\frac{1}{ED_{x,t}} \approx \frac{1}{\frac{d}{dx}E\hat{L}(x,t)} \approx \sqrt{x/t}. \tag{38}$$

Now consider pile a. This is first created at time t_0 such that

$$a \approx 2\sqrt{t_0}. \tag{39}$$

And the top card of pile a at time t has label x such that $a \approx 2\sqrt{tx}$ and so

$$x \approx \frac{a^2}{4t} \approx \frac{t_0}{t}.$$

So by (38), the rate of growth of pile a at time $t > t_0$ is approximately $\frac{t_0^{1/2}}{t}$, and so the size of pile a at time n is approximately

$$\begin{aligned} S(a,n) &\approx \int_{t_0}^{n} \frac{t_0^{1/2}}{t}\, dt = t_0^{1/2} \log(n/t_0) \\ &\approx \frac{a}{2} \log\left(\frac{n}{(a/2)^2}\right) \text{ by (39)} \\ &= a\ \log\left(\frac{2\sqrt{n}}{a}\right) \end{aligned}$$

as stated in the conjecture.

xxx prove Theorem 17

4 Continuous Limits and the Leftmost Piles

xxx section 4 essentially unchanged from old notes.

4.1 Another particle process

To give results on the number of cards in a fixed pile we need a different, but related, particle process. There are a fixed number k particles, at positions $0 < X_1(t) < X_2(t) < \ldots < X_k(t) < \infty$ at time t. The process evolves according to the rules

Growth rule. Each particle moves deterministically to the right at exponentially increasing rate, i.e. according to $dx(t)/dt = x(t)$.

Fallback rule. There is a space-time Poisson(1) process of events. When an event happens at (x_0, t_0) with $X_{i-1}(t_0) < x_0 < X_i(t_0)$, particle i is moved from $X_i(t_0)$ to x_0.

xxx picture

It is easy to check this process can be obtained from the previous particle process by a deterministic transformation of time and space, as follows. Let

$$L^*(x,t) = \#\{i : X_i(t) \leq x\}$$

be the number of particles of this new process in $[0,x]$ at time t. Then

$$L^*(x,t) = \hat{L}(xe^t, e^t)$$

With this construction the initial condition is $X_i(0) = \infty$, but it is easy to see that $X_i(t) < \infty$ for $t > 0$.

In terms of patience sorting with n cards labeled $1, \ldots, n$, let $B(n,i)$ be the label on the card on top of pile i at the end of the game, with $B(n,i) = \infty$ if there is no such pile. Arguing as above (63) we see that *for fixed* t

$$(X_i(t); i = 1, \ldots, k) \stackrel{d}{=} (\Gamma(B(M_{e^t}, i)); i = 1, \ldots, k) \tag{40}$$

where (M_t) is Poisson(1) process of "times", and where $(\Gamma(1), \Gamma(2), \ldots)$ are the points of an independent Poisson(1) process on the "space" interval $[0,\infty)$, with $\Gamma(\infty) = \infty$. Conceptually, the purpose of the time-change is to obtain a process whose evolution rule is stationary in time: in patience sorting the growth rate of pile 1 after n cards is order $1/n$, and our exponential time-change makes this order 1.

Returning to patience sorting, recall that the position of cards 1 through b_k in the piles depends only on their initial relative order in the deck, so for

$1 = b_1 < b_2 < \ldots < b_k$ the probability $P(B(n,1) = b_1, \ldots, B(n,k) = b_k)$ is the same for all $n \geq b_k$. So

$$(B(n,i); i = 1, \ldots, k) \ \stackrel{d}{\to} \ (B(i); i = 1, \ldots, k) \text{ , say.} \tag{41}$$

Then by (40) there is a limiting distribution for the particle process

$$(X_1(\infty), \ldots, X_k(\infty)) \ \stackrel{d}{=} \ (\Gamma(B(1)), \ldots, \Gamma(B(k))) \tag{42}$$

which has some density $p_k(x_1, \ldots, x_k)$ say. From the evolution rules for the particle process we can write down the balance equation for p_k (i.e. the continuous analog of the balance equation $\pi(j) = \sum_i \pi(i)P(i,j)$ for the stationary distribution π of a discrete-time and -space chain)

$$x_k p_k = -p_k - \sum_{i=1}^{k} x_i \frac{\partial p_k}{\partial x_i} + \sum_{i=1}^{k} \int_{x_i}^{x_{i+1}} p(x_1, \ldots, x_i', \ldots, x_k) dx_i' \tag{43}$$

where $x_{k+1} = \infty$.

Note the elegant interplay here of combinatorial and process ideas. From the description of the particle process it is not quite obvious that a limit distribution exists, but the existence of the limit distribution $(B(1), \ldots, B(k))$ in (41) is obvious. Conversely, we cannot automatically write down a characterization of the latter distribution, whereas (43) is an automatic characterization of the former limit. Unfortunately it seems impossible to extract useful information from (43).

There is a natural coupling of two versions of the particle processes started at $(x_1, \ldots, x_k)$ and $(x_1', \ldots, x_k')$. The coupling simply uses the same space-time process of events to determine the jumps of each version. We make several uses of this coupling. Suppose first that $x_i' = x_i$ for $i = 1, \ldots, k-1$ and $x_k < x_k'$. Then the processes will couple at the first jump of $X_k(t)$, and this has mean time bounded by $1/(x_k' - x_k)$, so in particular is a.s. finite. An easy inductive argument shows

Lemma 33 *Two versions of the particle process with arbitrary initial configurations will couple in a.s. finite time.*

Lemma 33 and the existence of a stationary distribution imply the particle process is a *Harris-recurrent* Markov process

xxx refs

We want to use this machinery to study the number $S_n(i)$ of cards in pile i when patience sorting is played with n cards. As in section 3.3, we may use the "unending" variation of patience sorting, with the cards labelled by random real numbers. In terms of the particle process, we study

$$J_i(t) = \text{ number of jumps of } X_i(\cdot) \text{ before time } t \ .$$

For the *stationary* particle process we have

$$EJ_i(t) = c(i)t$$

where

$$c(i) = E(X_i(\infty) - X_{i-1}(\infty)) = E(B(i) - B(i-1)). \quad (44)$$

Persi ? can you see sharply why this $c(i)$ is the same as in Corollary 10?

xxx can we prove $c(i)$ increasing?

As a consequence of the general CLT for Harris-recurrent chains

xxx ref new Tweedie book?

we have

Proposition 34 *For the stationary particle process,*

$$t^{-1/2}(J_i(t) - c(i)t) \stackrel{d}{\to} \mathit{Normal}(0, \sigma^2(i))$$

where

$$\sigma^2(i) = c(i)\left(1 + 2\lim_{t\to\infty}(EJ_i^*(t) - c(i)t)\right)$$

where $(J_i^*(t))$ *is the process* $(J_i(t))$ *conditioned on* $X_i(\cdot)$ *having a jump at time* 0 *(this jump is not counted in* $J_i^*(t)$*).*

xxx argue $\sigma^2(i)$ not 0 or ∞.

We shall use this to prove Theorem 11,

$$\frac{S_n(i) - c(i)\log n}{\sqrt{\log n}} \stackrel{d}{\to} \text{Normal}(0, \sigma^2(i)) \text{ as } n \to \infty.$$

First, we can take "time 1" to be the time n when the first card is put into pile i. Then, by Lemma 35 below, we can switch to continuous time. But then, using the deterministic exponential time-change, the assertion of Theorem 11 is just the assertion of Theorem 34, applied to the particle process with a certain random initial configuration: but by Lemma 33 the initial configuration is immaterial.

xxx say above better.

We used the following elementary dePoissonization lemma.

Lemma 35 *Let* $(V(n); n = 1, 2, \ldots)$ *be a nondecreasing sequence of r.v.'s. Let* (M_t) *be a Poisson*(1) *counting process independent of* $(V(n))$*. If*

$$(V(M_t) - a(t))/b(t) \stackrel{d}{\to} Z$$

where $a(t) \uparrow \infty, b(t) \uparrow \infty$*,* $t^{1/2}a'(t)/b(t) \downarrow 0$ *then*

$$(V(n) - a(n))/b(n) \stackrel{d}{\to} Z.$$

4.2 Calculations for $k = 2$

xxx result inconsistent – I'll try to work backwards from the analysis in Theorem 12.

Our first goal is to use (42) to obtain a formula for the stationary density $p_2(x_1, x_2)$. Trivially $B(n, 1) = 1$, so $X_1(\infty)$ has exponential(1) distribution. Next,

$$P(B(n,2) = b) = \frac{1}{b(b-2)!}, \quad b = 2, 3, \ldots, n \tag{45}$$

(with the remaining probability $1/n!$ there is only one pile). To argue (45), the compatability property (section 1.1) implies that the cards with labels larger than b play no role, so we may suppose $n = b$. Then $B(n, 2) = b$ if and only if cards 1 through $b - 1$ occur in exactly reverse order, and card b is not the first card. And this has chance $\frac{1}{(b-1)!} \times \frac{(b-1)}{b}$.

From (45) we can obtain an expression for p_2. Given $B(n, 2) = b$, and given the Poisson process S_m has points at x_1 and $x_2 > x_1$, the chance that $X_1(\infty) = x_1$ and $X_2(\infty) = x_2$ is the chance that there are no points of the Poisson process in $[0, x_1)$ and there are exactly $(b - 2)$ points in (x_1, x_2). So

$$p_2(x_1, x_2) = e^{-x_1} \sum_{b=2}^{\infty} q(x_2 - x_1, b - 2) \frac{1}{b(b-2)!} \tag{46}$$

where $q(\lambda, \cdot)$ is the Poisson(λ) probability function. We can rewrite this as

$$p_2(x_1, x_2) = e^{-x_1} a(x_2 - x_1) \tag{47}$$

where

$$a(\lambda) = \sum_{i=0}^{\infty} q(\lambda, i) \left(\frac{1}{(i+1)!} - \frac{1}{(i+2)!}\right). \tag{48}$$

Remark. It would be nice if the limit $(X_i(\infty); i = 1, \ldots, k)$ were Markov in i, but this appears to be false.

Consider the intensity $\pi(x, y, z), 0 < x < y < z$ specified by

$$\pi(x, y, z)dxdydzdt$$

$$= P(X_1(0) \in [x, x + dx], X_2(0) \in [y, y + dy], X_2(-dt) \in [z, z + dz]).$$

Clearly

$$\pi(x, y, z) = p_2(x, z).$$

Now fix (x, y, z) and consider the natural coupling between the process $(X_1(t), X_2(t))$ with $(X_1(0) = x, X_2(0) = z)$ and the process $(X_1^*(t), X_2^*(t))$ with $(X_1^*(0) = x, X_2^*(0) = y)$. These processes couple at the first time that $X_2^*(\cdot)$ jumps. So there is an intensity $g_{x,y,z}(t)$ specified by

$$g_{x,y,z}(t)dt = \text{chance } X_2(\cdot) \text{ jumps during } [t, t + dt] \text{ but } X_2^*(\cdot) \text{ doesn't.} \quad (49)$$

Now the expression for $\sigma^2(2)$ in Proposition 34 can be rewritten as

$$\sigma^2(2) = (e - 1) - 2 \int\int\int \left(\int_0^\infty g_{x,y,z}(t)dt \right) \pi(x, y, z)dxdydz. \quad (50)$$

where $\int\int\int$ is the triple integral over $\{0 < x < y < z < \infty\}$.

Since the event counted in (49) can occur only if $X_2^*(\cdot)$ has not jumped before t, it can only occur when $X_2^*(t) = ye^t$, and a jump of $X_2(t)$ to w can only be counted if $ye^t < w$ and $w < X_2(t)$. Also, the largest that $X_2(t)$ can be is ze^t. So

$$g_{x,y,z}(t) = \int_{ye^t}^{ze^t} P(X_2^*(t) = ye^t, X_2(t) > w)\ dw. \quad (51)$$

Conditional on $\{X_2^*(t) = ye^t\}$, we have $X_2(t) > w$ iff no events of the space-time Poisson process occurred in the region $\{(s, v) : 0 < s < t, ye^s < v < we^{s-t}\}$. This has conditional probability

$$\exp(- \int_0^t (we^{-t} - y)e^s\ ds)$$

$$= \exp(-(we^{-t} - y)(e^t - 1)) = \exp(-(1 - e^{-t})w)\ \exp(y(e^t - 1)).$$

So the integrand in (51) is

$$\exp(-(1 - e^{-t})w)\ \exp(y(e^t - 1))\ P_{x,y}(X_2^*(t) = ye^t)$$

where $P_{x,y}$ emphasizes that X^* starts with $X_1^*(0) = x, X_2^*(0) = y$. Integrating over w in (51),

$$g_{x,y,z}(t) = \frac{\exp(-(e^t-1)y) - \exp(-(e^t-1)z)}{1-e^{-t}} \ \exp(y(e^t-1)) \ P_{x,y}(X_2^*(t) = ye^t). \quad (52)$$

Now define

$$h(x,t) = E_x \exp\left(\int_0^t X_1(s)ds\right). \quad (53)$$

Conditional on $(X_1(s) : 0 \le s \le t)$, we have $X_2^*(t) = ye^t$ if and only if no events of the space-time Poisson process occur in the region $\{(s,v) : 0 < s < t, X_1(s) < v < ye^s\}$, and this has conditional probability

$$\exp(-\int_0^t (ye^s - X_1(s)) \ ds) = \exp(-y(e^t-1)) \ \exp(\int_0^t X_1(s) \ ds).$$

Taking expectations,

$$g_{x,y,z}(t) = \frac{\exp(-(e^t-1)y) - \exp(-(e^t-1)z)}{1-e^{-t}} \ \ h(x,t). \quad (54)$$

Now substitute into (50) and integrate dy over $x < y < z$.

$$\sigma^2(2) = (e-1) - 2\int_0^\infty \frac{dt}{1-e^{-t}} \int_0^\infty dx \ h(x,t) \ \int_x^\infty dz \ p_2(x,z)$$

$$\left(\frac{\exp(-(e^t-1)x) - \exp(-(e^t-1)z)}{e^t-1} - (z-x)\exp(-(e^t-1)z)\right). \quad (55)$$

Recall (47) that $p_2(x,z) = e^{-x}a(z-x)$, for $a(\cdot)$ defined at (48). We calculate

$$\int_0^\infty a(\lambda) \ d\lambda = 1$$

$$\int_0^\infty a(\lambda)e^{-\theta\lambda} \ d\lambda = 1 + \theta - \theta\exp\left(\frac{1}{\theta+1}\right)$$

$$\int_0^\infty \lambda a(\lambda)e^{-\theta\lambda} \ d\lambda = -1 + \left(1 - \frac{\theta}{(\theta+1)^2}\right)\exp\left(\frac{1}{\theta+1}\right).$$

These enable us to integrate over z in (55) to get

$$\sigma^2(2) = (e-1) - 2\int_0^\infty dt \ e^{-t}\exp(e^{-t}) \int_0^\infty \exp(-e^t x)h(x,t)dx. \quad (56)$$

Turning to the function $h(x,t)$, it is routine to derive the equation

$$\frac{\partial h}{\partial t} = x\frac{\partial h}{\partial x} + \int_0^x h(x',t)dx' \tag{57}$$

with $h(x,0) = 1$. Since we need a transform in (56) anyway, we can consider

$$w(\theta,t) = \int_0^\infty h(x,t)\theta e^{-\theta x}\ dx.$$

Then (57) transforms to

$$\frac{\partial w}{\partial t} = \theta\frac{\partial h}{\partial \theta} + \frac{w}{\theta} \tag{58}$$

with $w(\theta,0) = 1$. I don't know a closed-form solution of (58), but we can write down the series expansion

$$w(\theta,t) = \sum_{i=0}^\infty w_i(1/\theta)t^i$$

where $w_0 \equiv 1$ and $w_i(y)$ is the degree i polynomial defined recursively by

$$(i+1)w_{i+1}(y) = y(w_i(y) + w_i'(y)).$$

In terms of these w_i, (56) becomes

$$\sigma^2(2) = (e-1) - 2\int_0^\infty dt\ e^{-2t}\exp(e^{-t})\sum_{i=0}^\infty w_i(e^{-t})\ t^i. \tag{59}$$

Numerically this works out as about 1.44.

4.3 Hammersley's process on the circle

xxx talk about Leticia's work

4.4 Other particle processes

The particle process of section xxx, restricted to the space interval $[0,1]$, has the "birth rule"

> Particles are born at time of a Poisson(1) process, at uniform random positions

and a certain "death rule". Altering the death rule would typically lead to completely different behavior. Let us mention two alternative rules which have been discussed in the literature (with motivations quite different from ours).

Alternate Rule 1 *[?]. After each birth, with probability* $1-p$ *there is no death, and with probability* p *both the newly-born particle and the nearest particle to its right (if any) are killed.*

For $p < 1/2$ there is a stationary distribution. For $p \geq 1/2$ the number of particle tends to infinity, and the normalized empirical distribution of particles tends to the uniform distribution.

Alternate Rule 2 *([?], section 2). There is an independent Poisson*(1) *process of death times, at which times the leftmost particle (if any) present is killed.*

This is a critical $M/M/1$ queue with pre-emptive priorities, and there is a stationary distribution with finitely many particles in $[0, 1-\varepsilon)$ for each $\varepsilon > 0$.

5 A large deviation argument

Our aim is to prove Proposition 3. It is enough to prove

$$t^{-1}\log P(L^{\nearrow}_{t,t} < c't) \to -\infty \text{ as } t \to \infty;\ c' < c \tag{60}$$

because for $N(n)$ with Poisson(n) distribution,

$$P(L^{\nearrow}_{t,t} < c't) \geq P(N(n) \leq n)P(L_n < c'n)$$

and $P(N(n) \leq n) \to 1/2$.

Given K define $D^{(K)} = \{(x_1, x_2) : |x_1 - x_2| \leq K\}$. Define $L^{/K/}_{t,t}$ to be the maximum number of points of the point process $\mathcal{N}$ on an up-right path from $(0,0)$ to (t,t), restricted to paths lying in $D^{(K)}$.

Lemma 36 *Given $\varepsilon > 0$ there exists K such that*

$$\limsup_t t^{-1}\log P(L^{/K/}_{t,t} < (c-\varepsilon)t) < 0.$$

Granted this lemma, fix $j \geq 1$. Then

$$L^{\nearrow}_{t,t} \geq \max_{1\leq i\leq j} M^{(i)}_t$$

where $M^{(i)}_t \stackrel{d}{=} L^{/K/}_{t-2iK,t-2iK}$ is the maximum number of points of the point process $\mathcal{N}$ on an up-right path from $(0, 2iK)$ to $(t-2iK, t)$, restricted to the region $\{(x_1, x_2) : (2i-1)K \leq x_2 - x_1 \leq (2i+1)K\}$.

xxx picture

These regions are disjoint as i varies, so the $M^{(i)}_t$ are independent, so

$$\log P(L^{\nearrow}_{t,t} < (c-\varepsilon)t) \leq j\ \log P(L^{/K/}_{t-2jK,t-2jK} < (c-\varepsilon)t).$$

It follows that

$$\limsup_t t^{-1}\log P(L^{\nearrow}_{t,t} < (c-\varepsilon)t) \leq j\ \limsup_t t^{-1}\log P(L^{/K/}_{t,t} < (c-\varepsilon/2)t).$$

Since j is arbitrary, Lemma 36 implies (60).

To work toward the proof of Lemma 36, we first state a standard large deviation result.

Lemma 37 *Let $(\eta_i; i \geq 1)$ be i.i.d. non-negative r.v.'s. Then*

$$\limsup_n n^{-1}\log P(\sum_{i=1}^n \eta_i < bn) < 0,\ b < E\eta_1.$$

Next, note that subadditivity implies

$$t^{-1}EL^{/K/}_{t,t} \to c_K \leq c \text{ as } t \to \infty$$

for some c_K. Also, by subadditivity

$$EL^{\nearrow}_{K,K} = EL^{/K/}_{K,K} \leq Kc_K$$

and so

$$c_K \to c \text{ as } K \to \infty. \tag{61}$$

Now regard the point process $\mathcal{N}$ as being defined on all R^2. Define $L^{/K/}_{\mathbf{a},\mathbf{b}}$ to be the maximum number of points of the point process $\mathcal{N}$ on an up-right path from $\mathbf{a}$ to $\mathbf{b}$, restricted to paths lying in $D^{(K)}$. Fix K and Q. For each $i \geq 1$ define

$$\eta_{K,Q,i} = \inf\{L^{/K/}_{\mathbf{a},\mathbf{b}} : \mathbf{a},\mathbf{b} \in D^{(K)}, a_1 + a_2 = (i-1)Q, b_1 + b_2 = iQ\}.$$

These r.v.'s are independent as i varies, and

$$\sum_{i=1}^{n} \eta_{K,Q,i} \leq L^{/K/}_{(-K/2,-K/2),(nQ+K/2,nQ+K/2)} \stackrel{d}{=} L^{/K/}_{nQ+K,nQ+K}.$$

So Lemma 37 implies

$$\limsup_n n^{-1} \log P(L^{/K/}_{nQ+K,nQ+K} < bn) < 0, \ b < E\eta_{K,Q,1}.$$

This in turn implies

$$\limsup_t t^{-1} \log P(L^{/K/}_{t,t} < bt) < 0, \ b < \frac{E\eta_{K,Q,1}}{Q}. \tag{62}$$

But

$$\eta_{K,Q,1} \geq L^{/K/}_{(K/2,K/2),(Q-K/2,Q-K/2)} \stackrel{d}{=} L^{/K/}_{Q-K,Q-K}$$

and so

$$\lim_{Q\to\infty} \frac{E\eta_{K,Q,1}}{Q} = c_K.$$

So Lemma 36 follows from (62) and (61).

6 Miscellaneous

6.1 Processes of uniform random permutations

There are many different constructions of uniform random permutation processes, i.e. many different possible joint distributions of $(\Pi_n; n \geq 1)$ where each Π_n is uniform of the set Σ_n of permutations of $\{1, 2, \ldots, n\}$. These are often useful in probabilistic analyses of quantities Q_n associated with Π_n, and in particular it is often helpful to choose a process for which Q_n is increasing (for each sample path).

If for each n we have a $n--1$ map $\Sigma_n \to \Sigma_{n-1}$ then the exists a process with $\Pi_{n-1} = f_n(\Pi_n)$, and this process can be specified "forwards" by

$$\text{given } \Pi_{n-1} = \pi, \text{ take } \Pi_n \text{ uniform on } \{\pi^* : f_n(\pi^*) = \pi\}.$$

We illustrate three processes of this type by a verbal description of the particular "forwards rule" and an illustration with our example

7 2 8 1 3 4 10 6 9 5

(a) Insert card n at a random position in the deck.

7 2 8 11 1 3 4 10 6 9 5

(b) Add a new card at the end of the deck, give it a random label j, and increase by 1 the label on cards previously labeled j through $n-1$.

8 2 9 1 3 4 11 7 10 5 6

(c) Put card n at the end of the deck, then interchange it with a randomly-chosen card.

7 2 8 1 11 4 10 6 9 5 3

There are other processes of this simple type, for instance the "Chinese restaurant process" (e.g. [**?**]) based on the cycle representation.

The longest increasing subsequence $l(\Pi_n)$ is increasing for processes (a) and (b), but not for (c). Of course process (b) is the natural "card-dealing" process implicit in our description of patience sorting and the Schensted correspondence . In terms of Hammersley's process $\mathbf{L}^{\nearrow}$, the embedded jump process of $\mathbf{L}^{\nearrow}_{1,t}$ is $(l(\Pi_n))$ for process (b), while the embedded jump process of $\mathbf{L}^{\nearrow}_{x,1}$ is $(l(\Pi_n))$ for process (a), But the embedded jump process of $\mathbf{L}^{\nearrow}_{t,t}$ (to which the subadditive ergodic theorem applies), is the $(l(\Pi_n))$ process for a certain random permutation process (Π_n) which does not have such a simple explicit description. Finally, note that in Lemma 6 it was easier to use process (a).

7 dump

To play patience sorting with n cards in the way described earlier, one must specify n in advance, in order to label the cards 1 through n. But for mathematical analysis, suppose instead that we have an infinite supply of cards and label each with a random real number, chosen uniformly from $[0, 1]$. Then we can imagine continuing the game forever. For each n, the order of the labels of the first n cards is a uniform random permutation, and so the pattern of piles we see after n cards are dealt is probabilistically the same as if we had specified n in advance.

A second alteration is more technical. Stochastic processes are often easier to study in *continuous* time rather than discrete time. Thus we imagine cards being dealt at random real times, the times of a Poisson(1) process. That is, in each time interval $[t, t + dt]$ there is chance dt that a card is dealt, independent of past events. Thus the number of cards dealt during time $[0, t]$ is a r.v. M_t, say, with Poisson(t) distribution, so in particular M_t has expectation t and s.d. $t^{1/2}$.

So at time t we will see the labels on the the top card on each pile, say 0.137, 0.355, 0.602, 0.884. We now switch terminology and talk about *particles* and *positions*. That is, at time t we have 4 particles at positions 0.137, 0.355, 0.602, 0.884 on the interval $[0, 1]$. A moment's thought shows this particle process evolves according to the following two rules.

Birth rule. New particles are born at times of a Poisson(1) process, and at uniform random positions on $[0, 1]$.

Death rule. When a particle is born, the nearest particle to its right (if any) is immediately killed.

Initially, there are 0 particles.

Let $\hat{L}(1, t)$ be the number of particles present at time t. Then

$$\hat{L}(1, t) = L_{M_t} \tag{63}$$

where $(L_n; n = 1, 2, 3, \ldots)$ is the number of piles after n cards when patience sorting is played forever, as described above.

We can extend the particle process to a process of particles on the semi-infinite line $[0, \infty)$. The birth process is now a *space-time* Poisson(1) process, $\mathcal{N}$ say. That is, in each time interval $[t, t+dt]$ and each space interval $[x, x+dx]$ there is chance $dt\ dx$ for a particle to be born. The death process remains the same. And (contrary to intuition, but appealing to the Kolmogorov extension theorem) it still makes sense to start at time 0 with 0 particles.

For this extended process, write $\hat{L}(x,t)$ for the number of particles at time t in the interval $[0,x]$. It is clear than the behavior of the process on a spatial interval $[a,b]$ is affected by the behavior on $[0,a]$ but is not affected by the behavior on $[b,\infty)$. It follows that the behavior of the extended process on the spatial interval $[0,1]$ is exactly the same as the process previously defined on $[0,1]$, so we don't lose anything by studying the extended process. But note that the extended process has no natural interpretation in terms of patience sorting, so we really have gotten some new structure.