Feature allocations, probability functions, and paintboxes

Tamara Broderick
UC Berkeley
(MIT starting 2015)

Clustering/Partition

$$\pi_7 = \{\{1, 2, 7\}, \{3, 5\}, \{4\}, \{6\}\}$$

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Document 1

Document 2

Document 3

Document 4

Document 5

Document 6

Document 7

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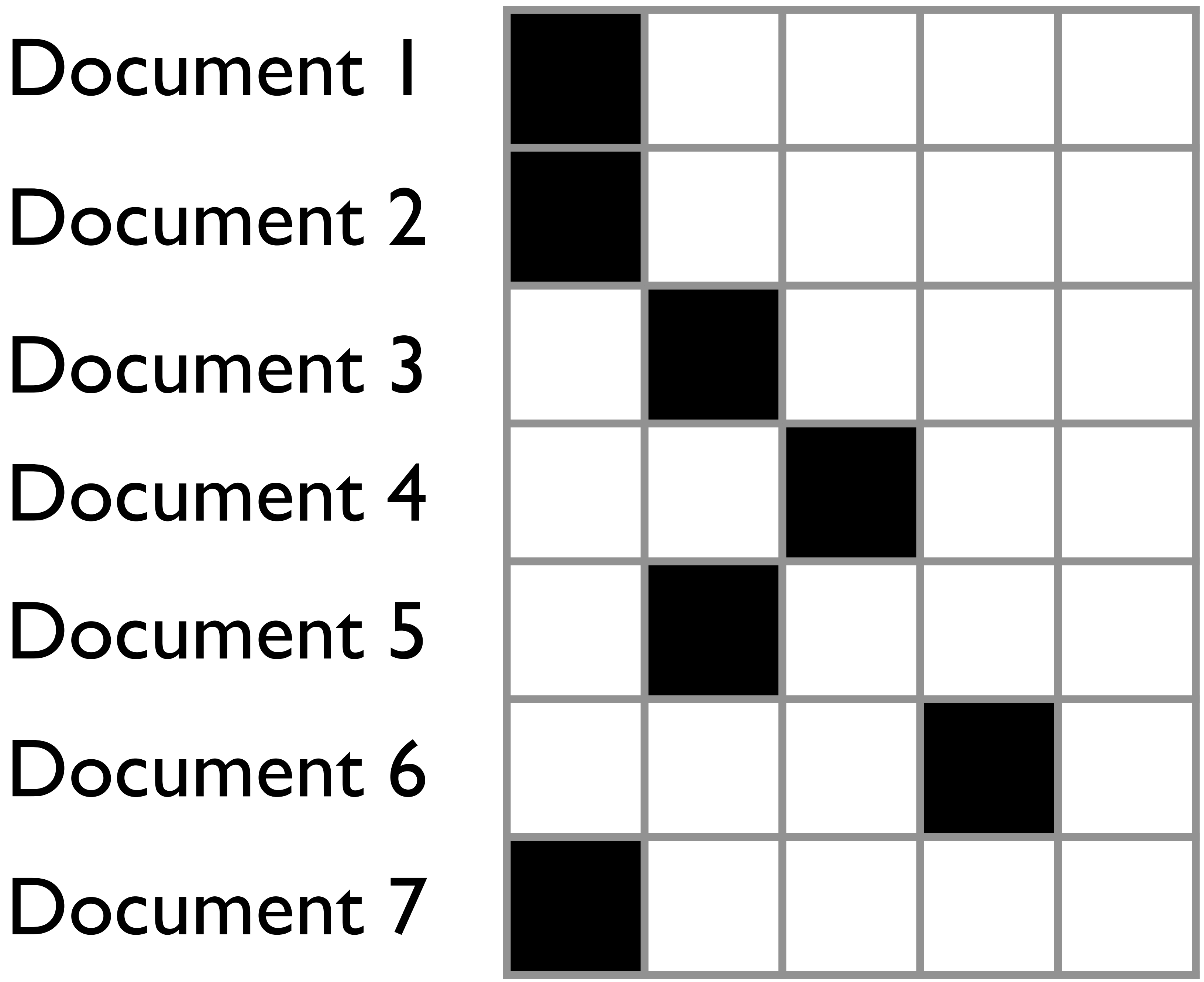

Clustering/Partition

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Document 1
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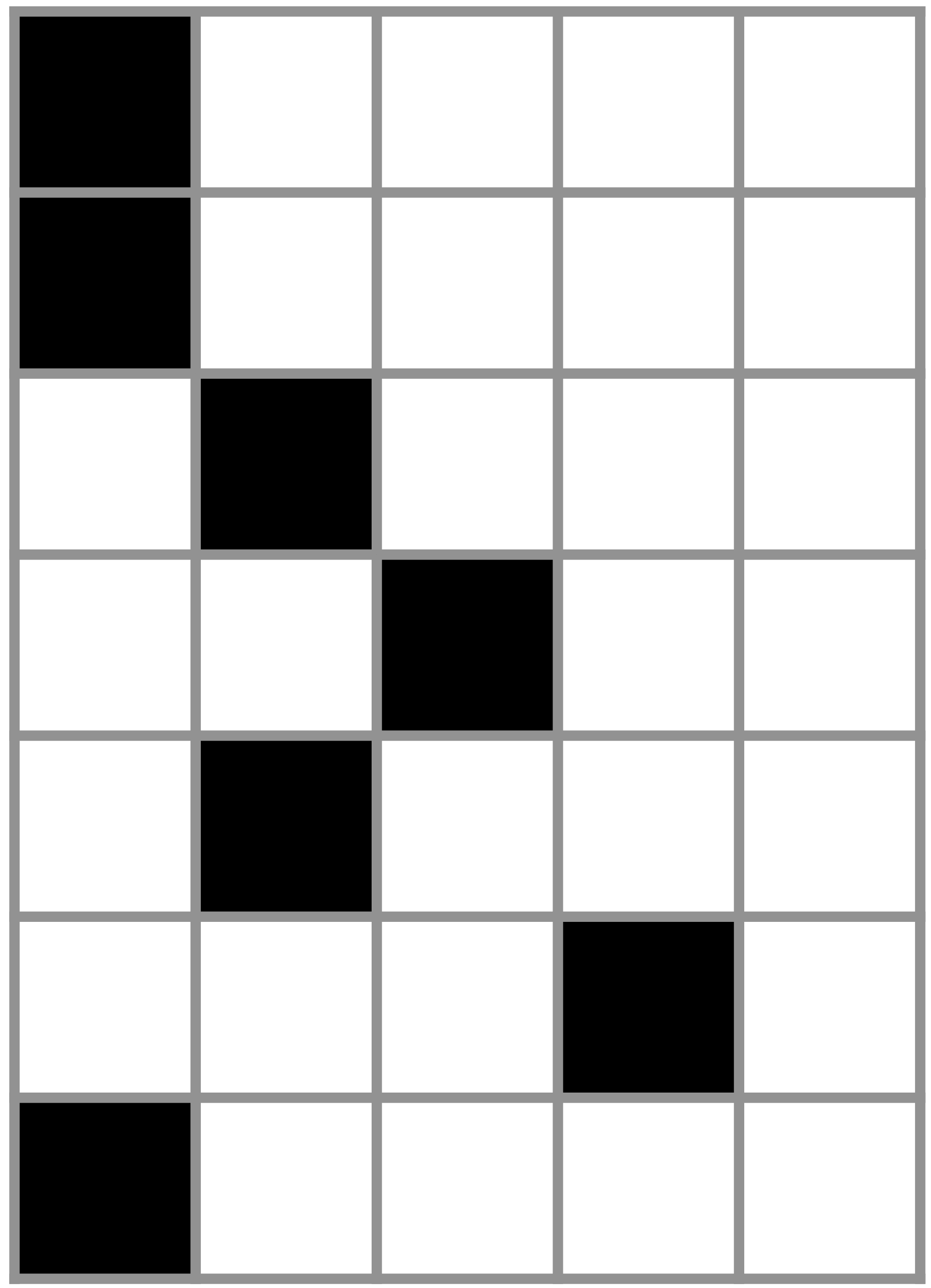

"clusters", "blocks (of a partition)"

Clustering/Partition

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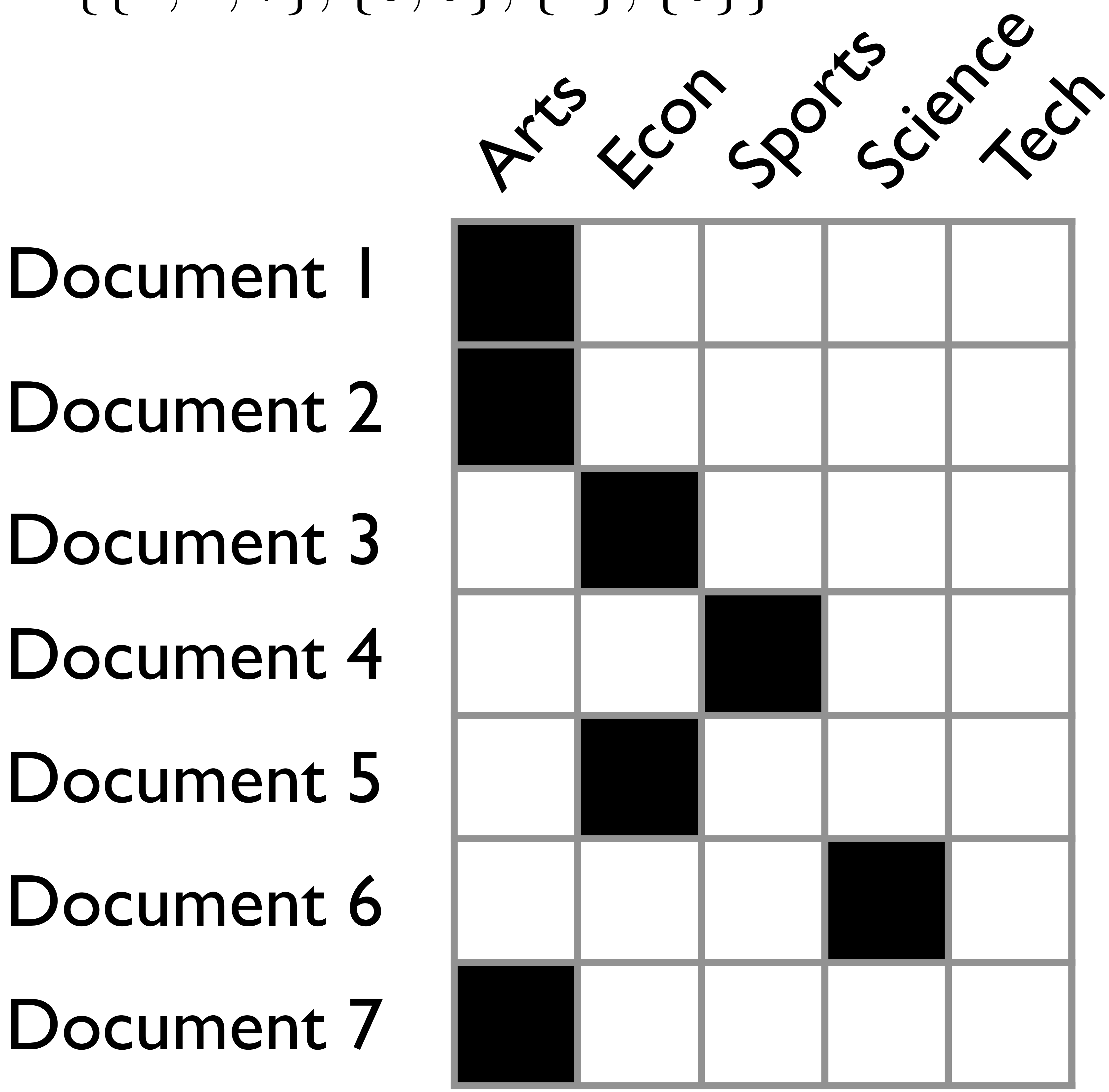

“clusters”,
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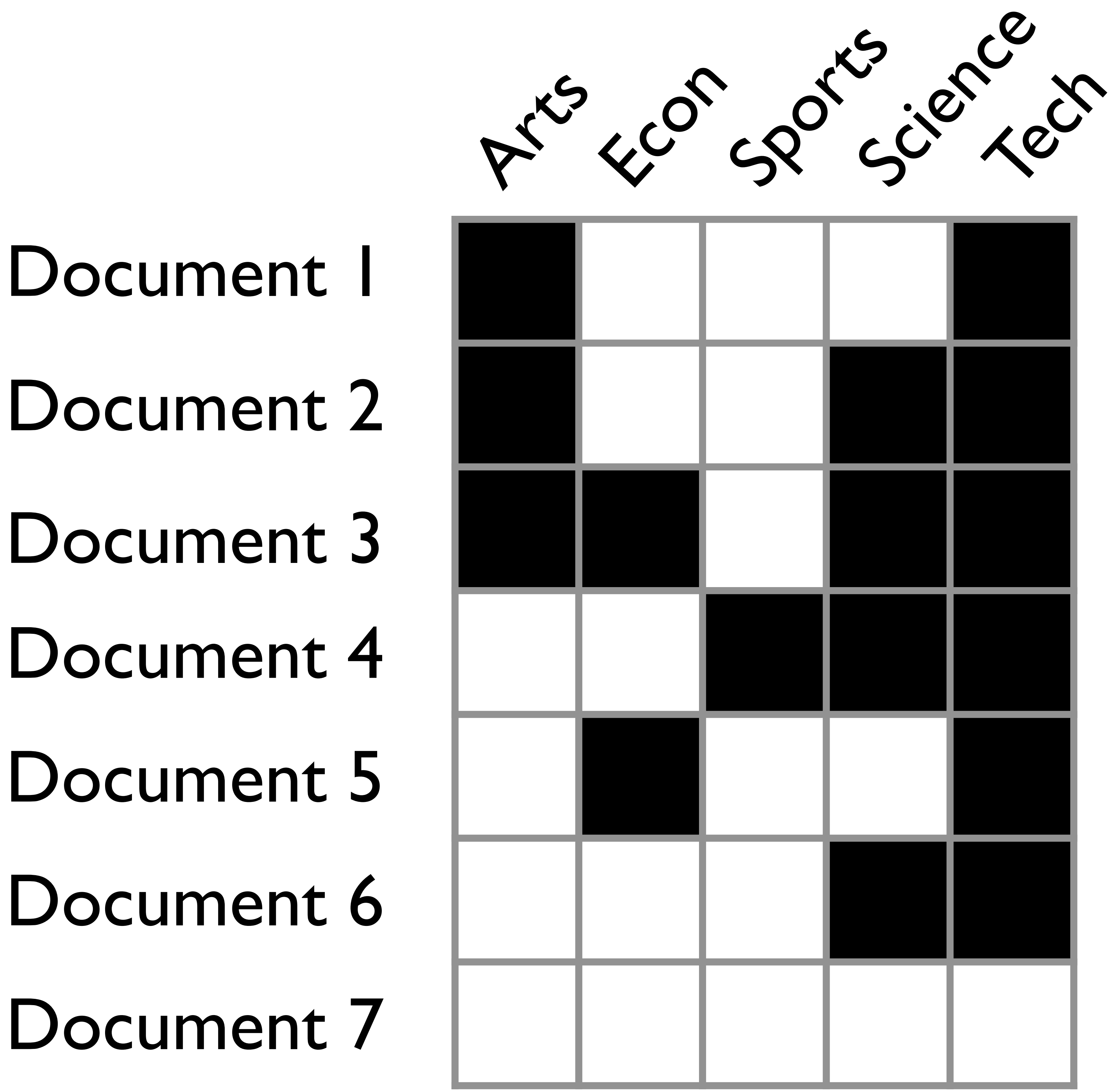
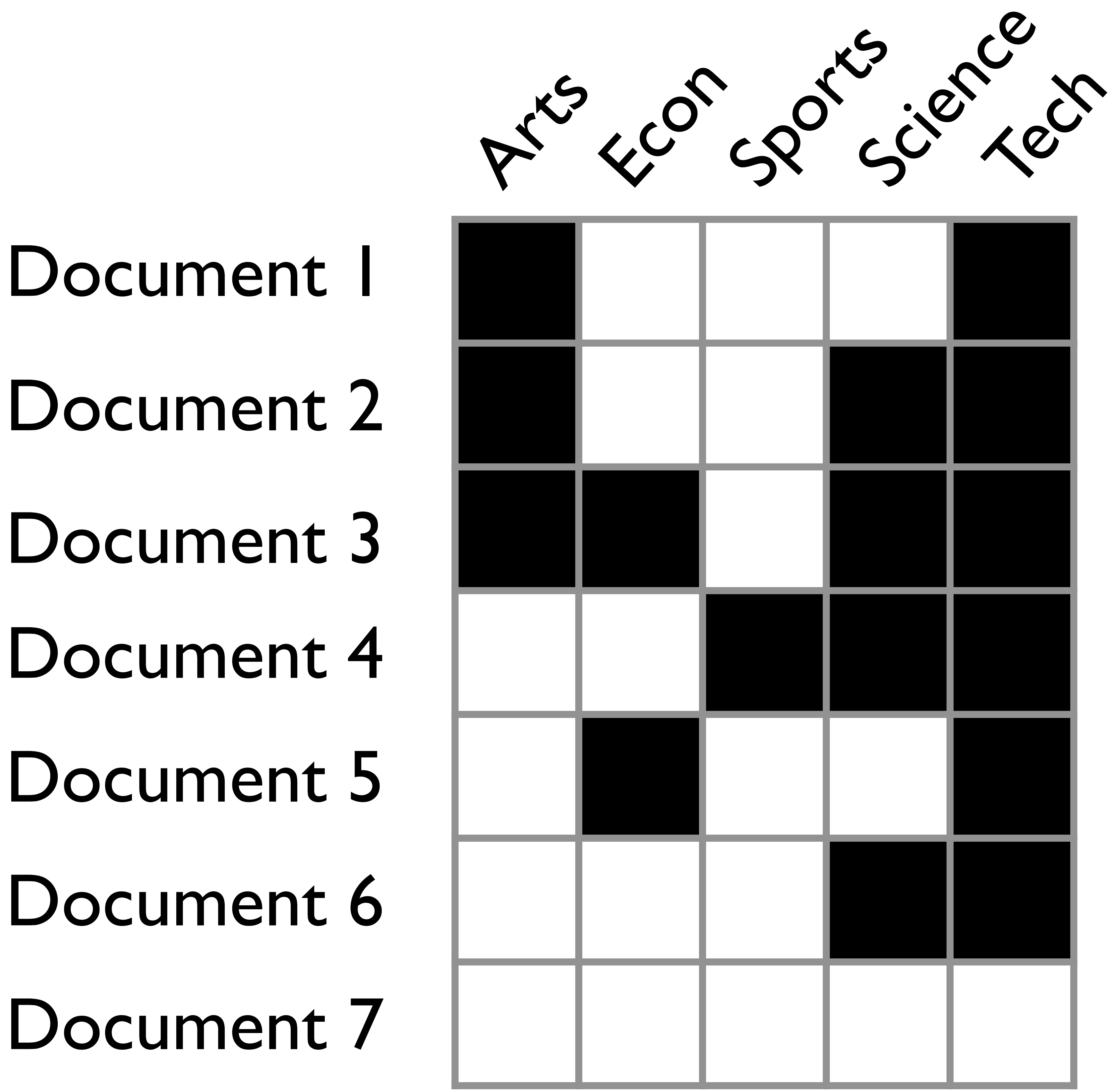
Latent feature allocation



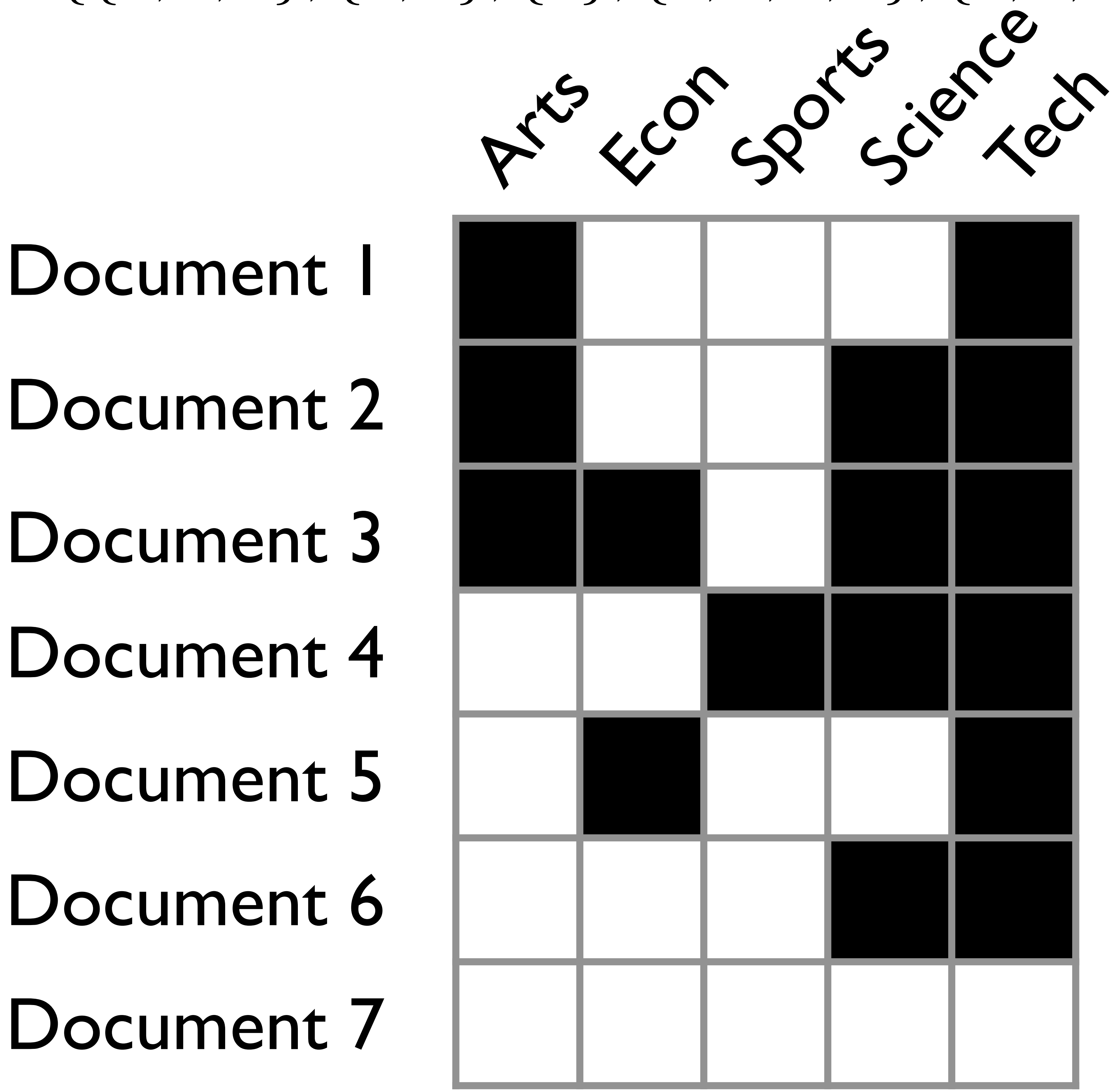
Latent feature allocation



“features”,
“topics”

Latent feature allocation

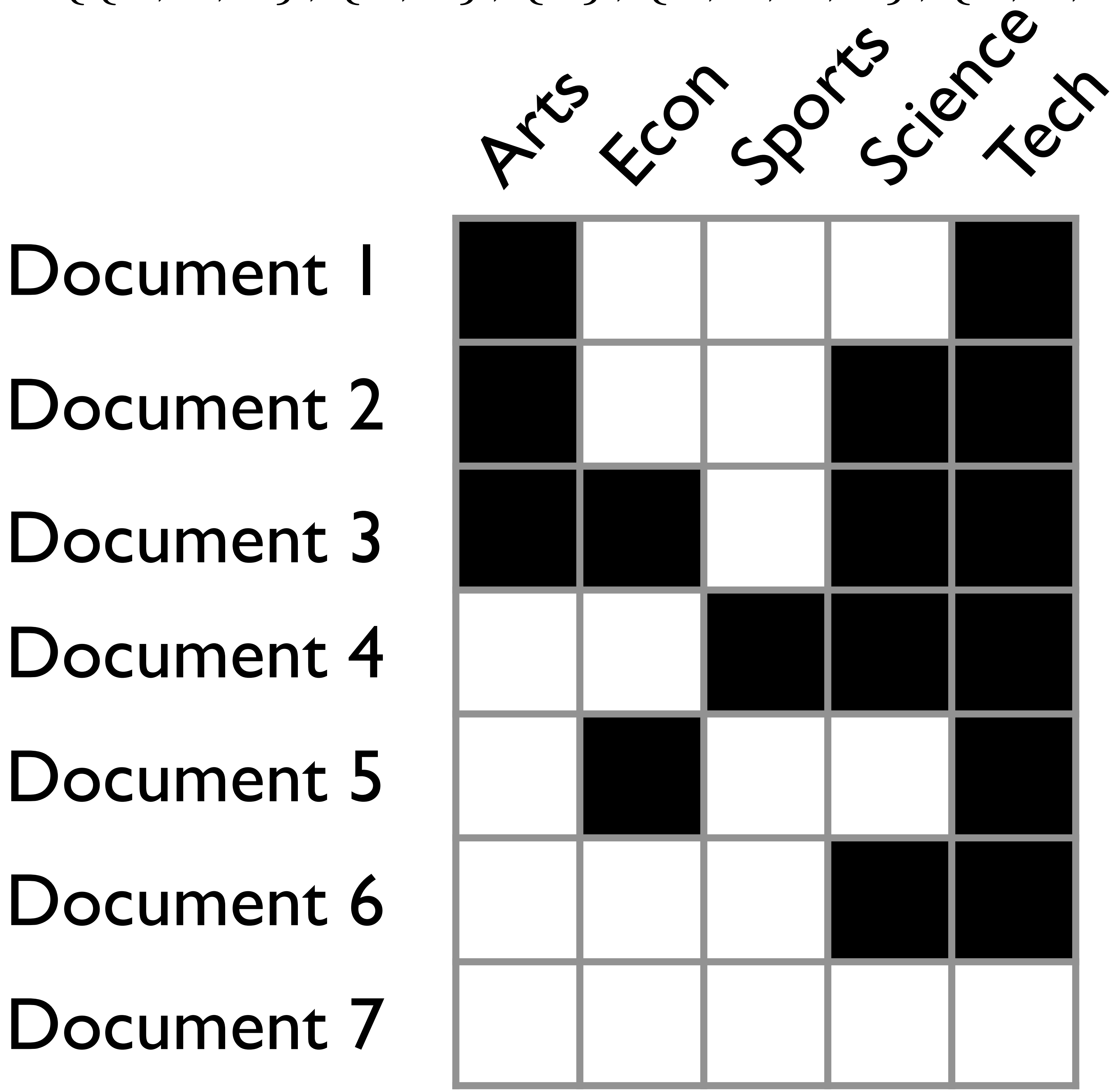
$$f_7 = \{\{1, 2, 3\}, \{3, 5\}, \{4\}, \{2, 3, 4, 6\}, \{1, 2, 3, 4, 5, 6\}\}$$



"features",
"topics"

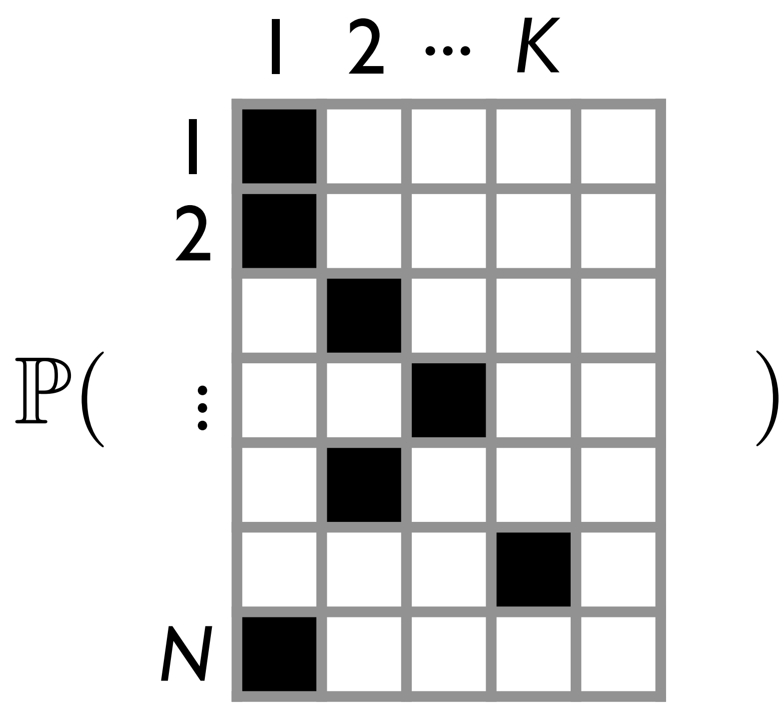
Latent feature allocation

$f_7 = \{\{1,2,3\},\{3,5\},\{4\},\{2,3,4,6\},\{1,2,3,4,5,6\}\}$



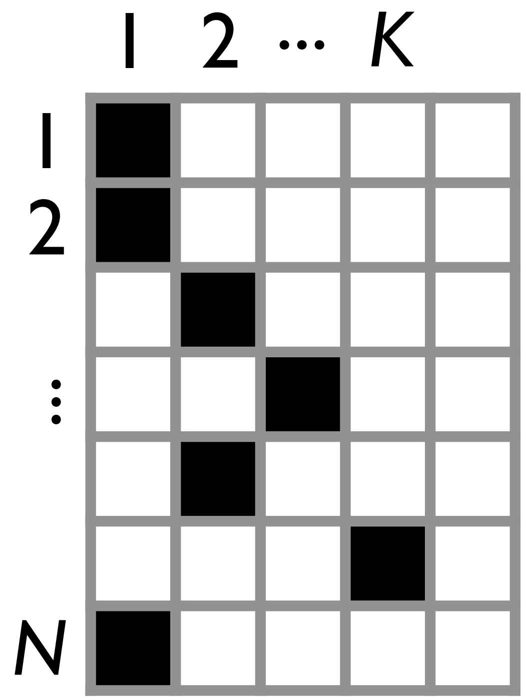
<u>Question</u>:
Can we characterize exchangeable feature distributions?

Exchangeable probability functions



Exchangeable probability functions

$\mathbb{P}($

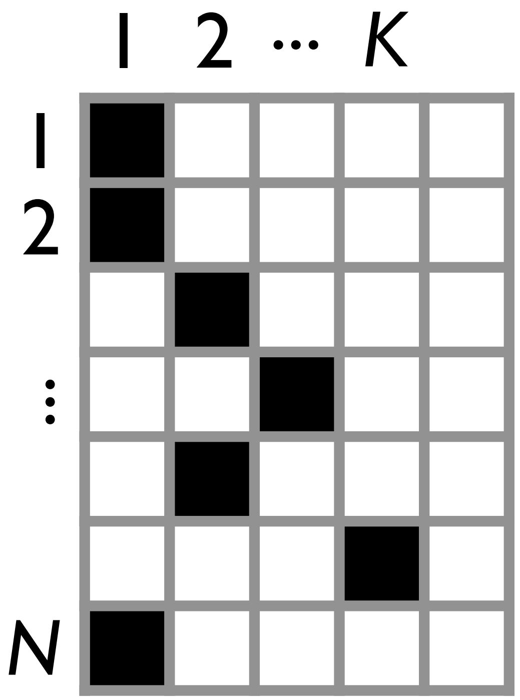


$) = p\left(S_{N,1}, \ldots, S_{N,K}\right)$

[Pitman 1995]

Exchangeable probability functions

Size of *K*th cluster

$\mathbb{P}($ 

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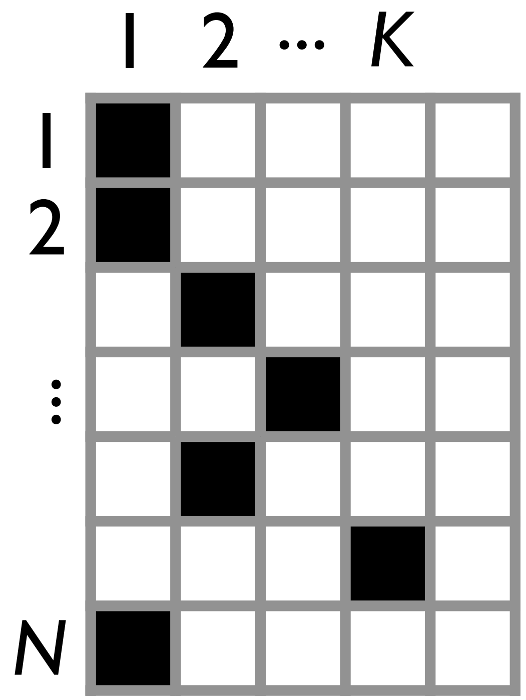
Exchangeable probability functions

Exchangeable partition probability function (EPPF)



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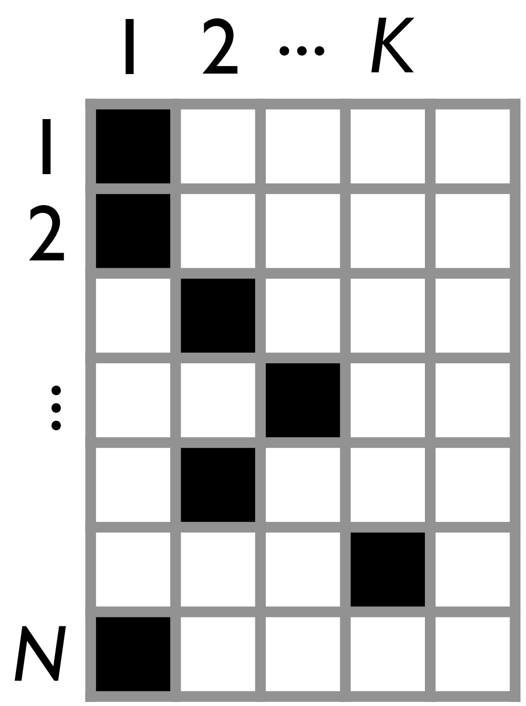

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"Exchangeable **feature** probability function" (EFPF)?

Example: Indian buffet process

[Griffiths, Ghahramani 2006]

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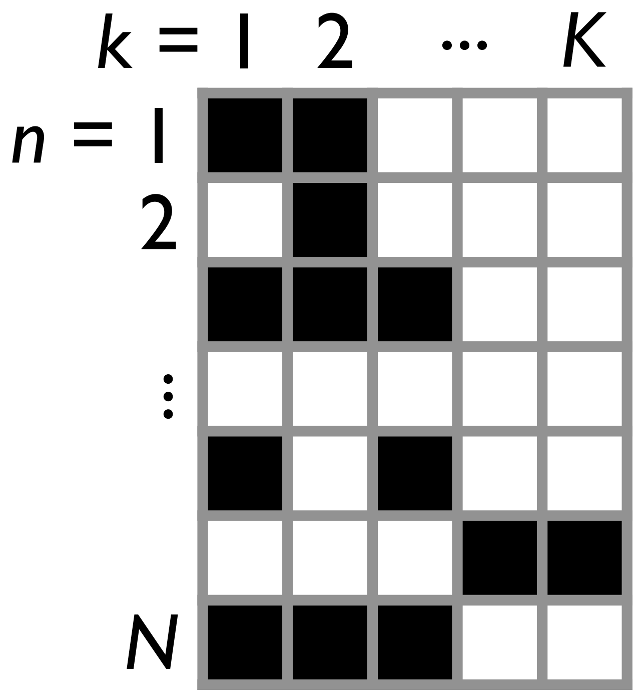

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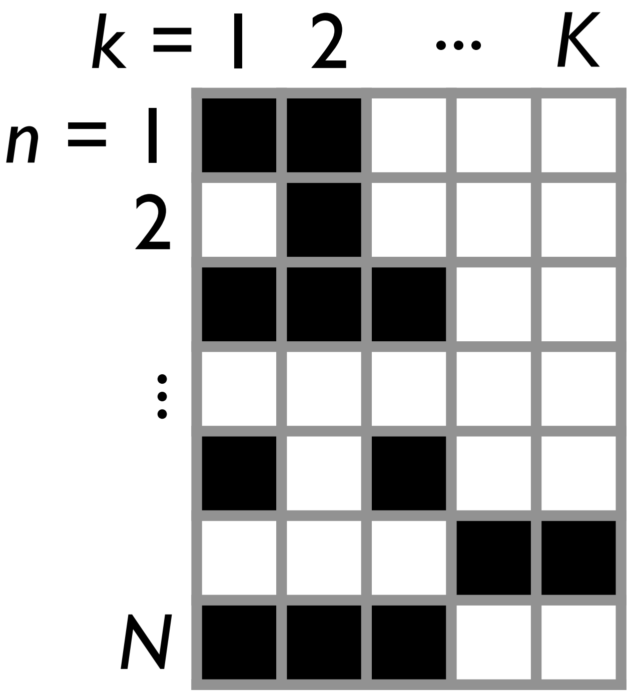

For n = 1, 2, ..., N

[Griffiths, Ghahramani 2006]

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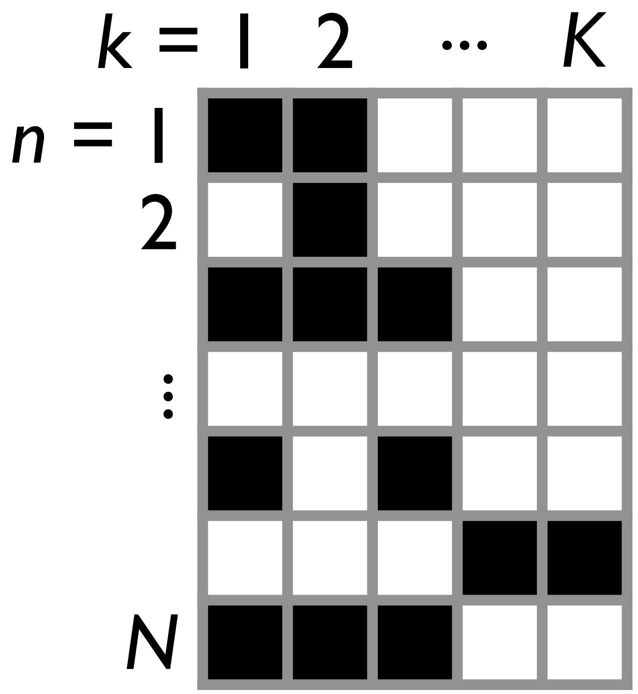

For n = 1, 2, ..., N

1. Data point n has an existing feature k that has already occurred $S_{n-1,k}$ times with probability $\frac{S_{n-1,k}}{\theta + n - 1}$

[Griffiths, Ghahramani 2006]

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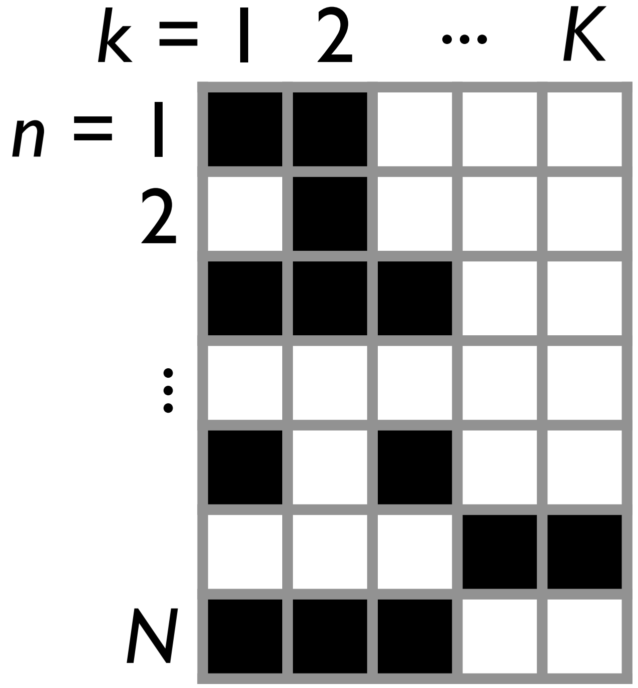

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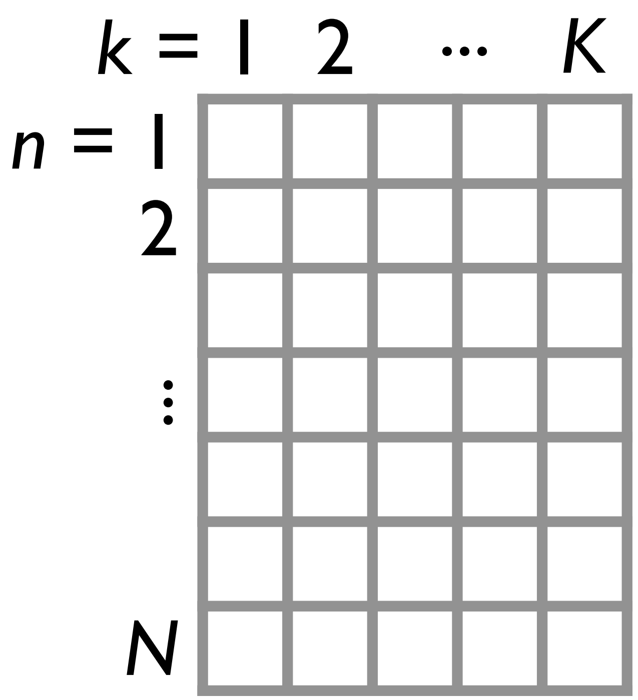

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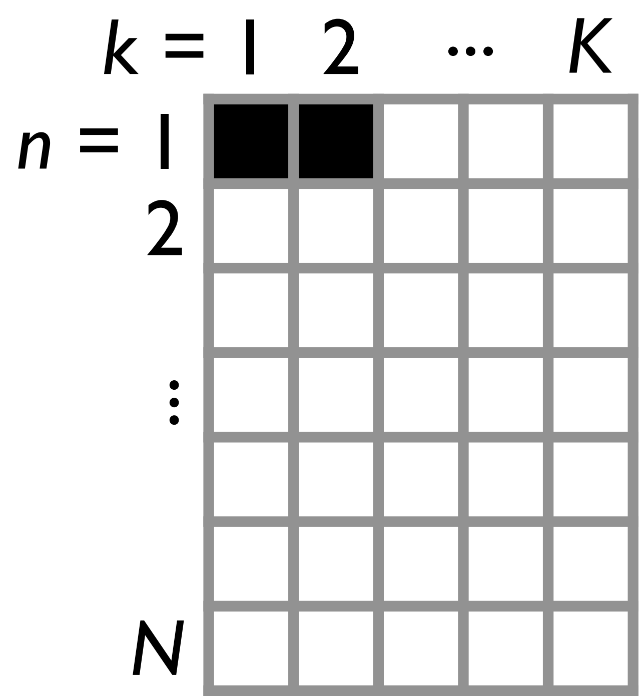

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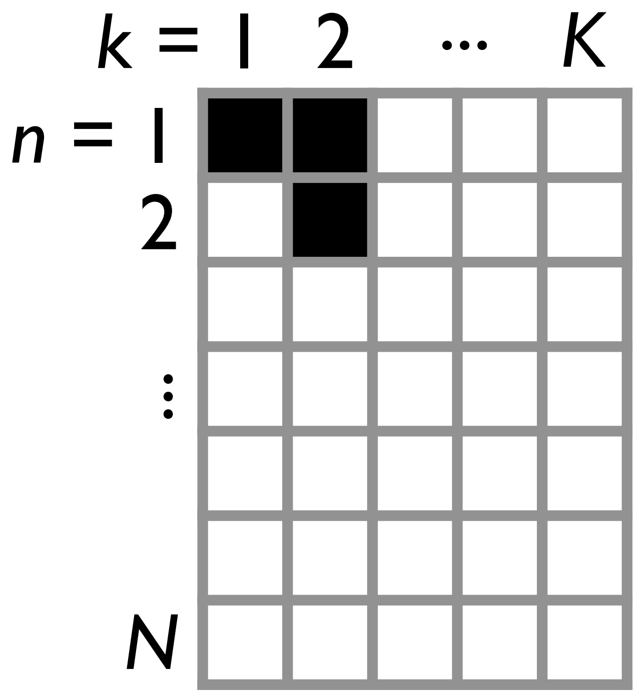

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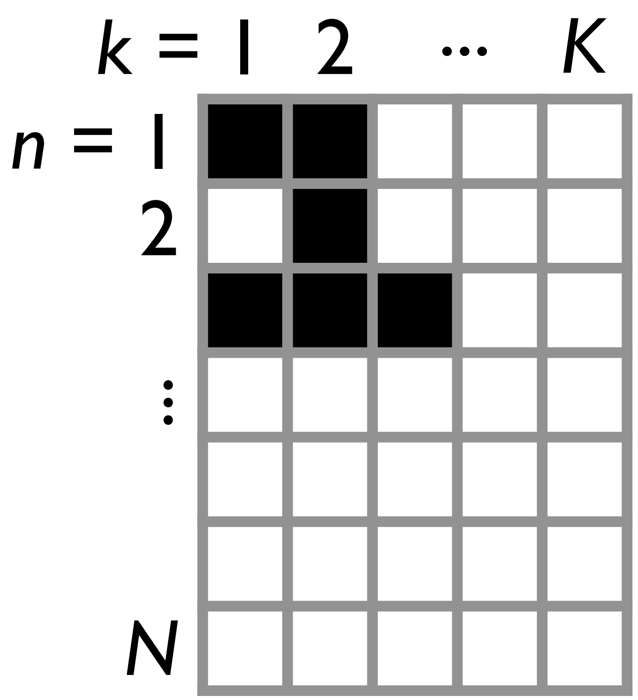

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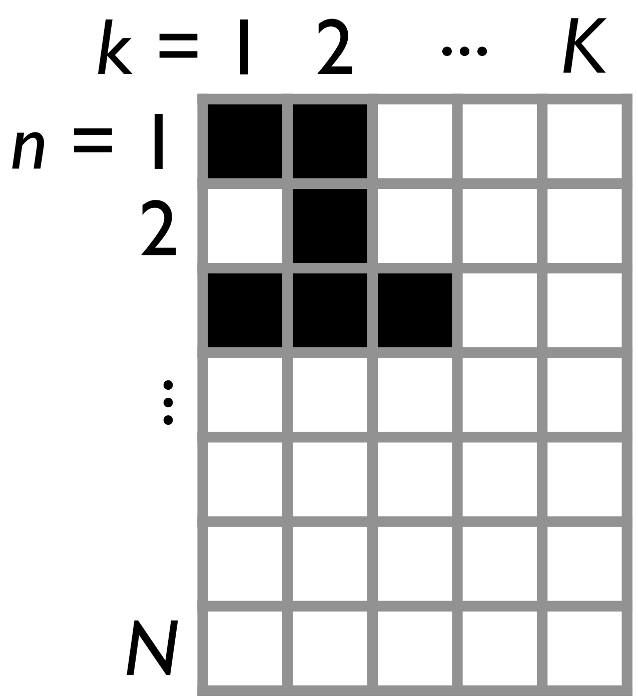

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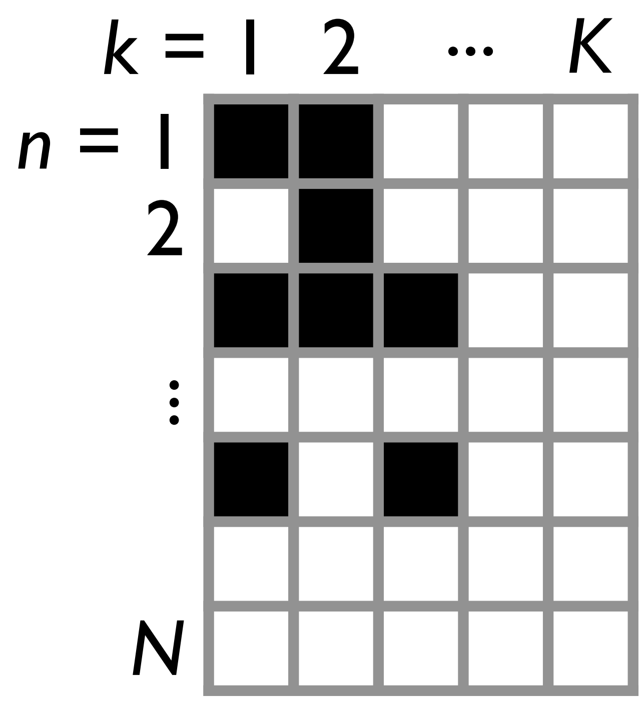

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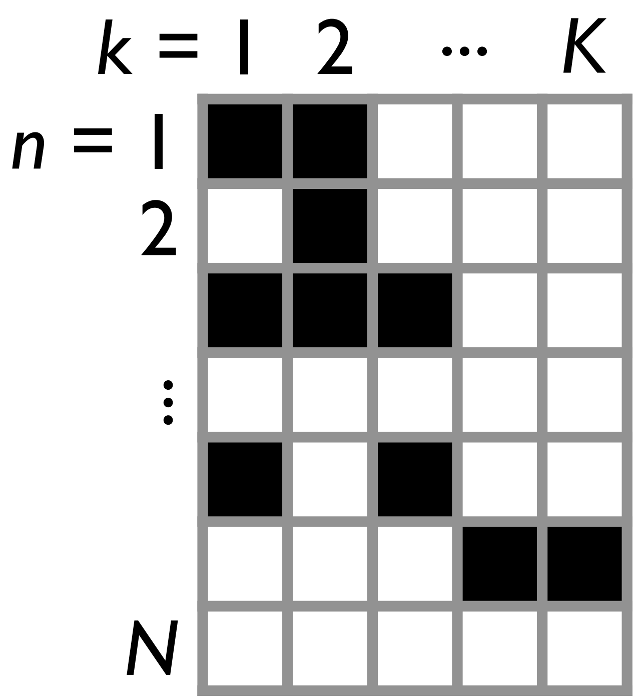

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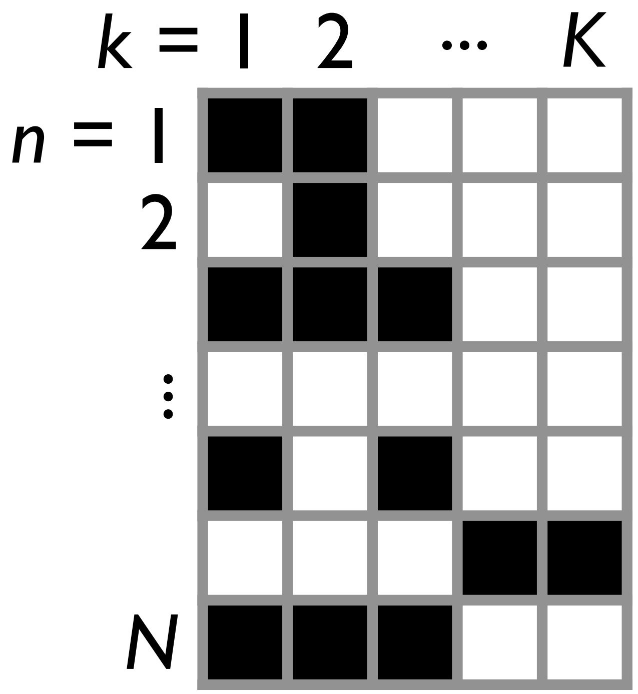

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Exchangeable probability functions

"Exchangeable feature probability function" (EFPF)?

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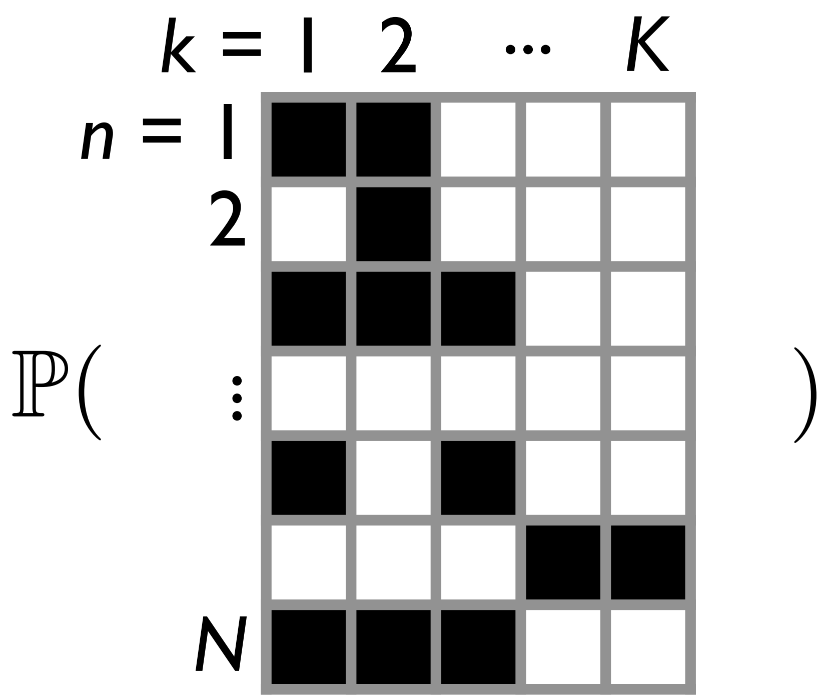
Example: Indian buffet process (IBP)



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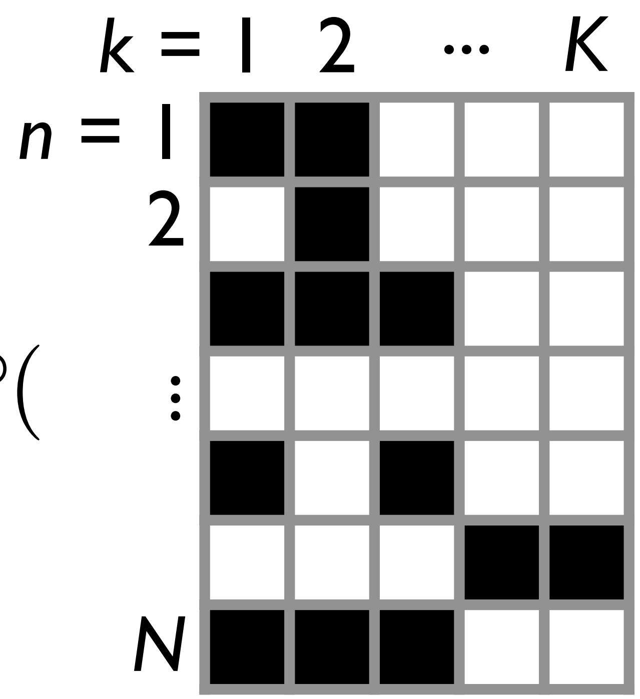

$)$

$$= \frac{1}{K_N!}(\theta\gamma)^{K_N} \exp\left(-\theta\gamma \sum_{n=1}^{N} (\theta + n - 1)^{-1}\right) \prod_{k=1}^{K_N} \frac{\Gamma(S_{N,k})\Gamma(N - S_{N,k} + \theta)}{\Gamma(N + \theta)}$$

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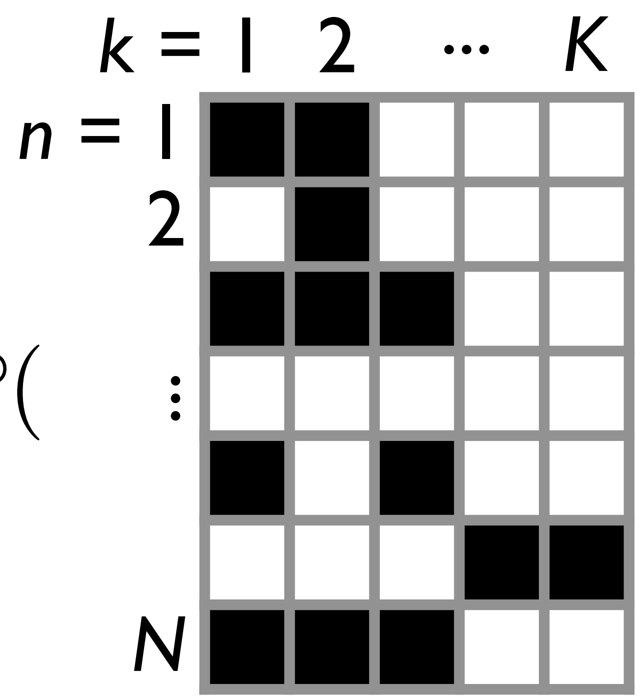
$\mathbb{P}($  $)$

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Size of *k*th feature

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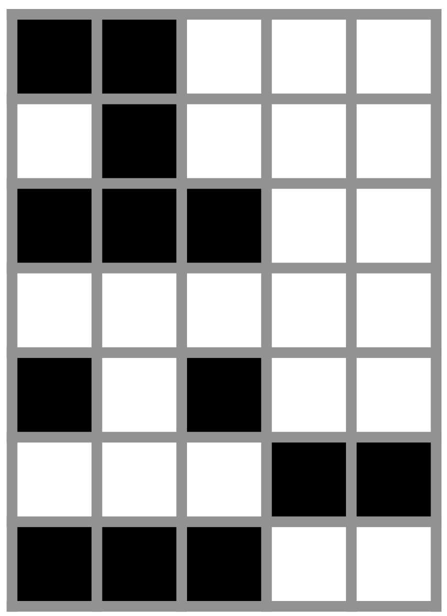
Example: Indian buffet process (IBP)

k = 1 2 ... K

n = 1, 2, ⋮, N

$\mathbb{P}($  $)$

Number of features

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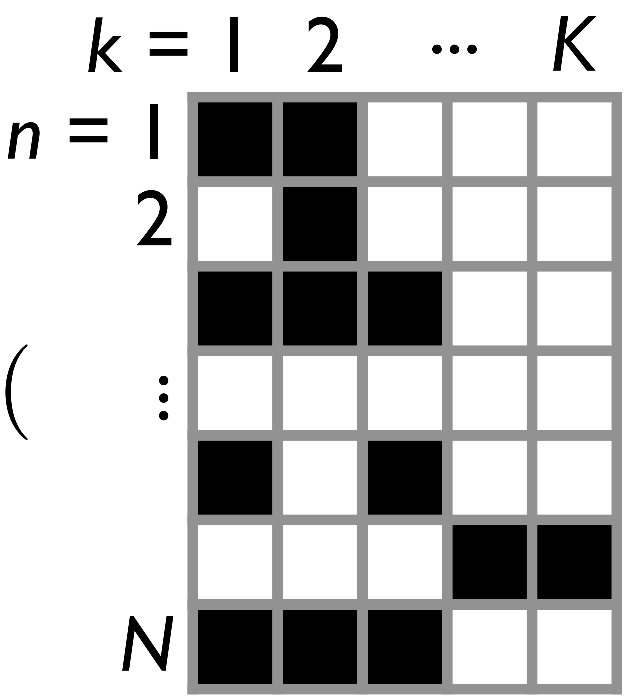
$\mathbb{P}($  $)$

Number of data points

Size of kth feature

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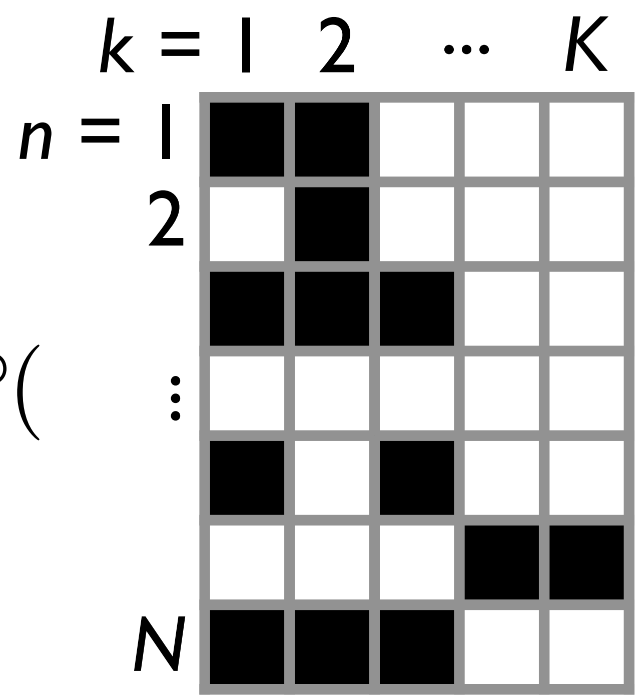
$\mathbb{P}($ 

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$$= p(N; S_{N,1}, S_{N,2}, \ldots, S_{N,K})$$

[Broderick, Jordan, Pitman 2013]

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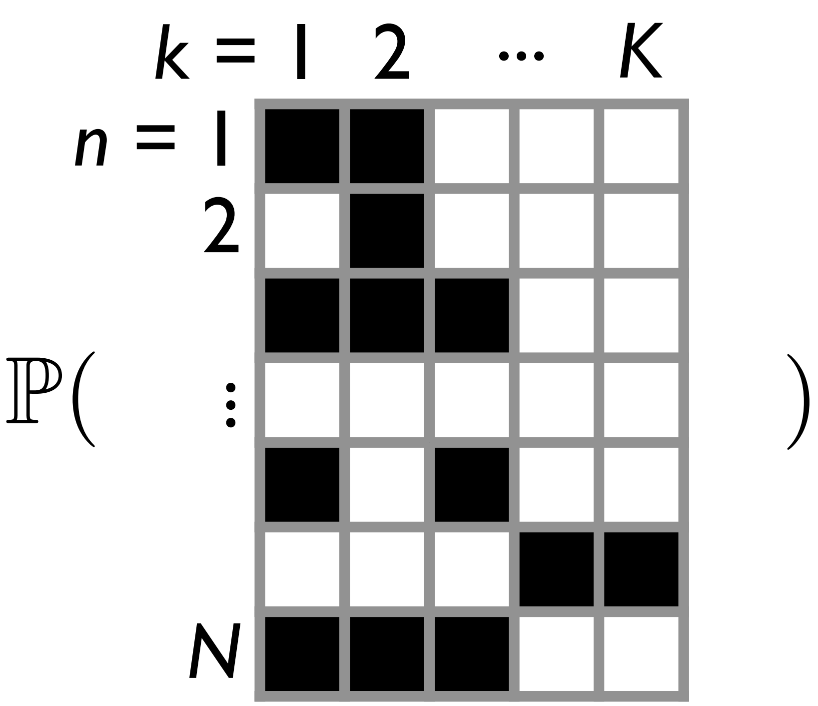

Number of data points

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"EFPF"

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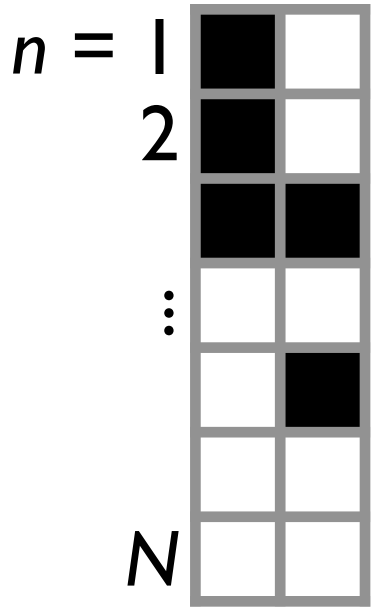
Counterexample



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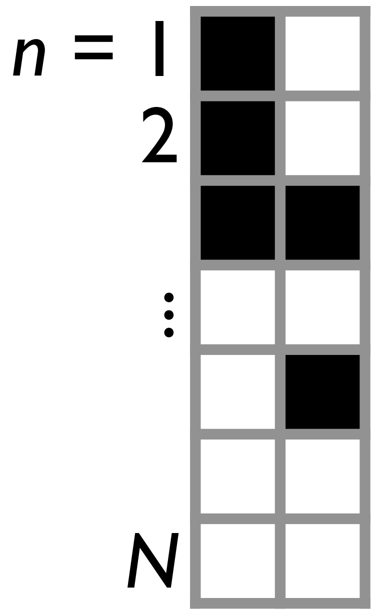

$\mathbb{P}(\text{row} =$  $) = p_1$

$\mathbb{P}(\text{row} =$  $) = p_2$

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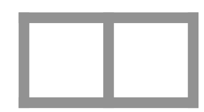
$\mathbb{P}(\text{row} =$  $) = p_4$

[Broderick, Jordan, Pitman 2013]

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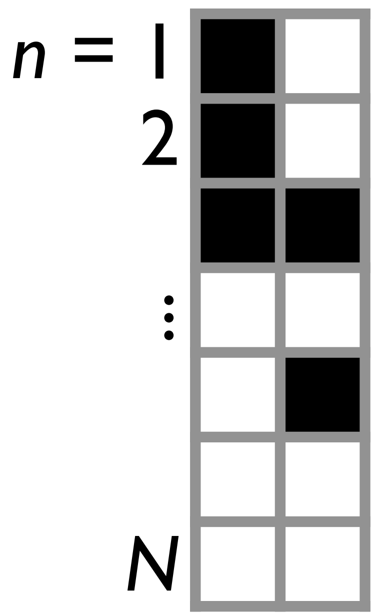

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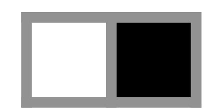
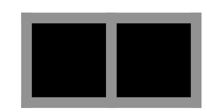
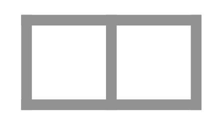
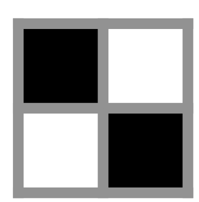
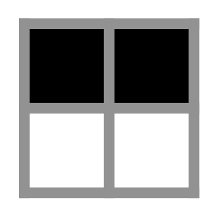
$\mathbb{P}(\text{row} =$  $) = p_2$

$\mathbb{P}(\text{row} =$  $) = p_3$

$\mathbb{P}(\text{row} =$  $) = p_4$





[Broderick, Jordan, Pitman 2013]

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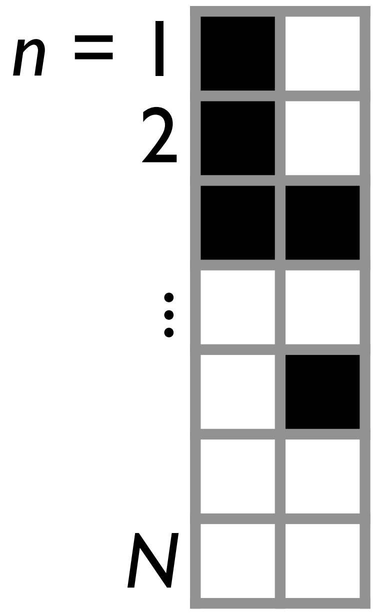

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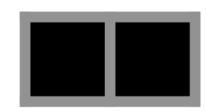
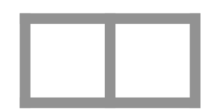
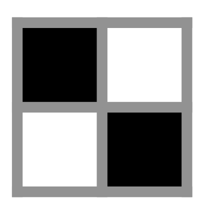
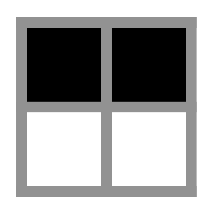
$\mathbb{P}(\text{row} =$  $) = p_3$

$\mathbb{P}(\text{row} =$  $) = p_4$

$\mathbb{P}($  $)$ $\mathbb{P}($  $)$

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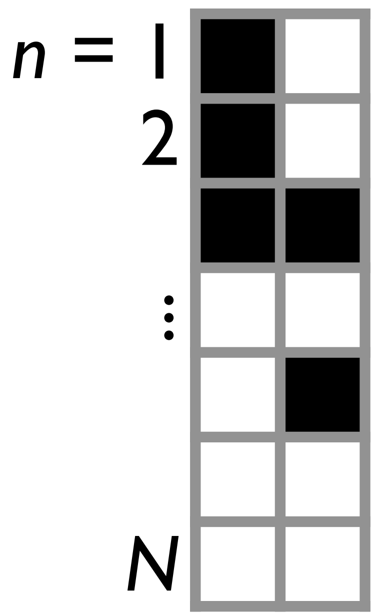

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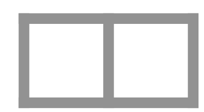
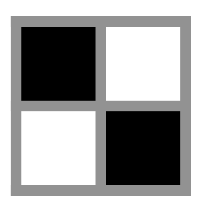
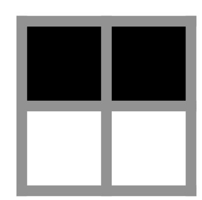
$\mathbb{P}(\text{row} =$  $) = p_4$

$\mathbb{P}($  $)$ $\quad$ $\mathbb{P}($  $)$

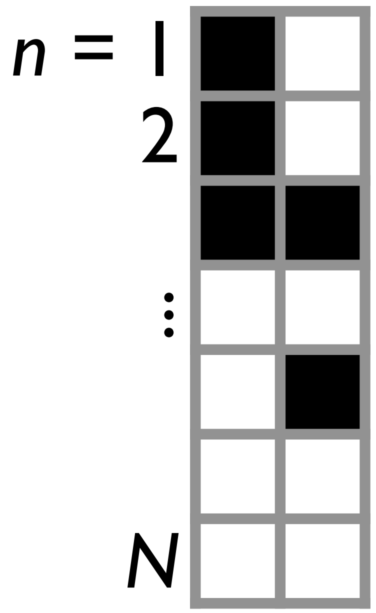
$p_1 p_2$ $\quad$ $p_3 p_4$

[Broderick, Jordan, Pitman 2013]

Exchangeable probability functions

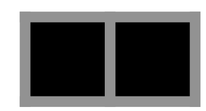
"Exchangeable feature probability function" (EFPF)?

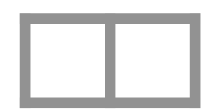
Counterexample

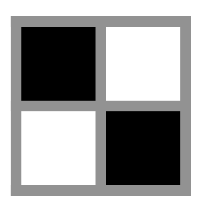
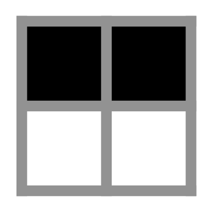


$\mathbb{P}(\text{row} =$  $) = p_1$

$\mathbb{P}(\text{row} =$  $) = p_2$

$\mathbb{P}(\text{row} =$  $) = p_3$

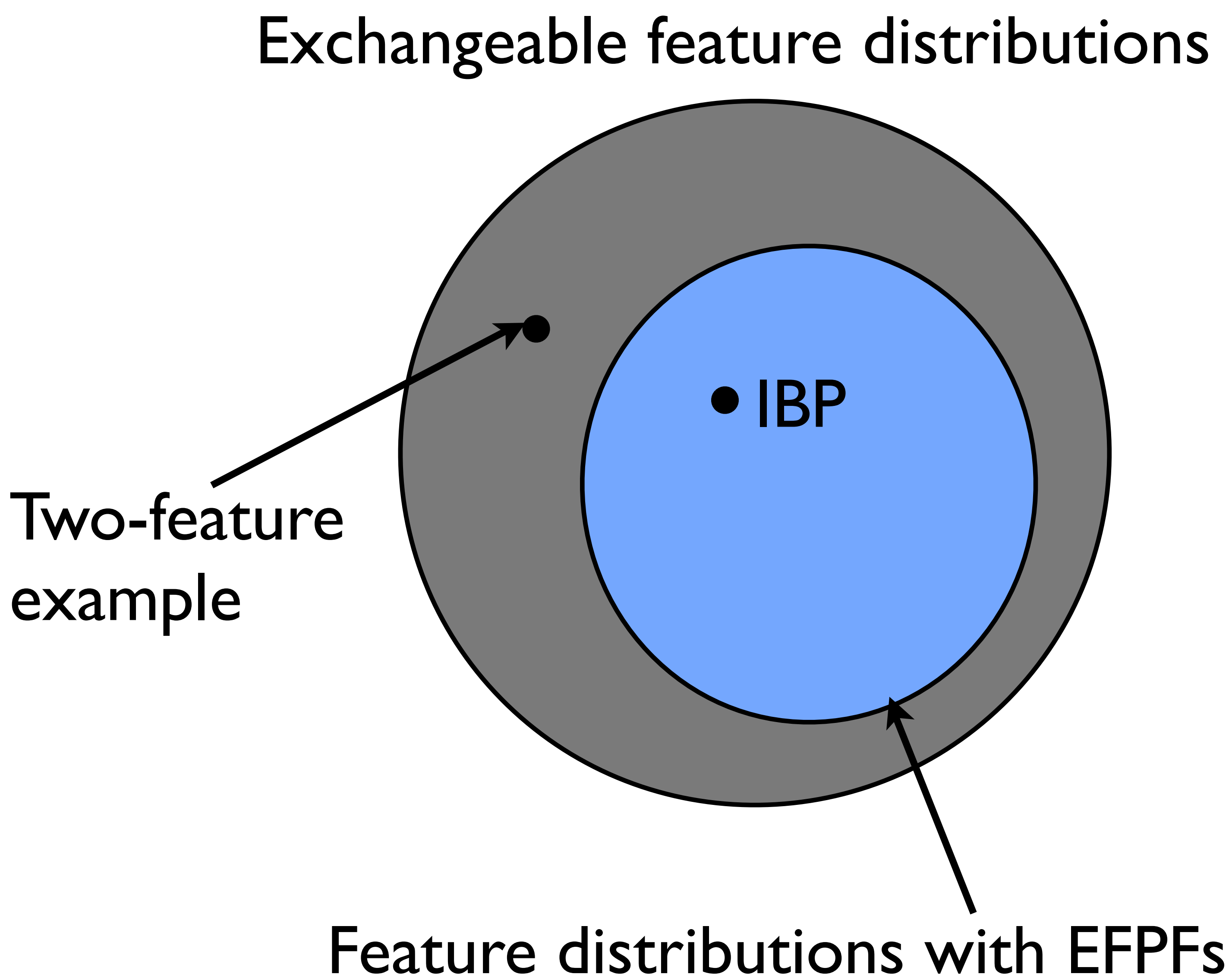
$\mathbb{P}(\text{row} =$  $) = p_4$

$\mathbb{P}($  $) \neq \mathbb{P}($  $)$

$$p_1 p_2 \neq p_3 p_4$$

[Broderick, Jordan, Pitman 2013]

Exchangeable probability functions



[Broderick, Jordan, Pitman 2013]

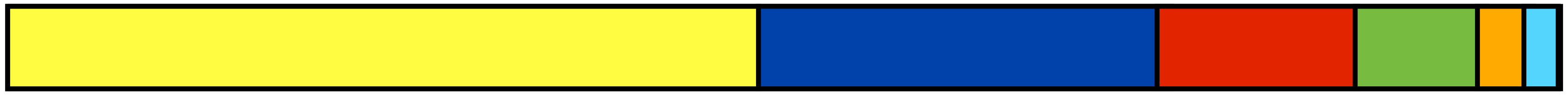
Paintboxes

Exchangeable partition: Kingman paintbox

[Kingman 1978]

Paintboxes

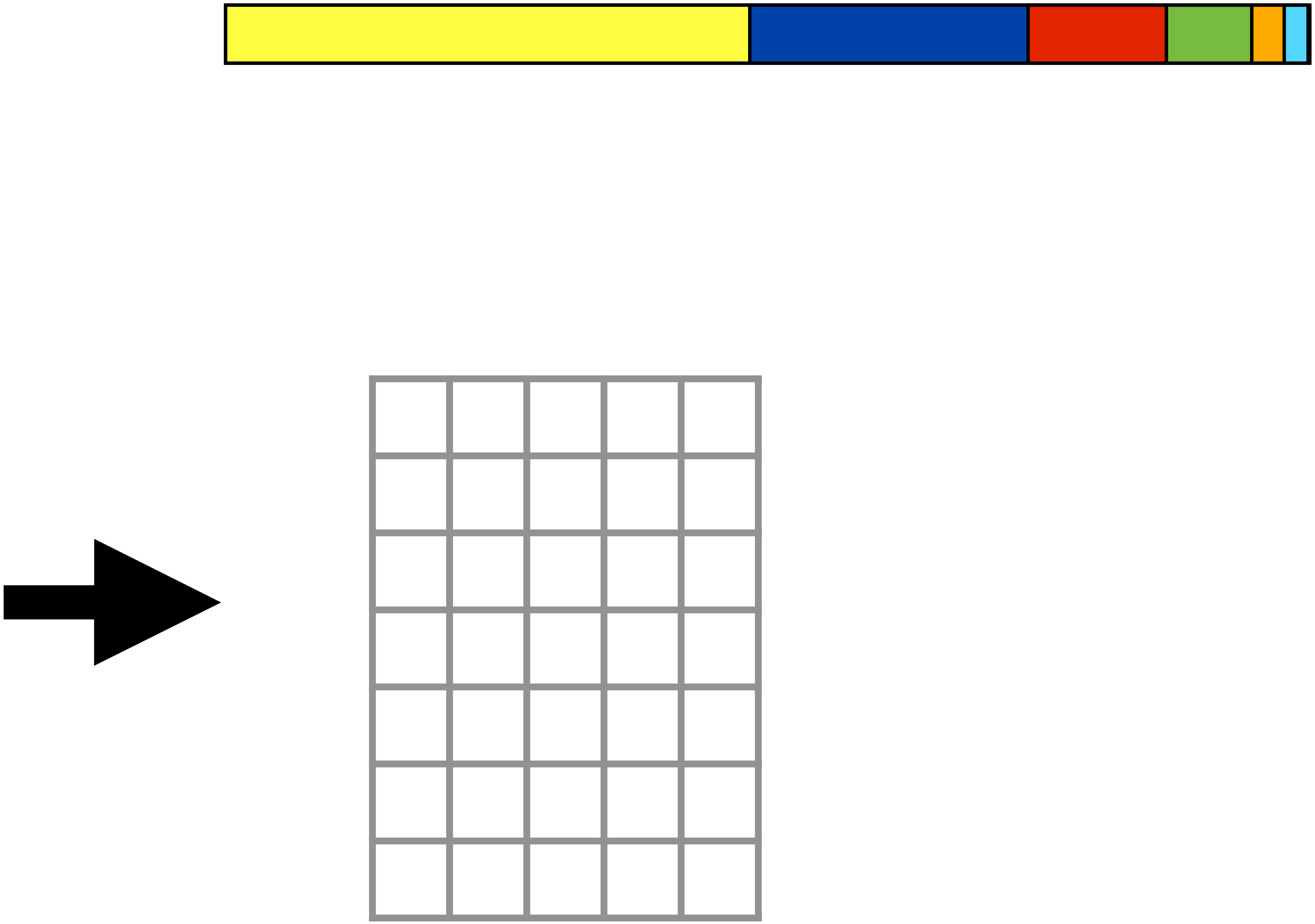
Exchangeable partition: Kingman paintbox



[Kingman 1978]

Paintboxes

Exchangeable partition: Kingman paintbox



[Kingman 1978]

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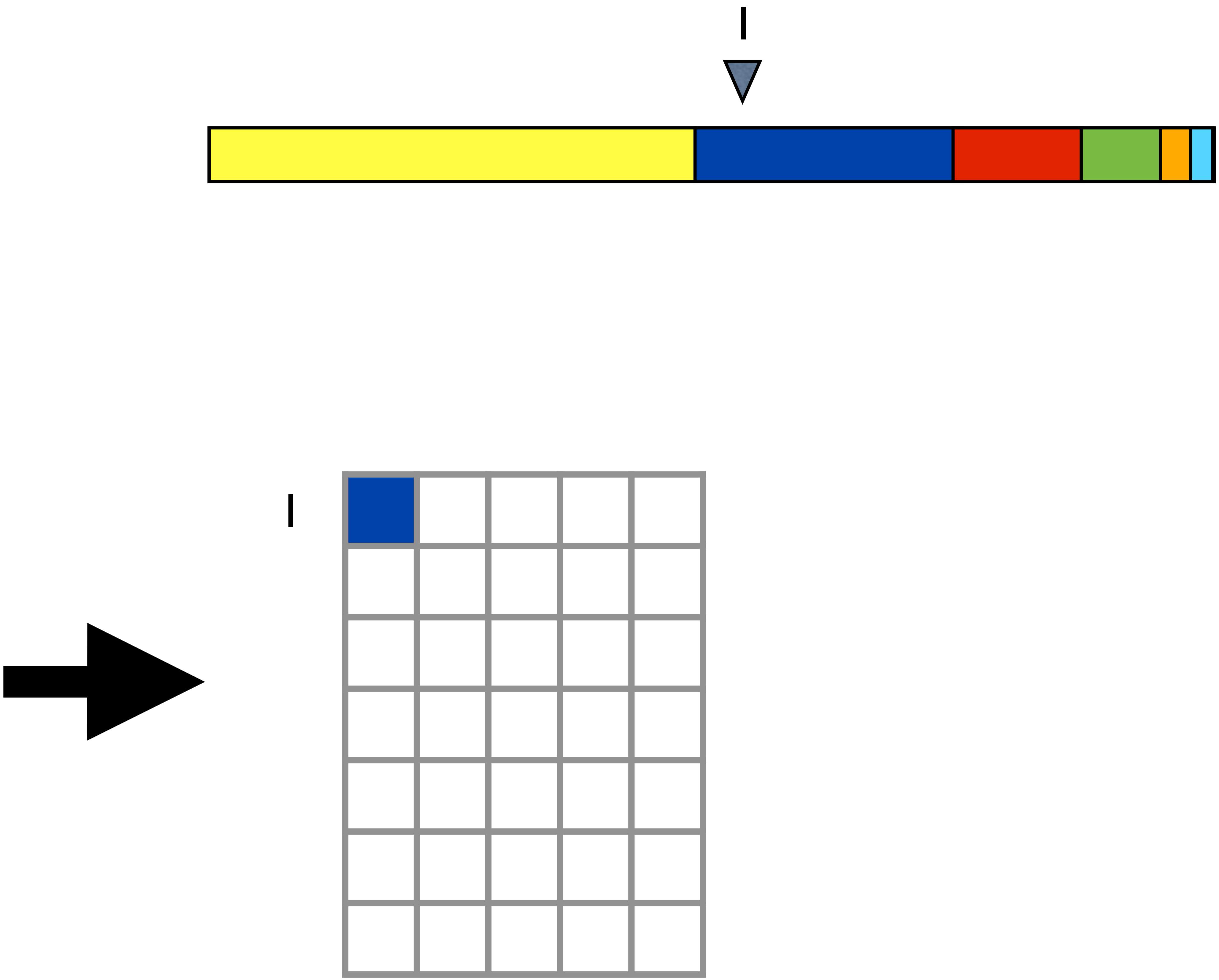
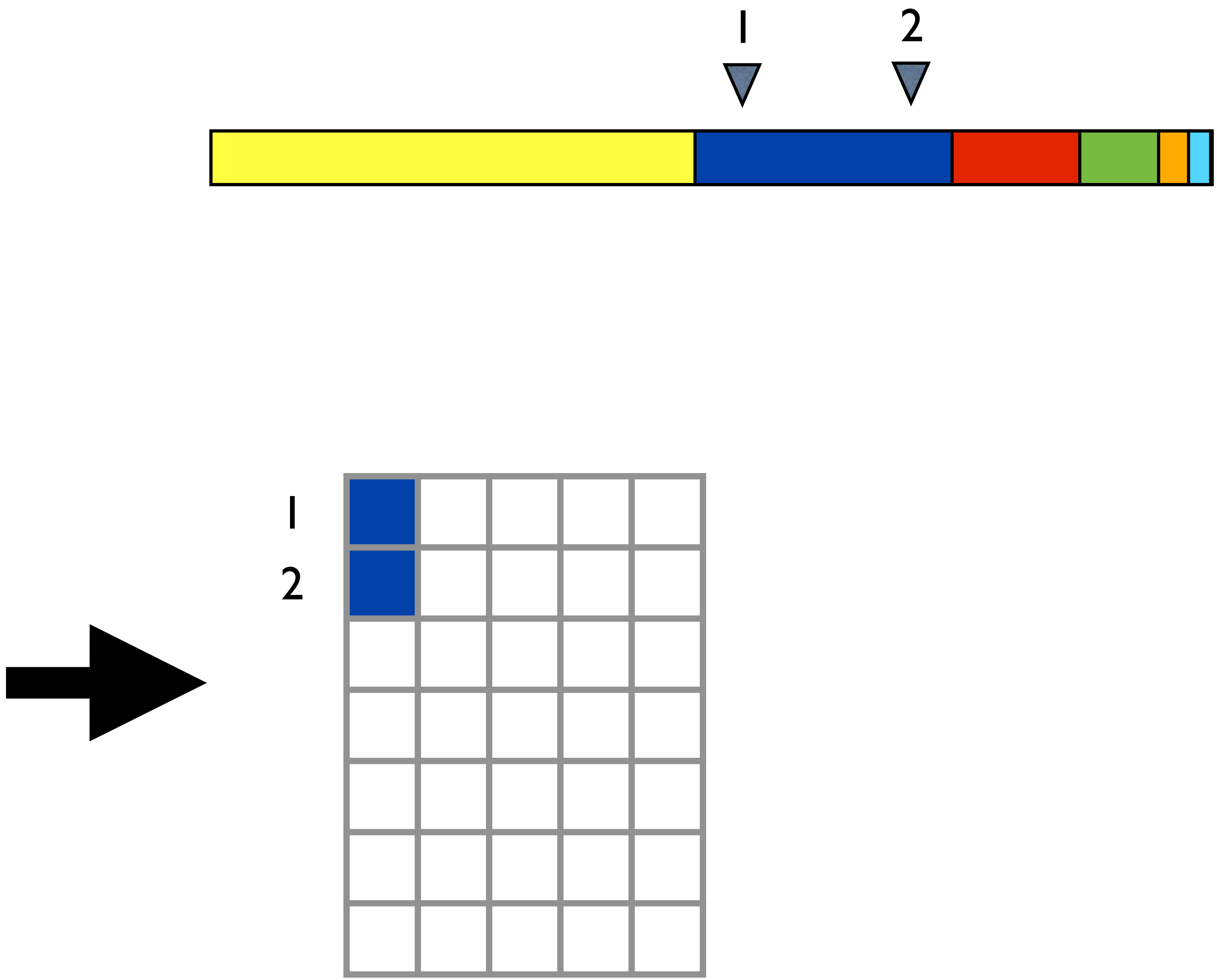
Exchangeable partition: Kingman paintbox



[Kingman 1978]

Paintboxes

Exchangeable partition: Kingman paintbox



[Kingman 1978]

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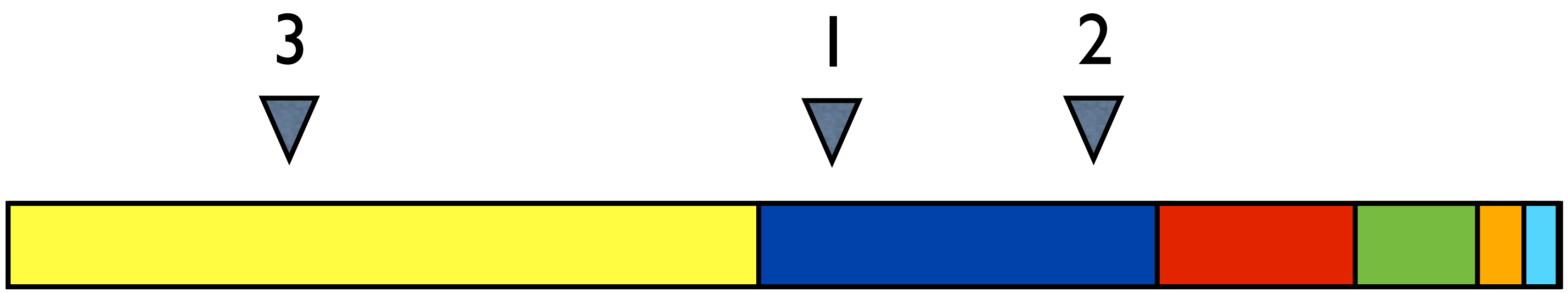
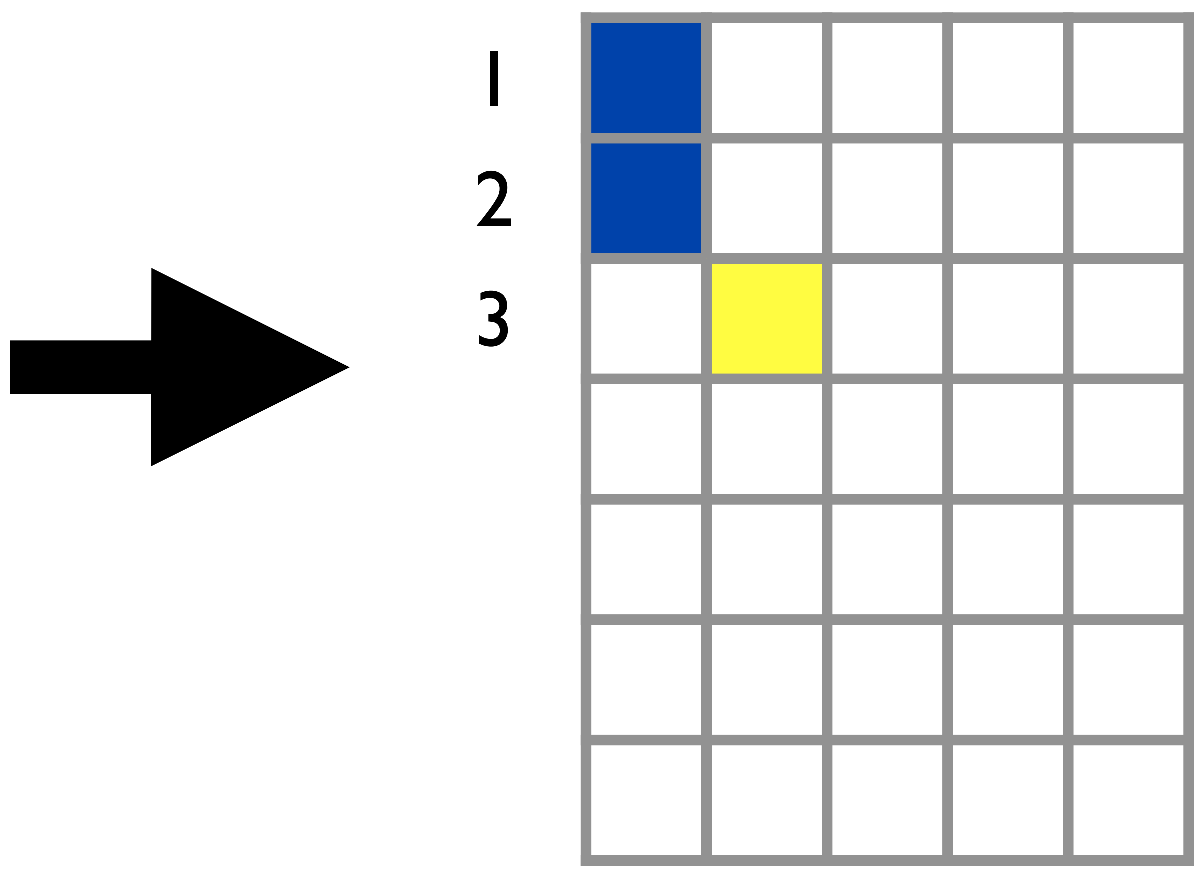
Exchangeable partition: Kingman paintbox





[Kingman 1978]

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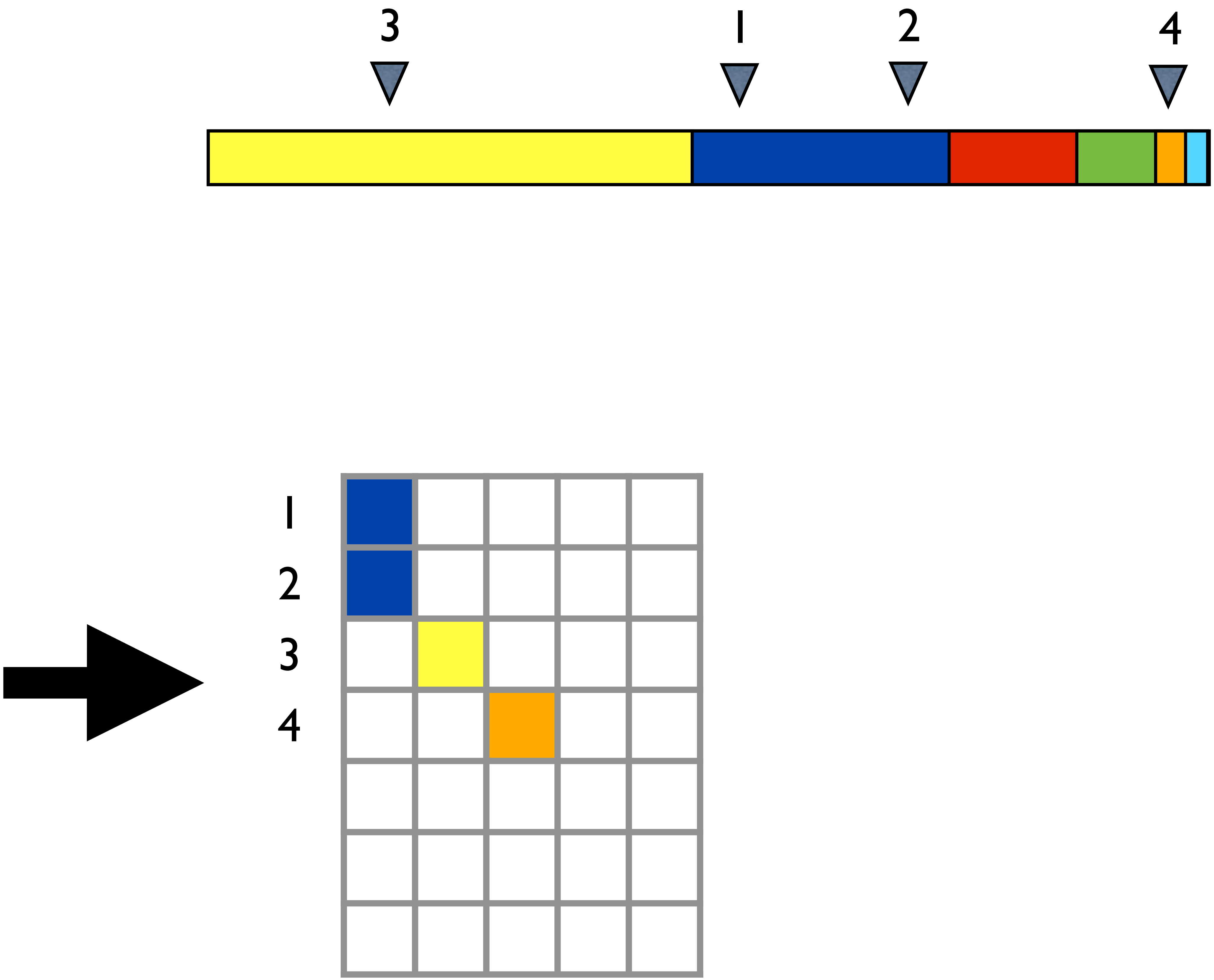

[Kingman 1978]

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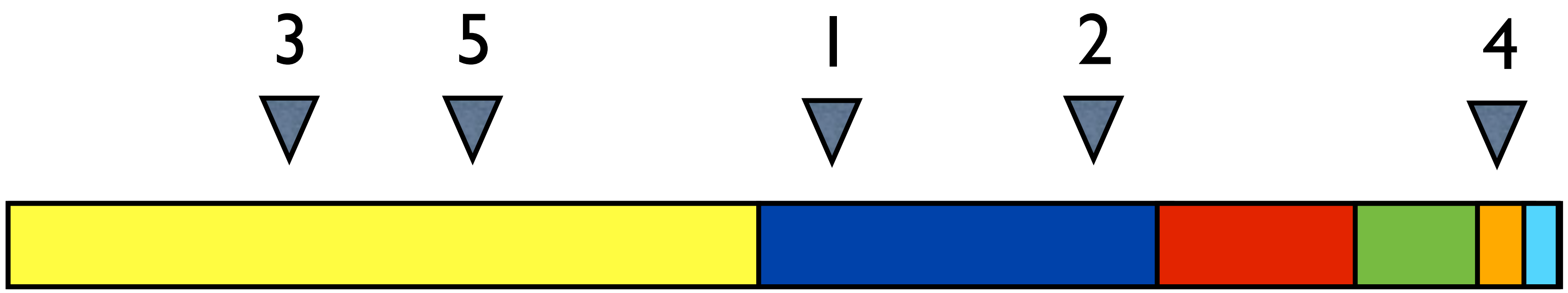
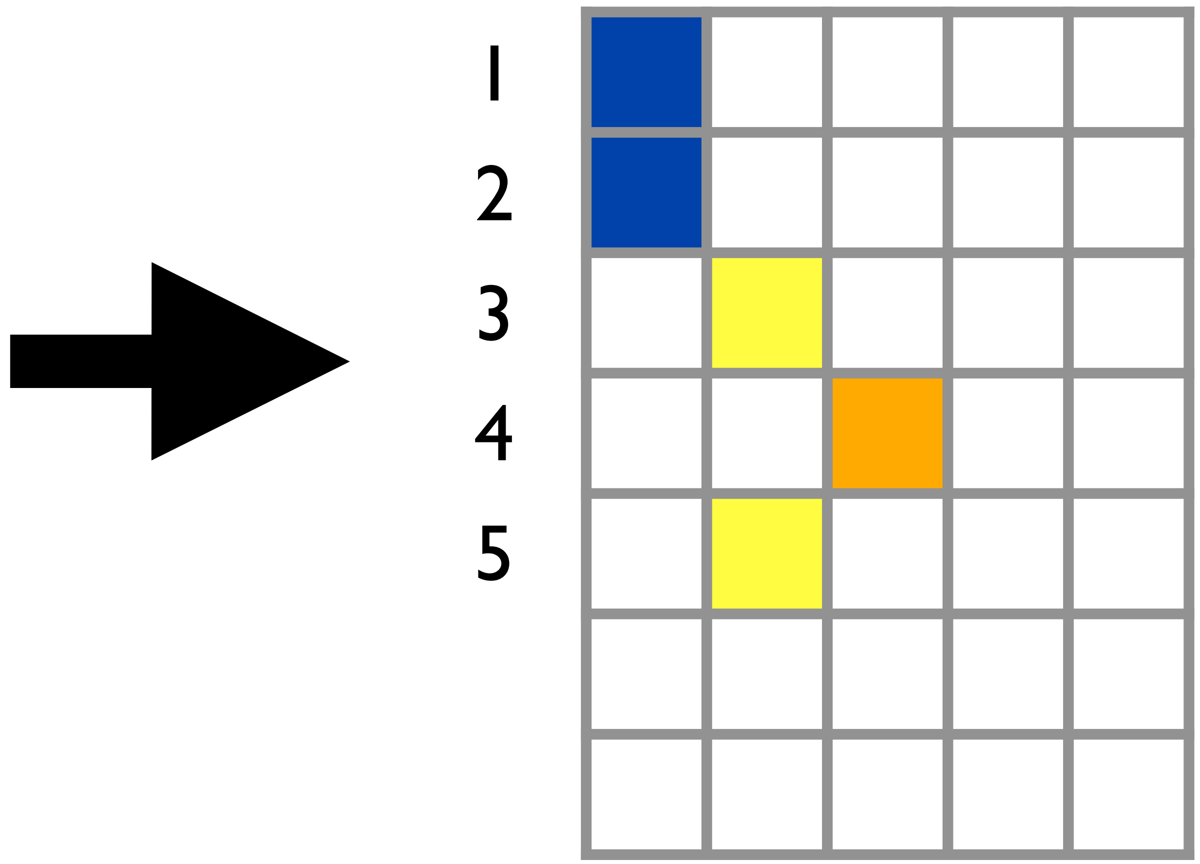
Exchangeable partition: Kingman paintbox





[Kingman 1978]

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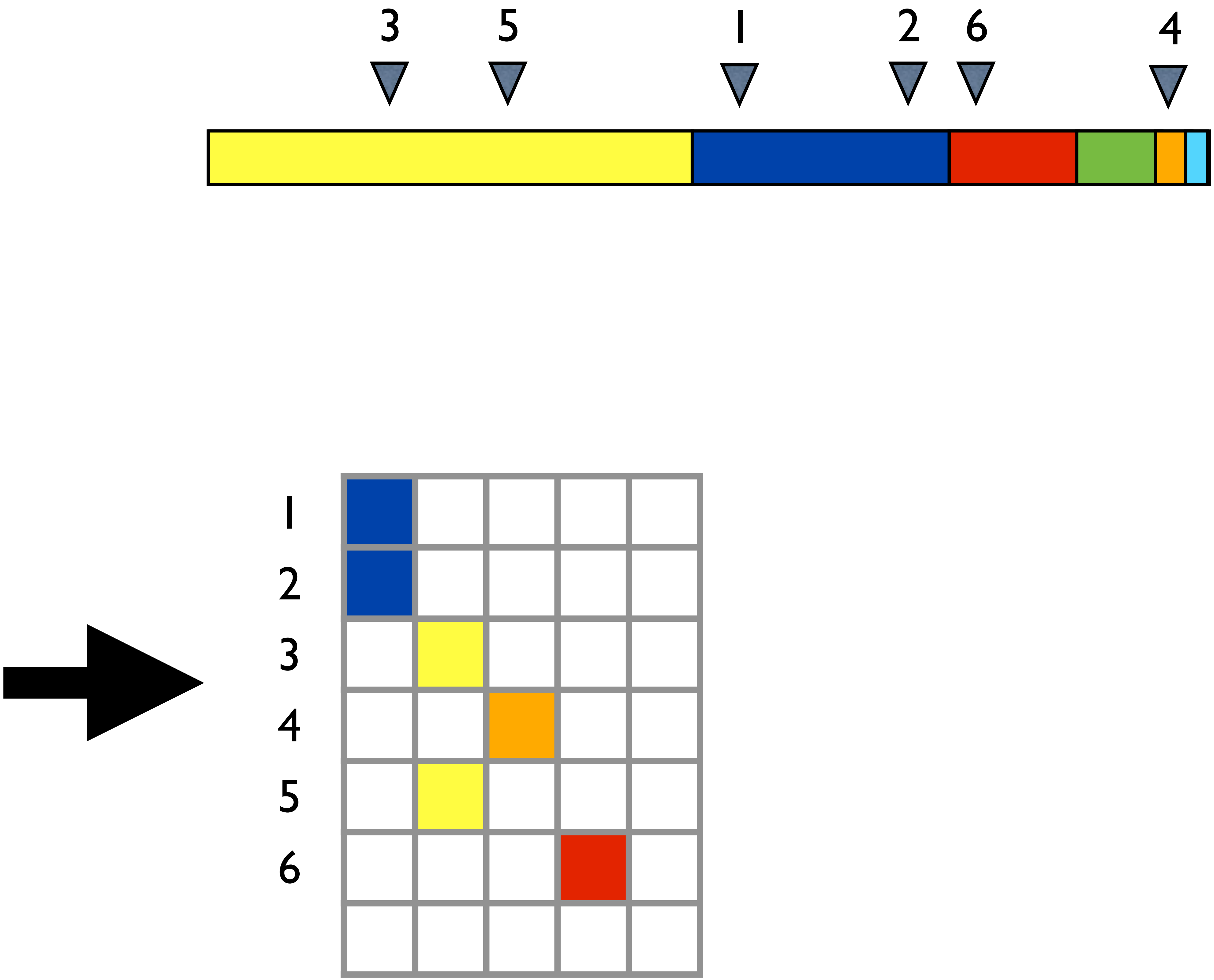

[Kingman 1978]

Paintboxes

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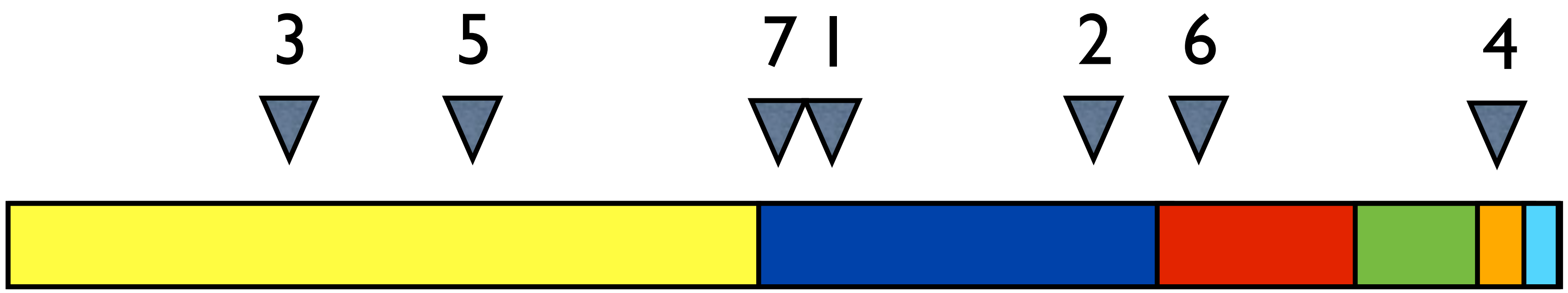

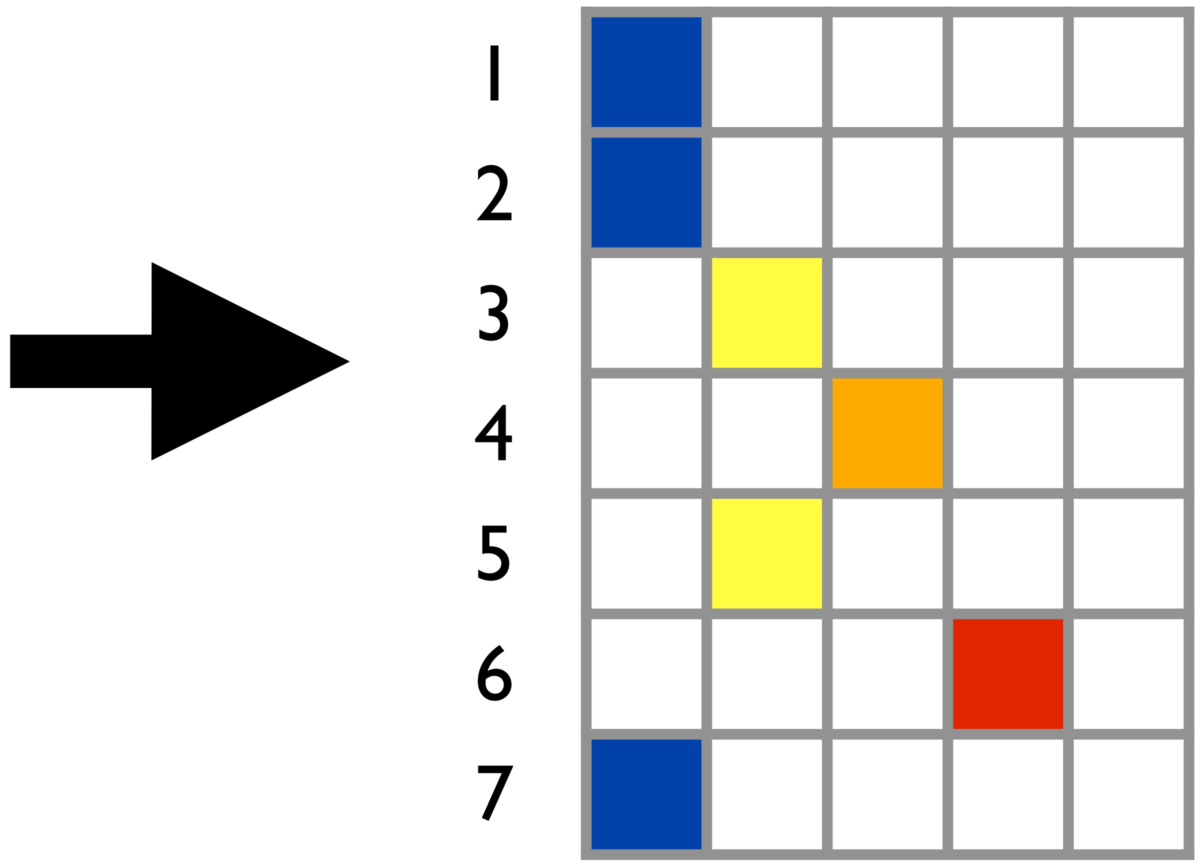

[Kingman 1978]

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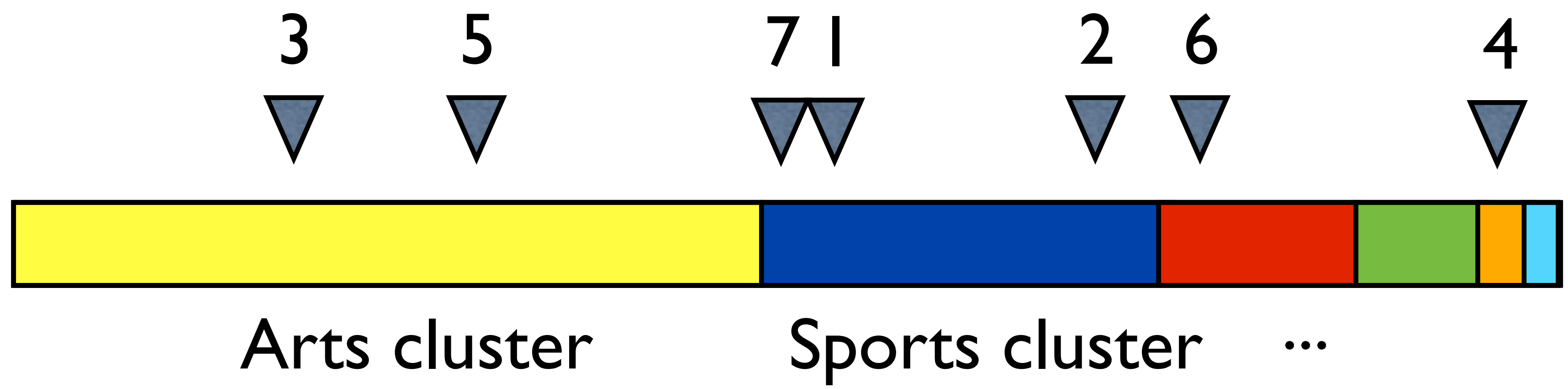

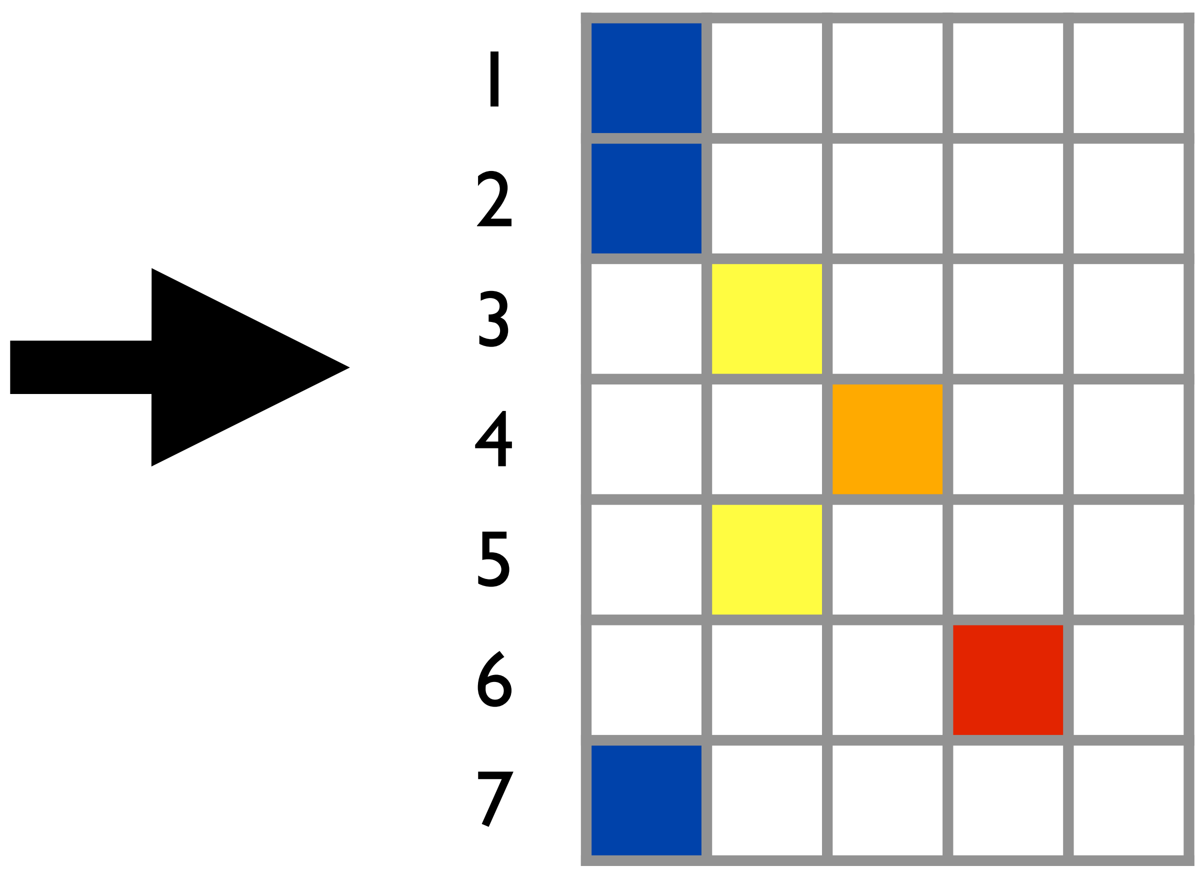

[Kingman 1978]

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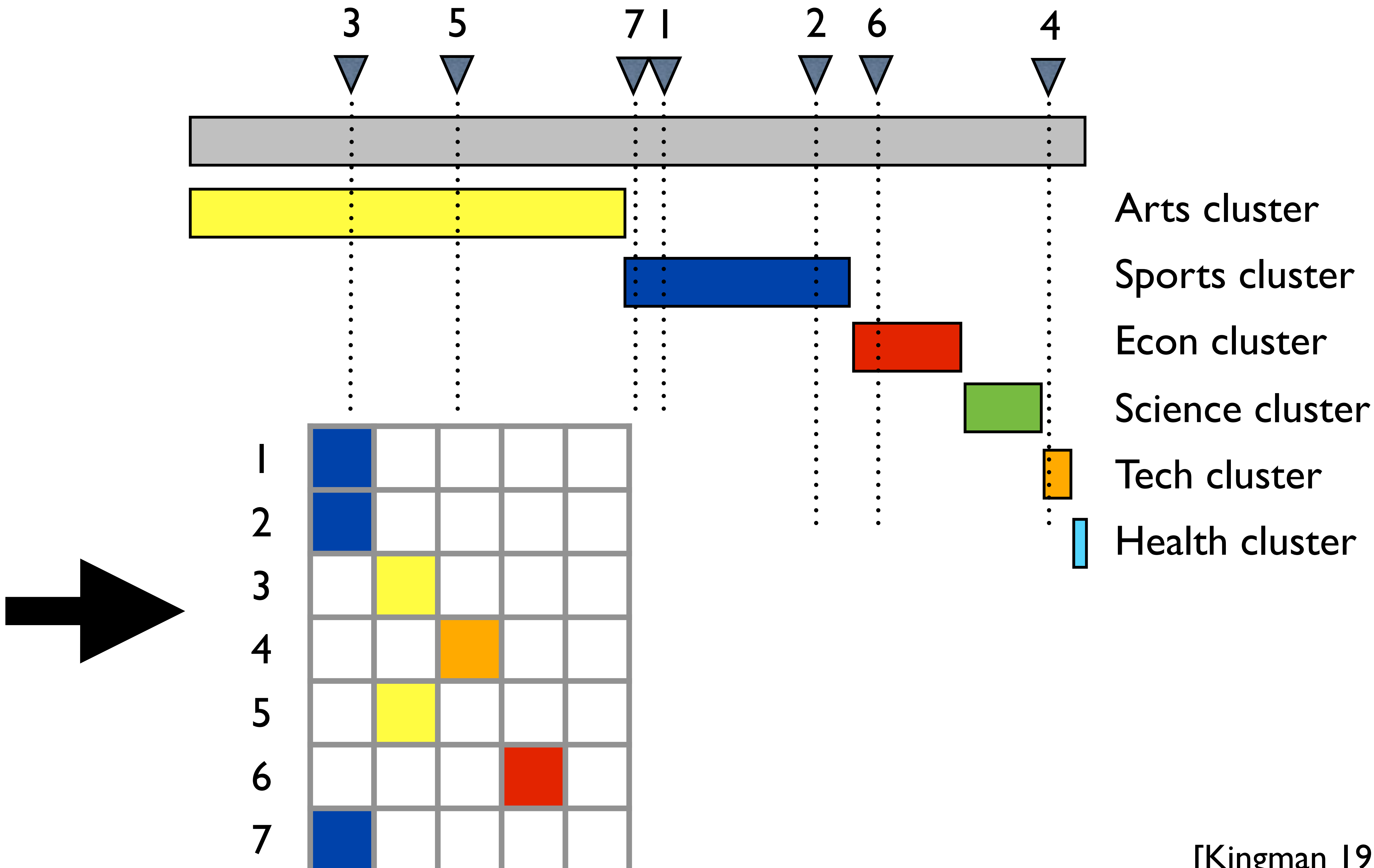
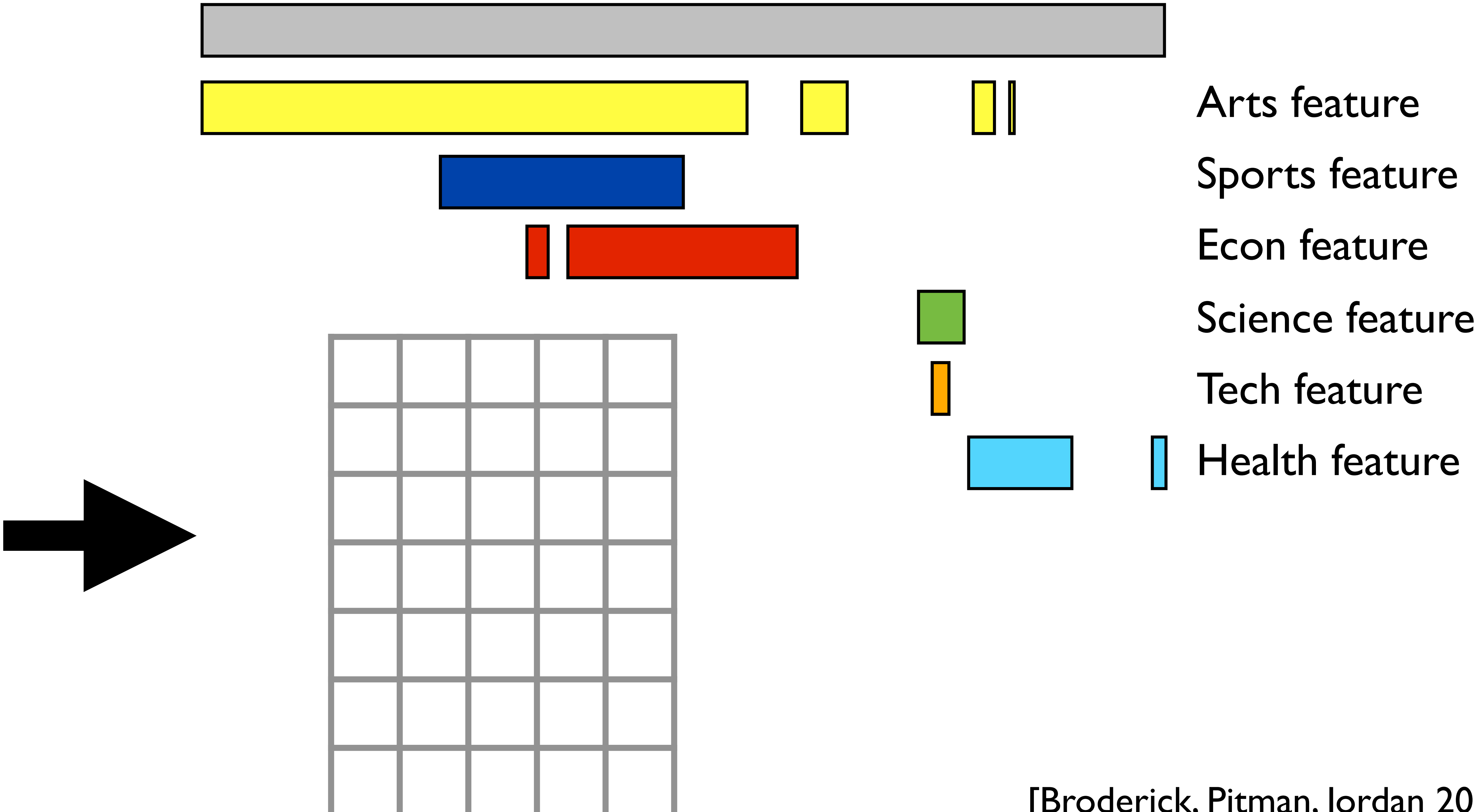


[Kingman 1978]

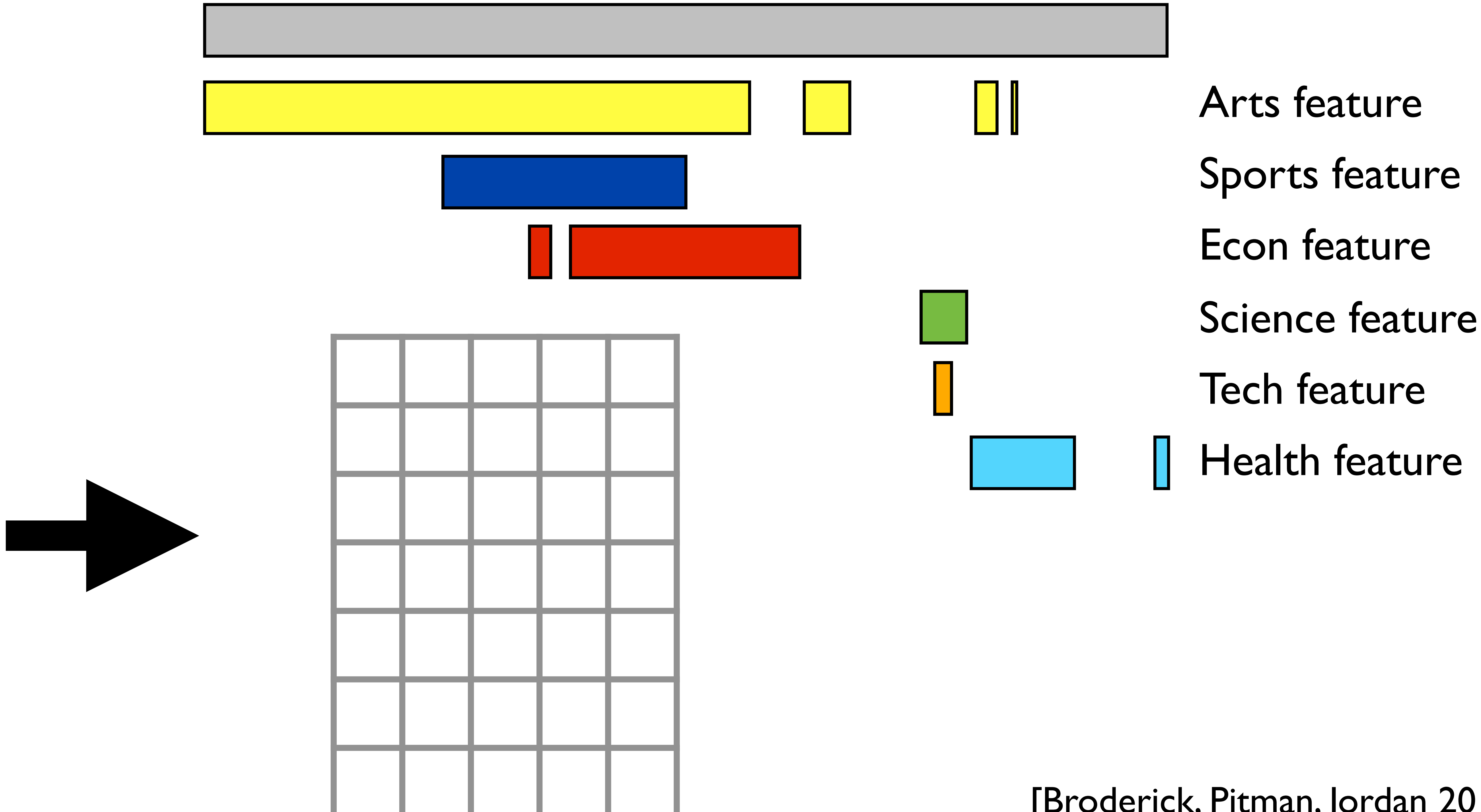
Paintboxes



[Broderick, Pitman, Jordan 2013]

Paintboxes

Exchangeable feature allocation: feature paintbox



[Broderick, Pitman, Jordan 2013]

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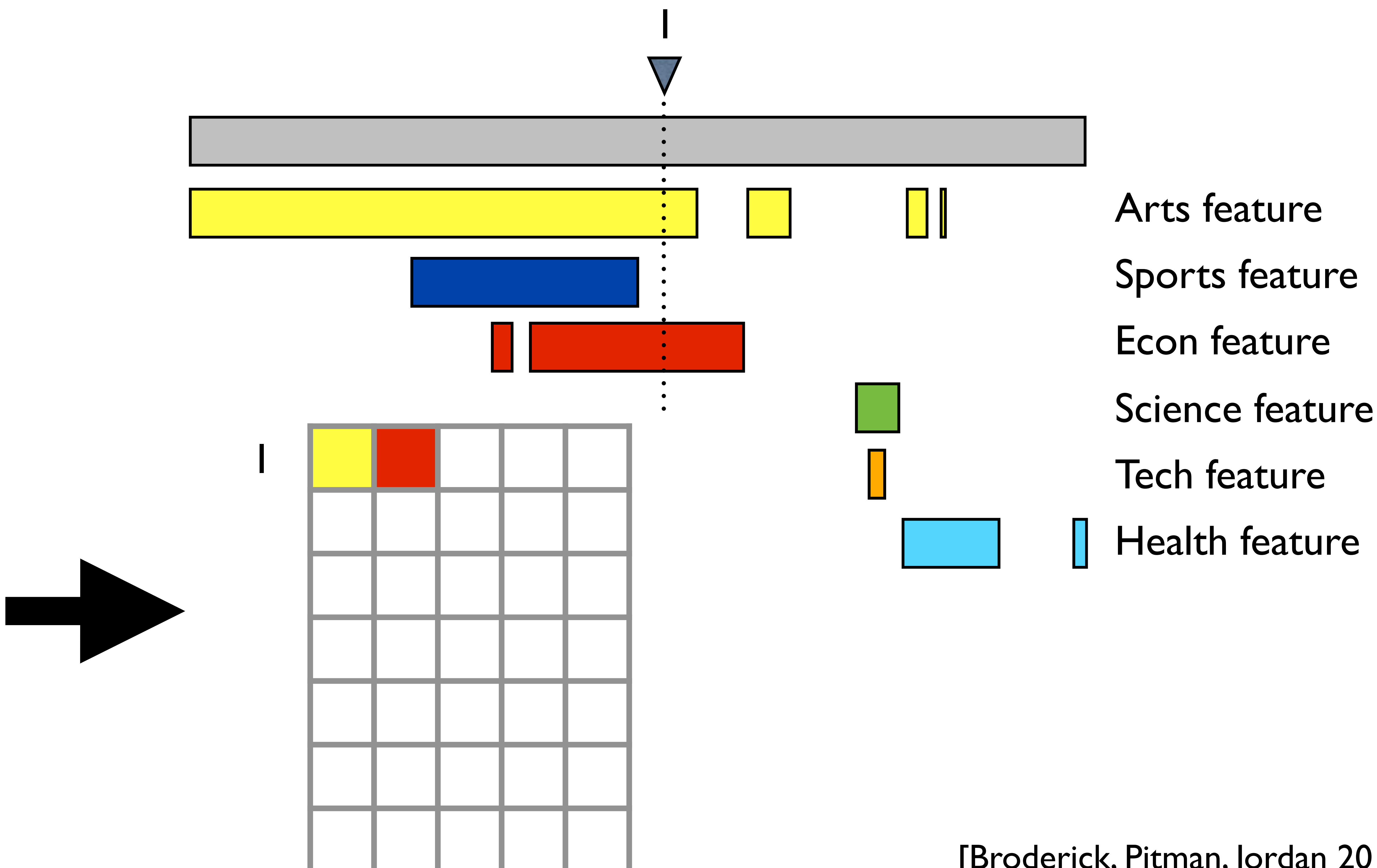
Exchangeable feature allocation: feature paintbox



[Broderick, Pitman, Jordan 2013]

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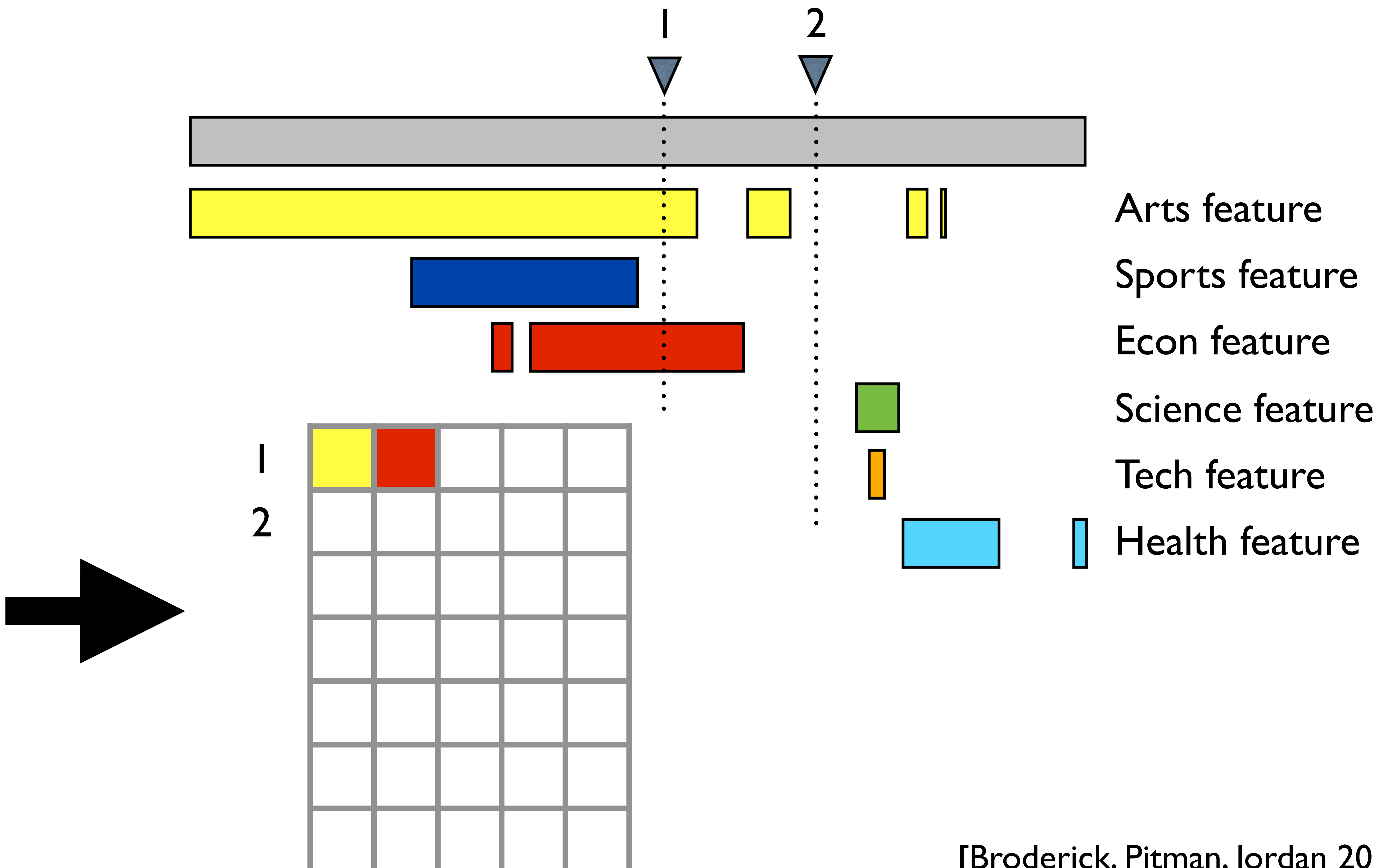
Exchangeable feature allocation: feature paintbox



[Broderick, Pitman, Jordan 2013]

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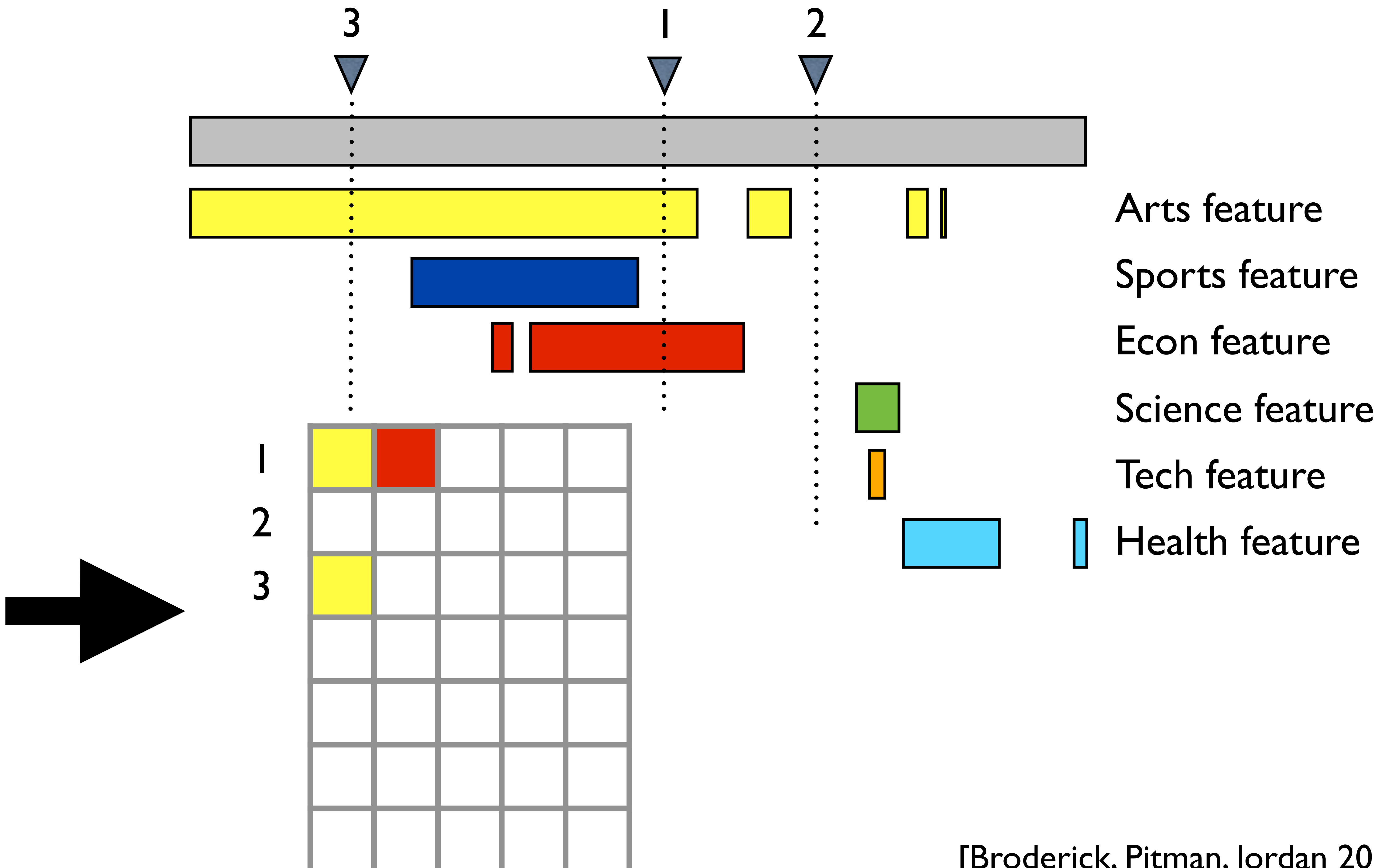

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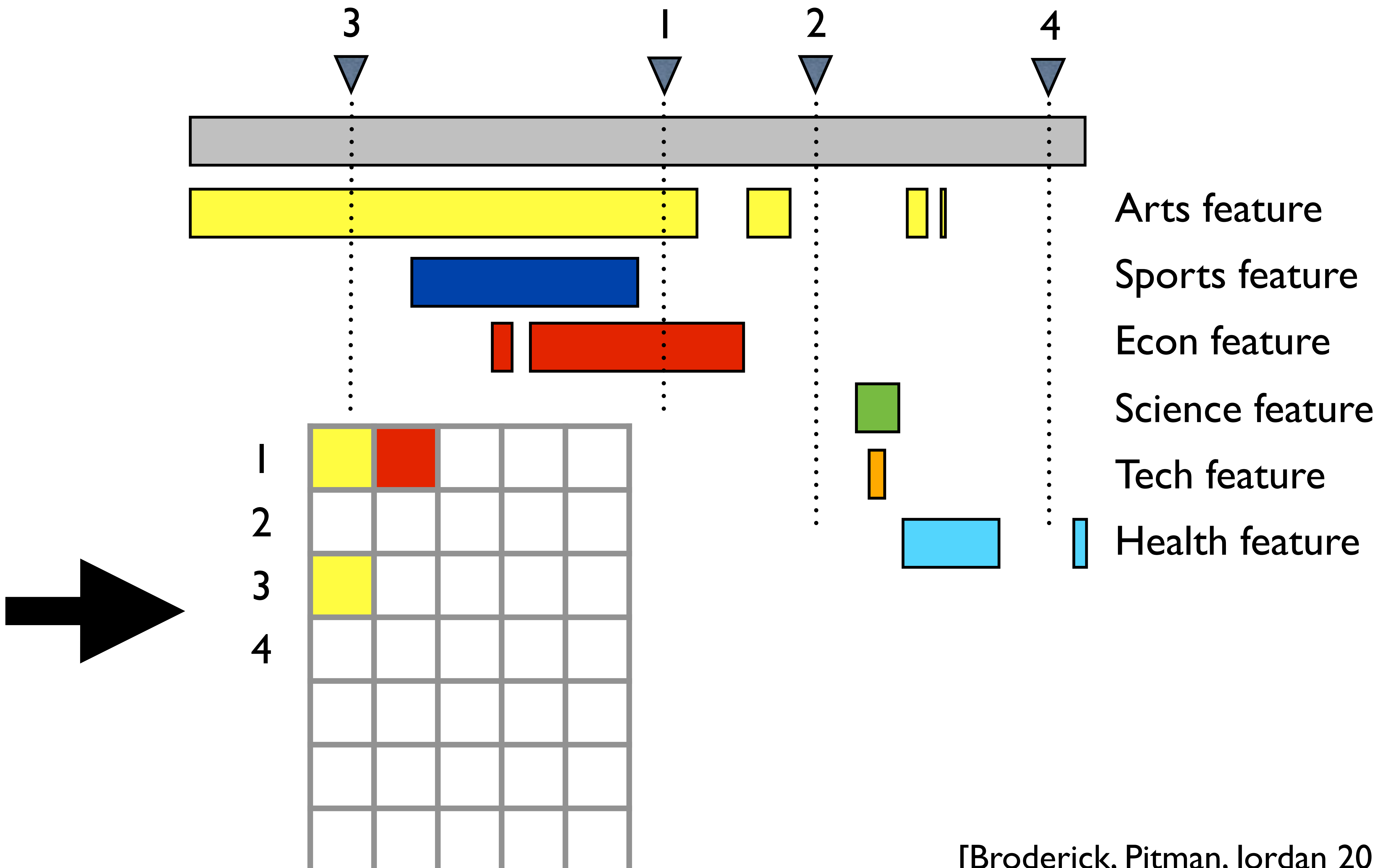

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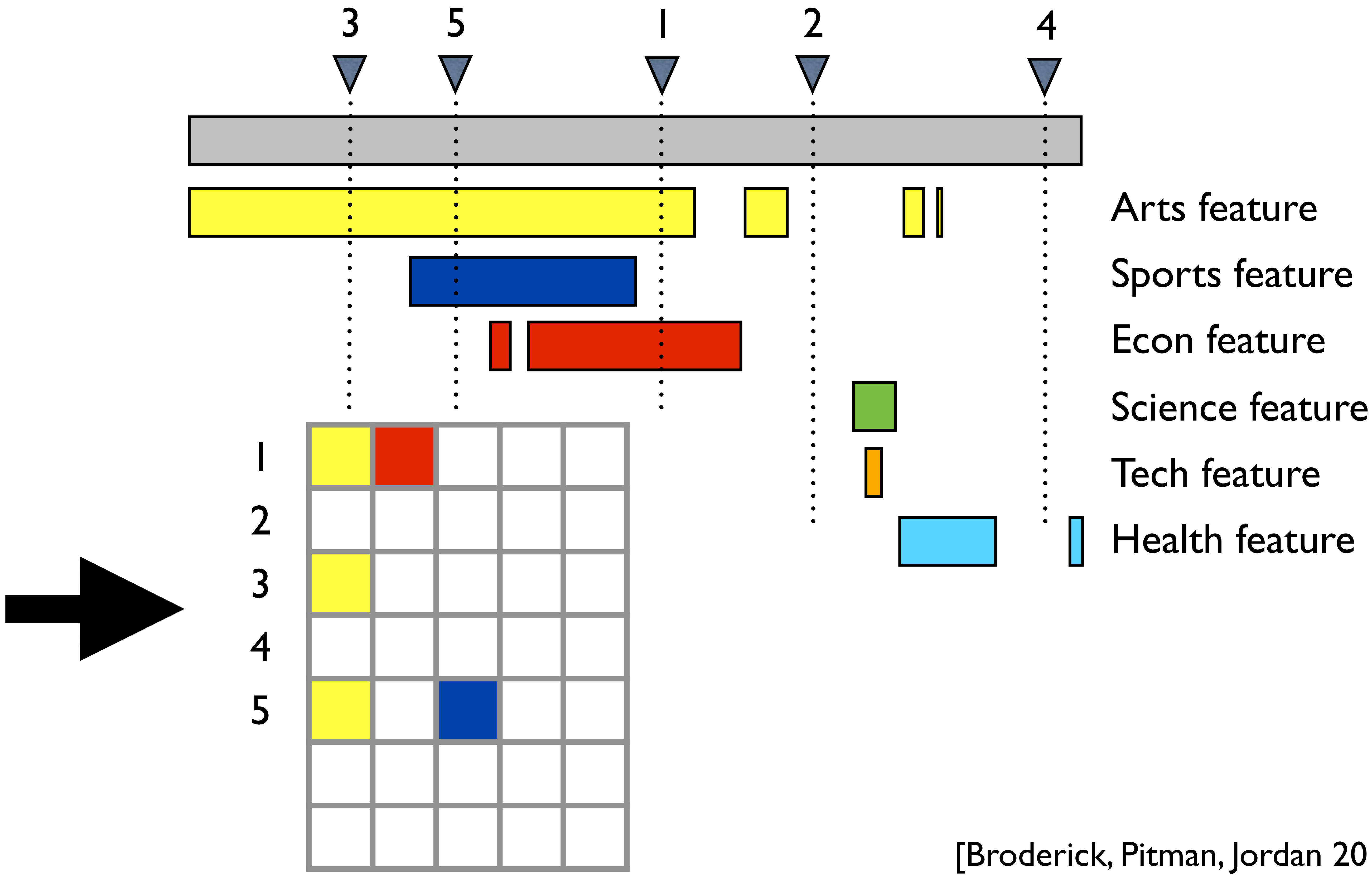

[Broderick, Pitman, Jordan 2013]

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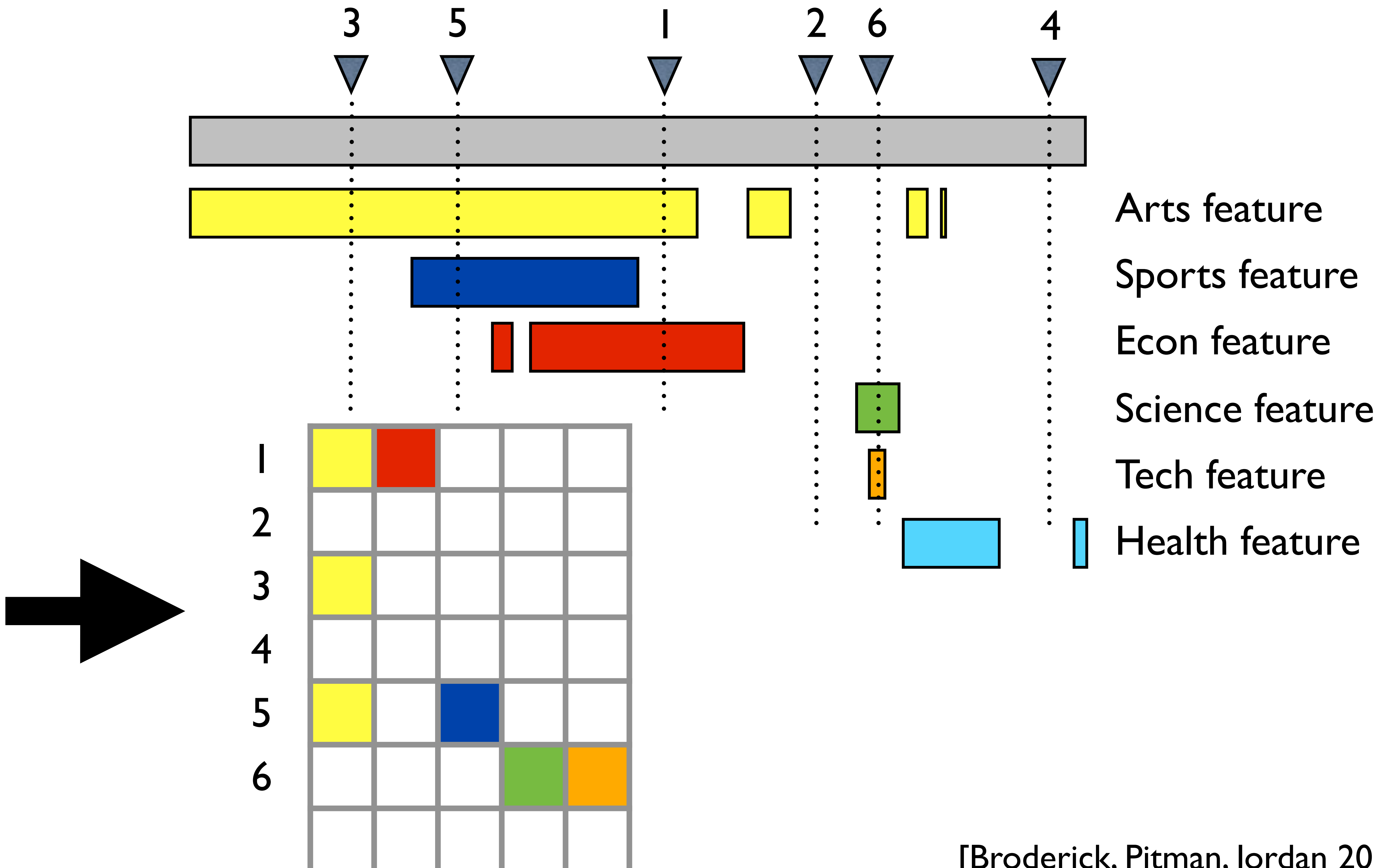

[Broderick, Pitman, Jordan 2013]

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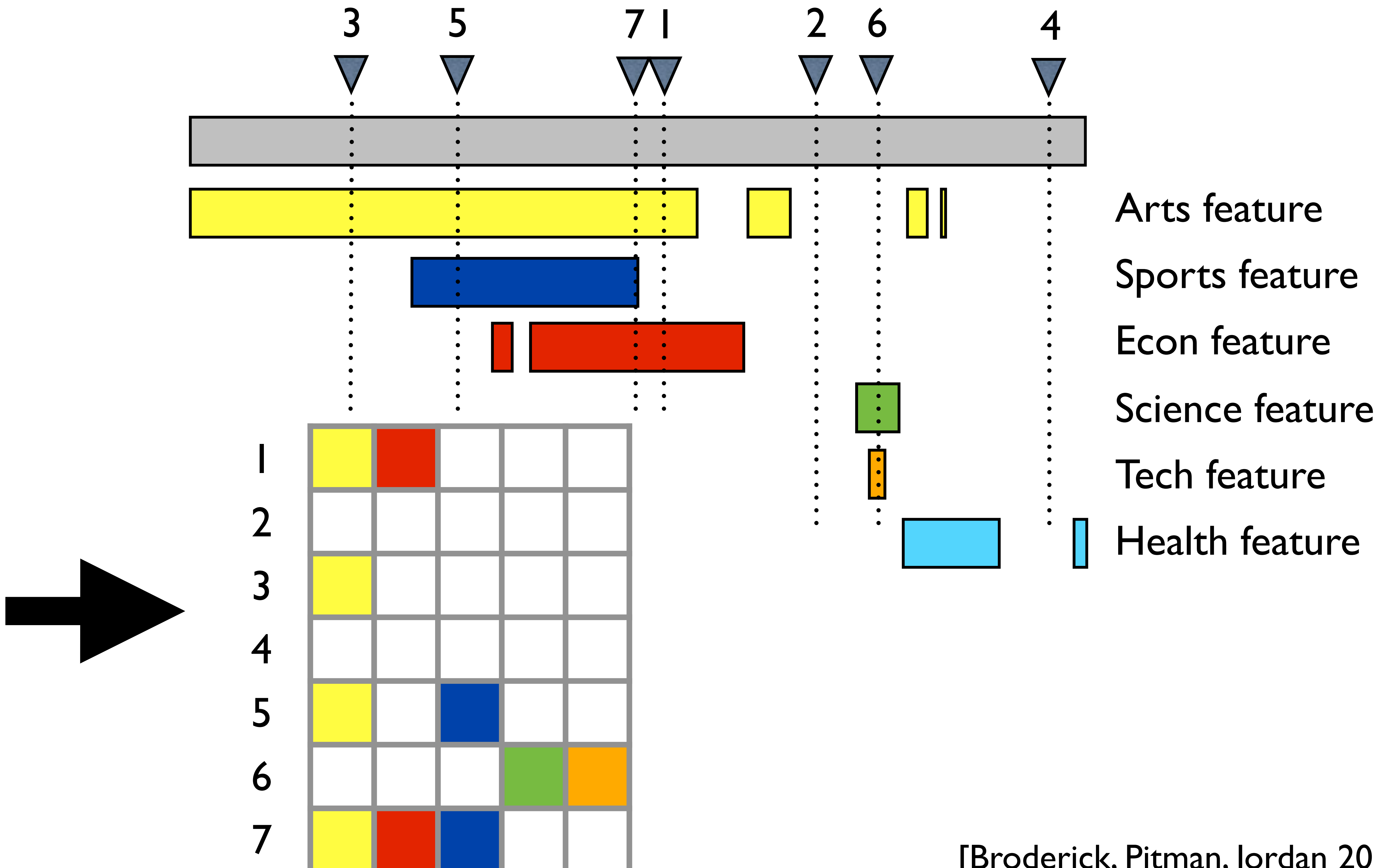
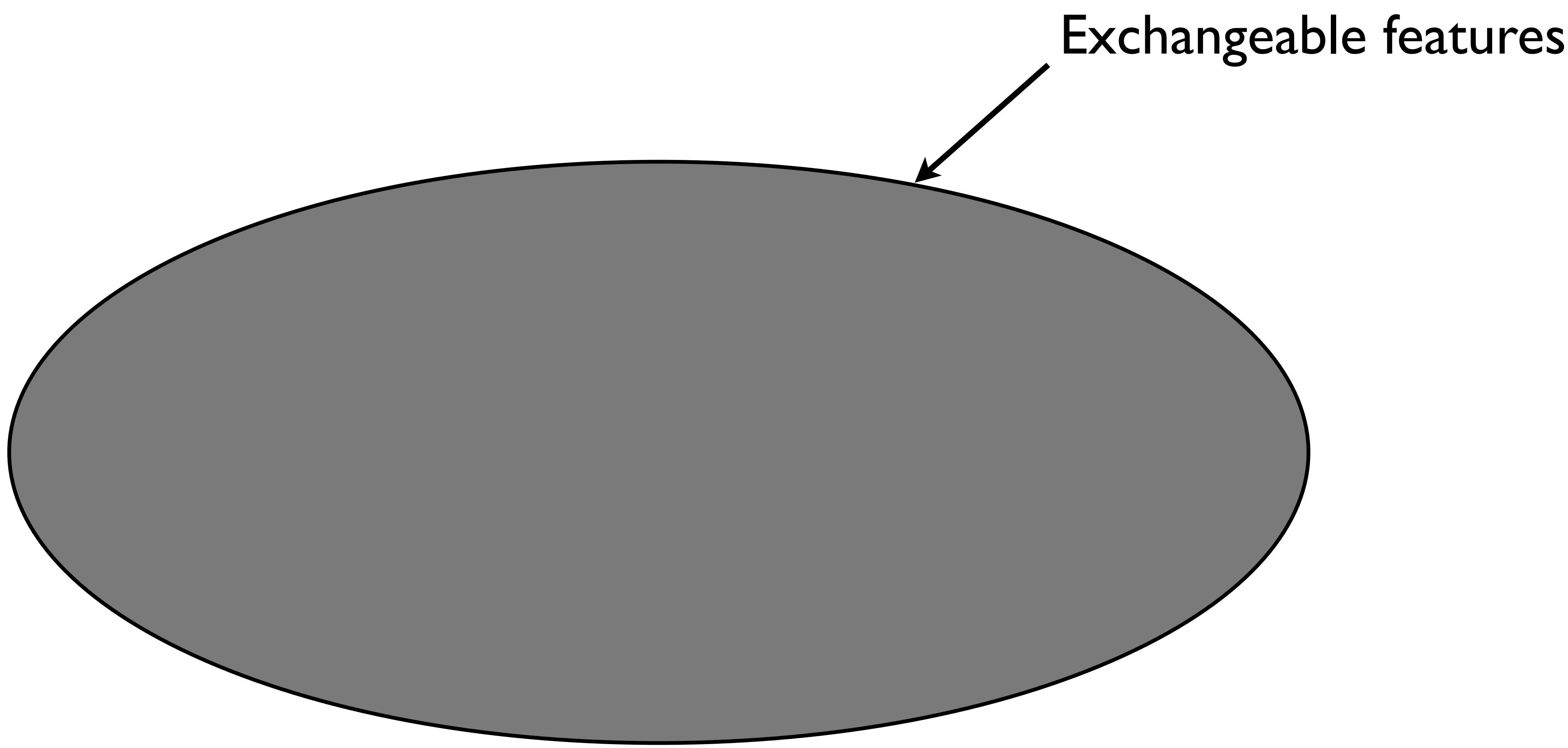


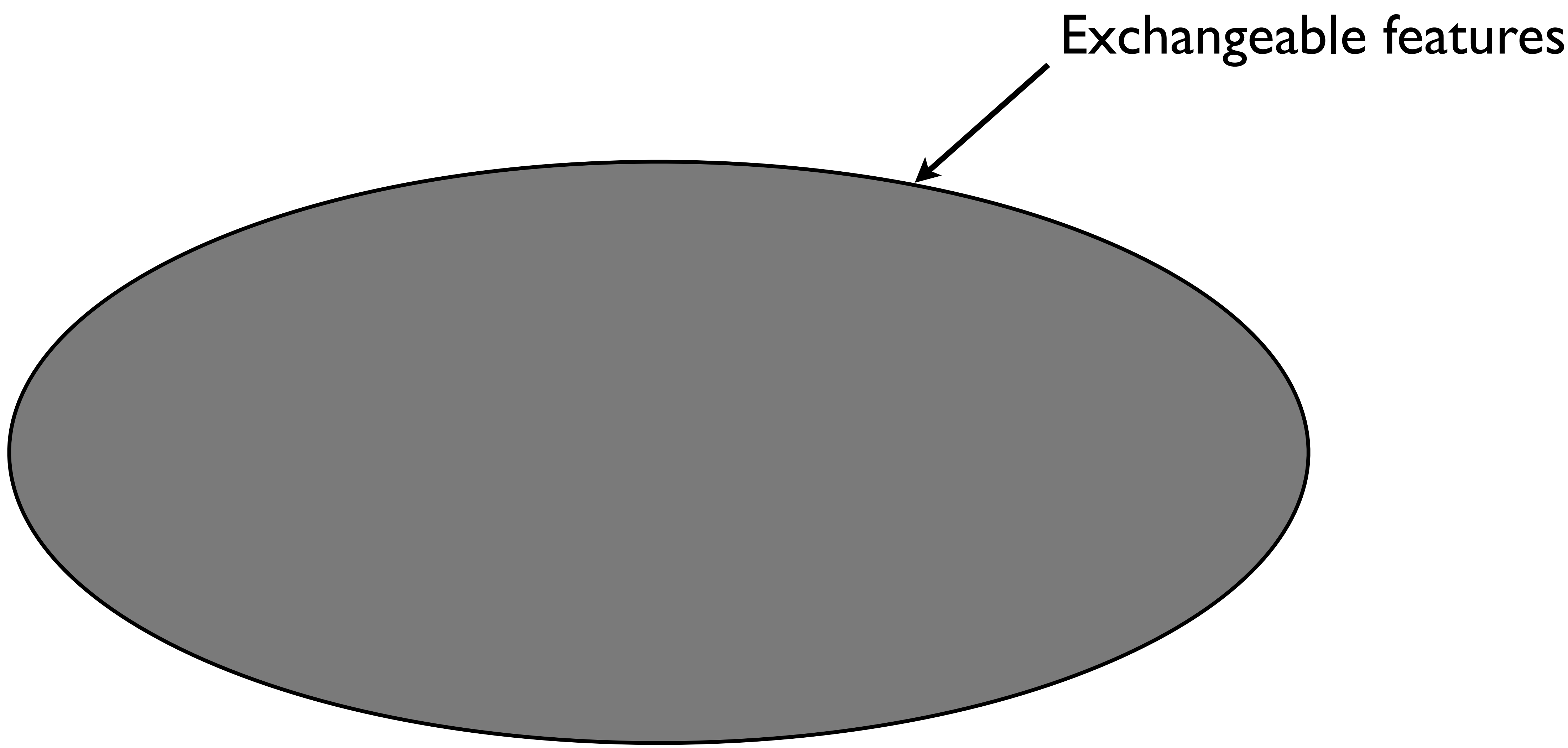
[Broderick, Pitman, Jordan 2013]

Conclusions



Conclusions

- Feature paintbox: characterization of exchangeable feature models



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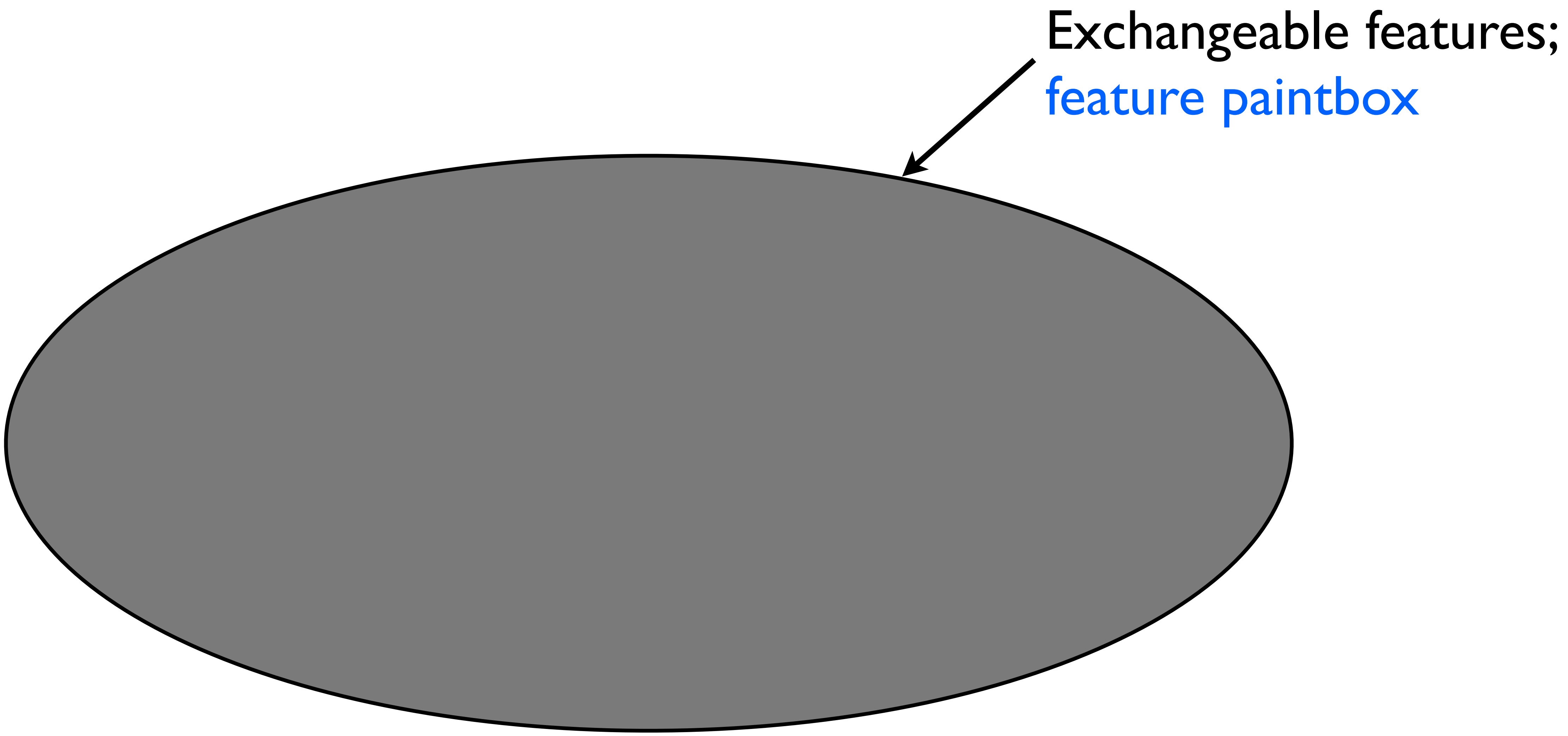

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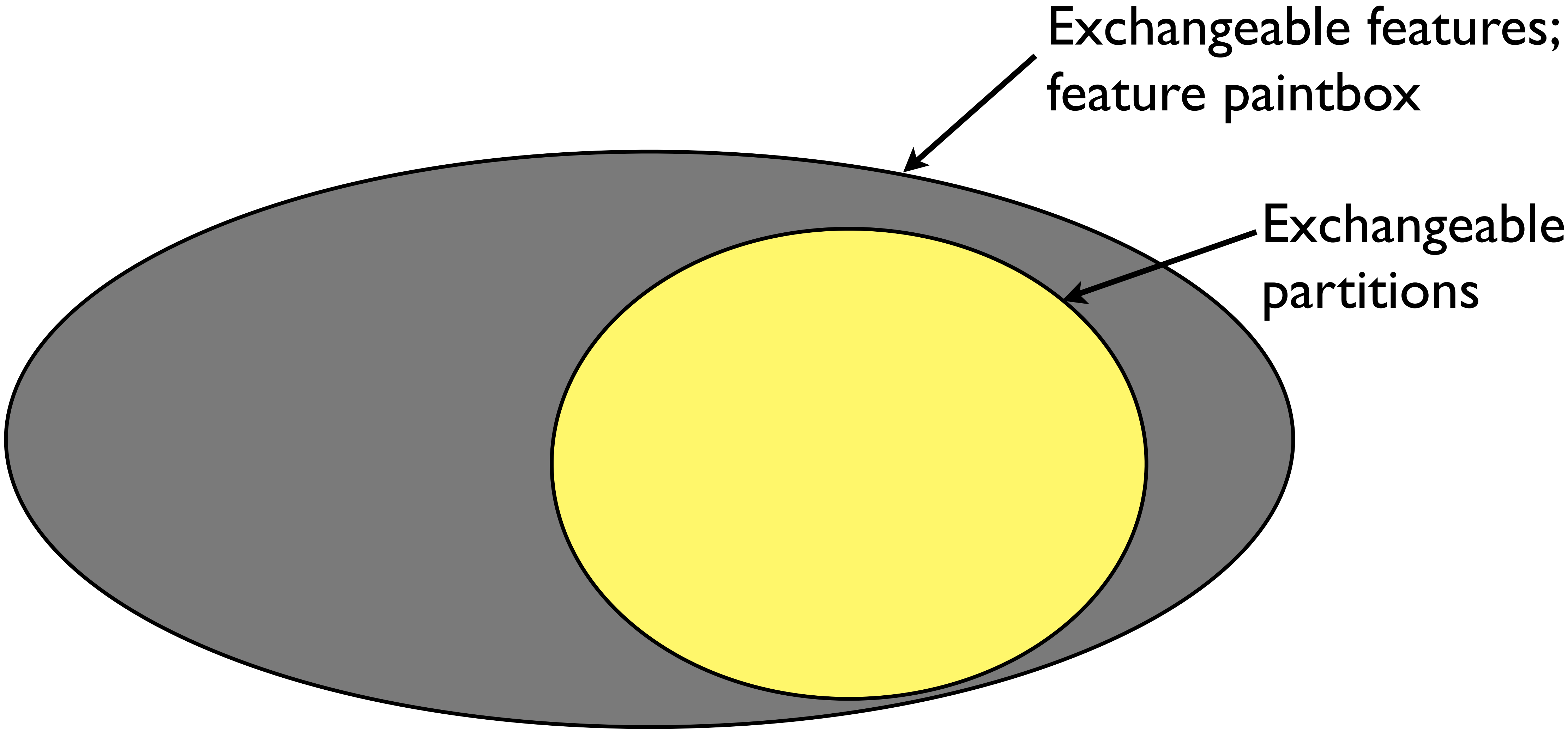

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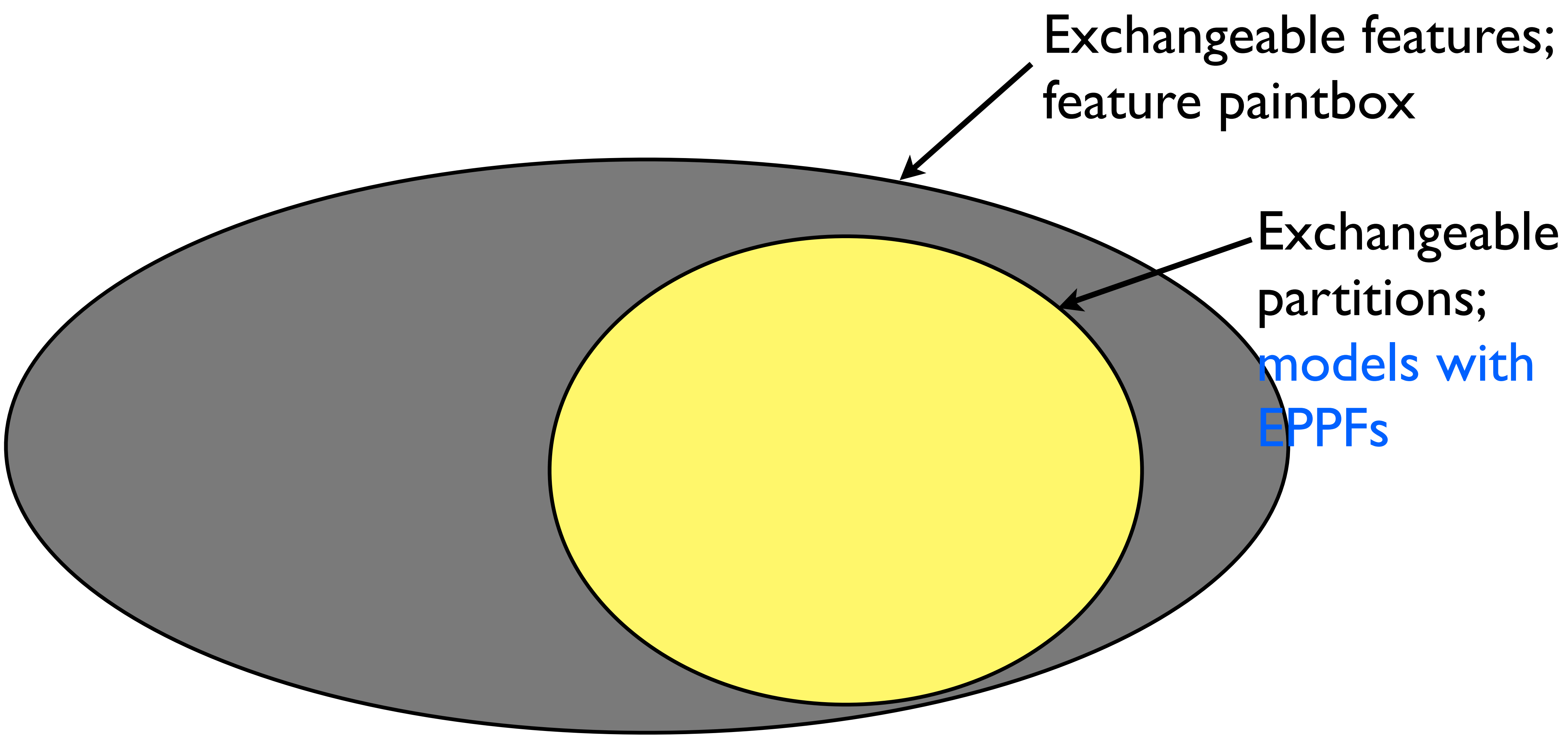

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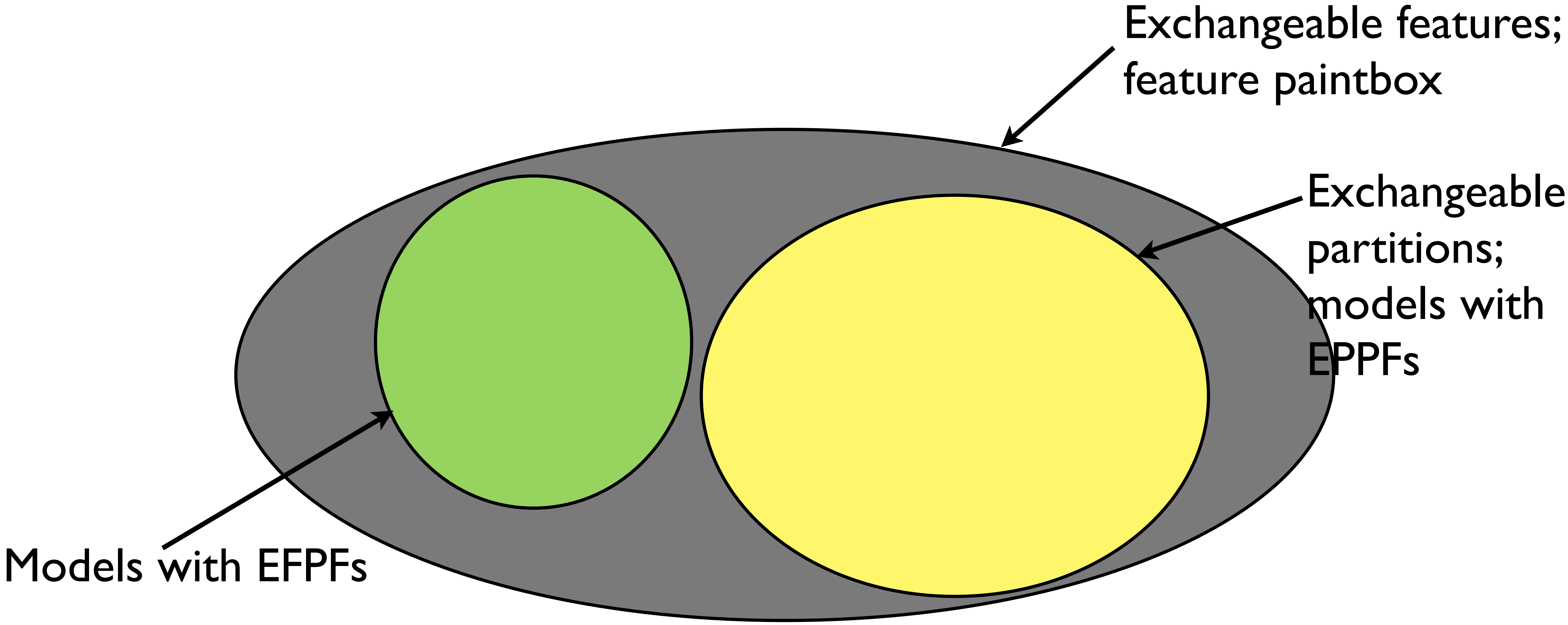
- Feature paintbox: characterization of exchangeable feature models



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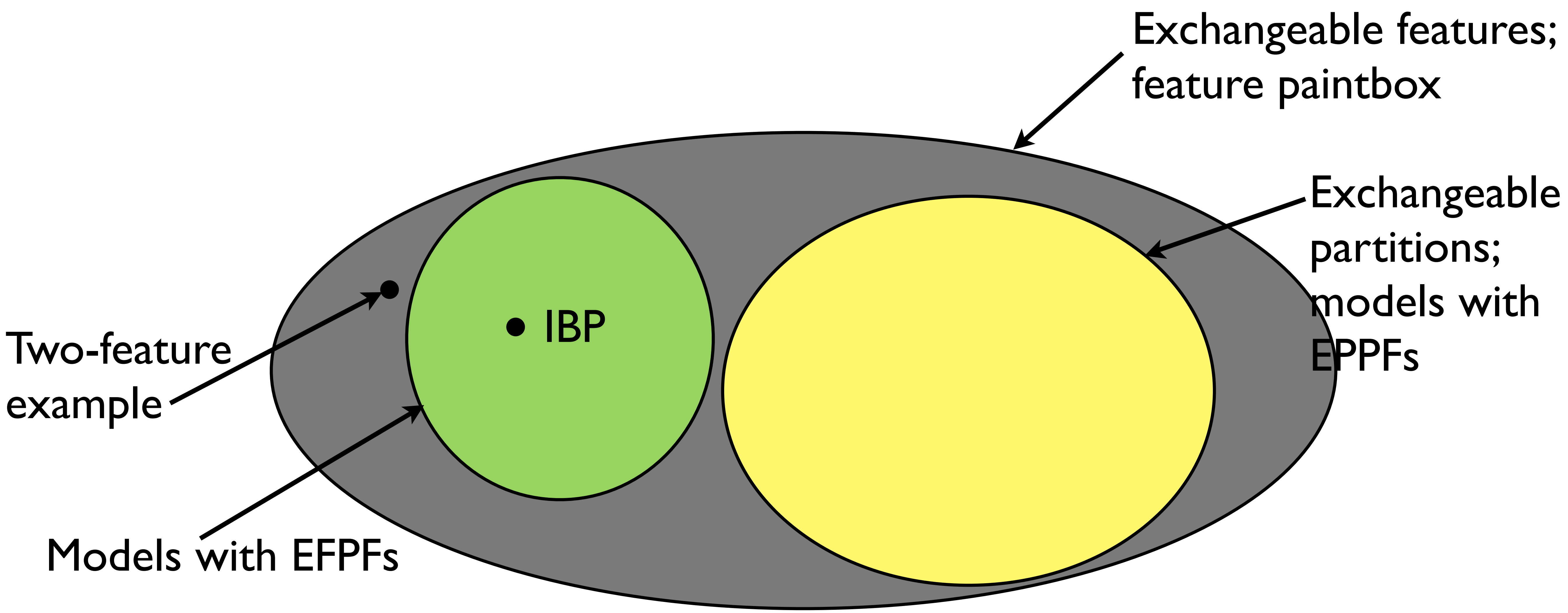
- Feature paintbox: characterization of exchangeable feature models



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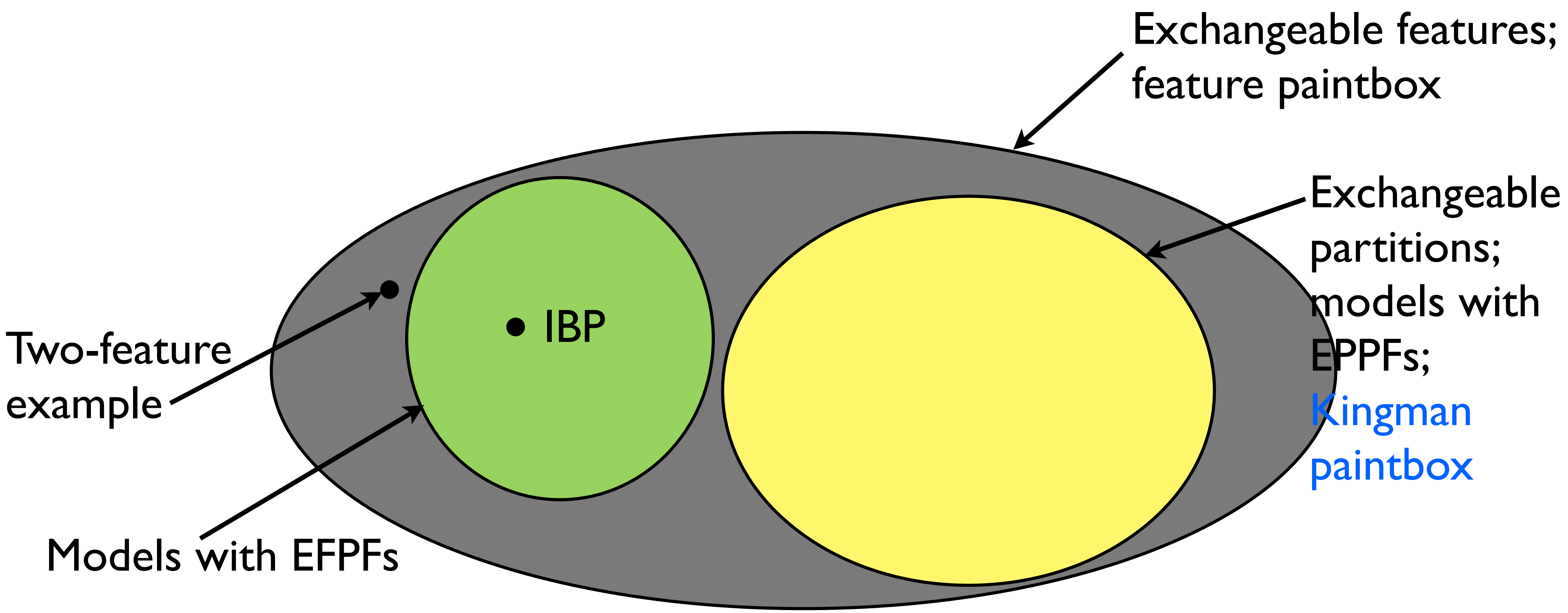
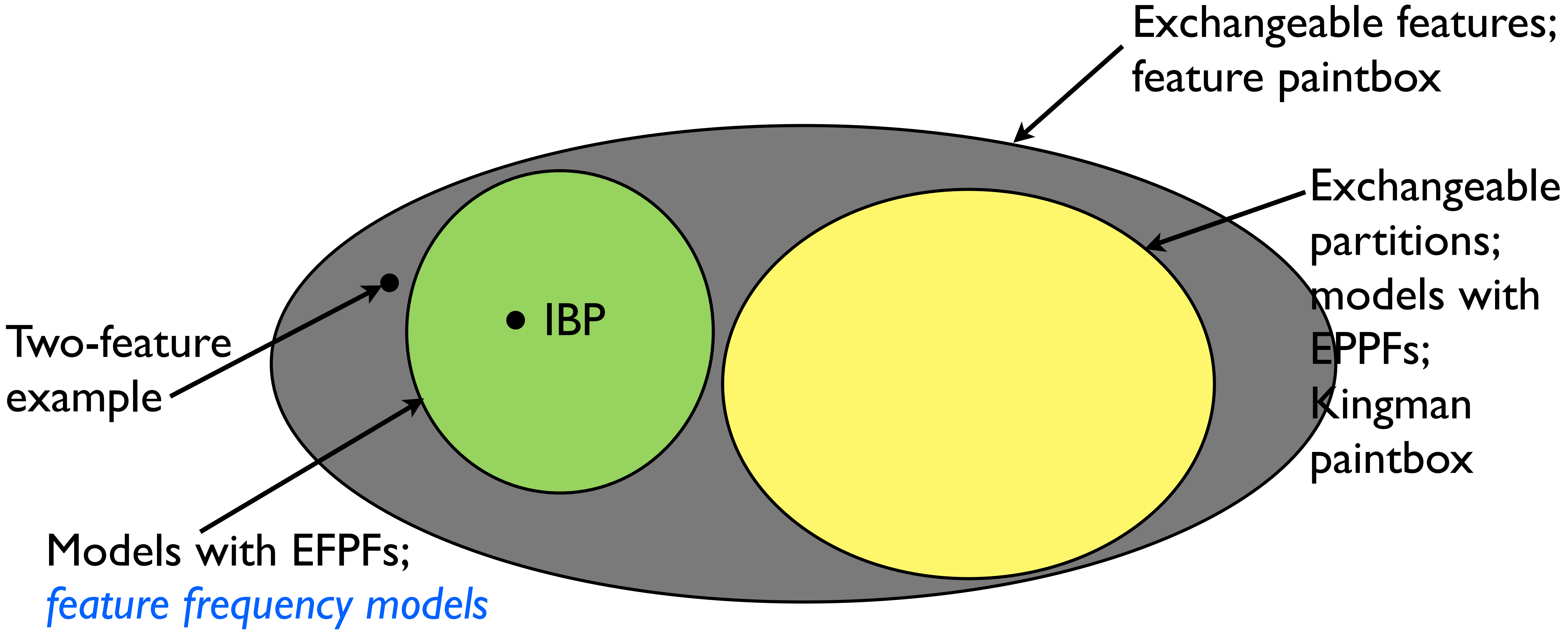
- Feature paintbox: characterization of exchangeable feature models



Conclusions

- Feature paintbox: characterization of exchangeable feature models



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Further References

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