Take integer parameters (T, N). Take discrete state space $\{-N, -N+1, \ldots, N-1, N\}$. We will define a discrete time process $(X_s,\ s = 0, 1, 2, \ldots, T)$ which is a martingale and a time-inhomogeneous Markov chain. The process has

$$X(0) = 0; \quad X(T) = N \text{ or } -N. \tag{1}$$

The process is designed to be the maximum entropy process satisfying (1) and the martingale property.

We can define the transition probabilities $p_s(i, j) = P(X_{s+1} = j | X_s = i)$ by backwards induction. Clearly for $s = T - 1$ we must have

$$p_{T-1}(i, N) = \tfrac{i+N}{2N}, \quad p_{T-1}(i, -N) = \tfrac{N-i}{2N}.$$

Define

$$e_{T-1}(i) = -\tfrac{i+N}{2N} \log \tfrac{i+N}{2N} - \tfrac{N-i}{2N} \log \tfrac{N-i}{2N}$$

that is the entropy of the distribution $p_{T-1}(i, \cdot)$.

Now inductively for $s = T-2, T-3, \ldots, 0$, for each i we define $p_s(i, \cdot)$ as the distribution $q(\cdot)$ on integers $[-N, N]$ which maximizes

$$-\sum_j q(j) \log q(j) + \sum_j q(j) e_{s+1}(j) \tag{2}$$

subject to having mean $= i$, and let $e_s(i)$ be the corresponding maximized value of (2). So this construction inductively specifies the maximum entropy process, starting at state i at time s, satisfying (1) and the martingale property.

Rather than try to study this process $(X_s,\ t = 0, 1, 2, \ldots, T)$ for fixed (T, N), let us consider the natural rescaling

$$X^*_t = N^{-1} X_{tT}$$

so that the time interval becomes $[0, 1]$ and the range becomes $[-1, 1]$. Intuitively, if we take limits as $T, N \to \infty$ in some appropriate way we should get a limit process – or perhaps a one-parameter family of processes – which will be time-inhomogeneous martingale diffusions, and therefore specified by the variance rate $\sigma^2(t, x)$.

Can we calculate $\sigma^2(t, x)$ heuristically? Copying the argument above, there should be some function $e(t, x)$ representing "normalized entropy for the process started at position x at time t" and we expect some PDE for the function $e = e(t, x)$ and an expression for the function σ^2 in terms of the function e.

Below I give a heuristic argument that the PDE is

$$e_t = \tfrac{1}{2}\log(-e_{xx}) \tag{3}$$

with the obvious boundary conditions

$$e(t,\pm 1) = 0,\ \ 0 \le t < 1; \quad e(1,x) = 0, -1 < x < 1;$$

and that

$$\sigma^2(t,x) = \frac{-1}{e_{xx}(t,x)} \tag{4}$$

The heuristic argument.

Fix large K and consider $N \to \infty$. We expect the entropy function $e_s(i)$ to scale, for fixed $0 \le s \le K-1$, as

$$e_s(i) \approx e_K(s, i/N) + (K-s)\log N \tag{5}$$

for some function $e_K(s,x),\ -1 \le x \le 1$. And we expect the step distribution $p_s(i,\cdot)$ to scale as

$$p_s(i,\cdot) \approx \mathrm{Normal}(i, N^2\sigma_K^2(s,i/N))$$

for some function $\sigma_K^2(s,x),\ -1 \le x \le 1$. Now (2) says that $\sigma_K^2(s,x)$ is the value of σ^2 that maximizes

$$\mathrm{entropy}(NZ) + \mathbb{E}e_{s+1}(xN + NZ) \tag{6}$$

where $Z =_d \mathrm{Normal}(0,\sigma^2)$. To calculate (6), the $\mathrm{Normal}(0,\sigma^2)$ density $f_\sigma(u)$ has

$$-\log f_\sigma(u) = \log(2\pi) + \log\sigma + \tfrac{x^2}{2\sigma^2}$$

and therefore has entropy $c + \log\sigma$ for $c = \log(2\pi) + \frac{1}{2}$. So the first term in (6) is $c + \log N + \log\sigma$. Next, use (5) to write the second term of (6) as

$$(K-s-1)\log N + \mathbb{E}e_K(s+1, x+Z) \approx (K-s-1)\log N + e_K(s+1,x) + \tfrac{\sigma^2}{2}e_K''(s+1,x)$$

where e_K'' is second derivative w.r.t. x. So the quantity (6) is

$$c + (K-s)\log N + e_K(s+1,x) + \log\sigma + \tfrac{\sigma^2}{2}e_K''(s+1,x).$$

This is maximized by

$$\sigma_K^2(s,x) = \frac{-1}{e_K''(s+1,x)} \tag{7}$$

and the maximized value is

$$c - \tfrac{1}{2} + (K-s)\log N + e_K(s+1,x) - \tfrac{1}{2}\log(-e''_K(s+1,x)).$$

This maximized value is, by definition, supposed to equal $e_s(xN)$, so from (5)

$$e_K(s,x) \approx c - \tfrac{1}{2} + e_K(s+1,x) - \tfrac{1}{2}\log(-e''_K(s+1,x)).$$

To study what happens as $K \to \infty$, we look for a solution of the form

$$e_K(s,x) \approx (K-s)(c - \tfrac{1}{2} - a_K) + Kf(s/K,x)$$

for some function $f(t,x)$ and some constants a_K. Setting $t = s/K$ this becomes

$$K\left(f(t,x) - f(t+\tfrac{1}{K},x)\right) + a_K = -\tfrac{1}{2}\log(-Kf_{xx}(t,x)).$$

So set $a_K = -\frac{1}{2}\log K$ to get

$$K\left(f(t,x) - f(t+\tfrac{1}{K},x)\right) = -\tfrac{1}{2}\log(-f_{xx}(t,x)).$$

This leads to (3), and (7) leads to (4).

Issues:

- Is the calculation in the heuristic argument for (3, 4) correct (as a heuristic)?
- Has (3) ever been studied before? If not, can you solve it?
- Possible methods of proof? – there must be similar "weak limits of dynamic programming" work somewhere.