Rigidity in Markovian maximal couplings.

Sayan Banerjee

(Joint work with Wilfrid S. Kendall).

University of Warwick

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Maximal Couplings

A coupling of Markov processes X and Y with laws μ and ν, with coupling time T, is called a **Maximal Coupling** if

$$P(T > t) = ||\mu_t - \nu_t||_{TV},$$

for all $t > 0$, where

- μ_t and ν_t are distributions of X_t and Y_t respectively.
- $||\cdot||_{TV}$ is the total variation distance between measures.

Existence

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- Pitman's construction simulates the *meeting point* first and then constructs the *forward* and *backward* chains.
- The coupling *cheats* by looking into the future.

Markovian couplings

- A coupling of Markov processes X and Y starting from x_0 and y_0 is called **Markovian** if

$$(X_{t+s}, Y_{t+s}) \mid \mathcal{F}_s$$

is again a coupling of the laws of X and Y starting from (X_s, Y_s). Here $\mathcal{F}_s = \sigma\{(X_{s'}, Y_{s'}) : s' \leq s\}$.

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- The coupling is not allowed to look into the future.

Question

When is it possible for two Markov processes to have a Markovian maximal coupling (MMC)?

We investigate this question for diffusions of the form

$$dX_t = b(X_t)dt + dB_t.$$

Known Examples

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- Kuwada (2009): Brownian motion on a homogeneous Riemannian manifold can be coupled by MMC if and only if the manifold has a reflection structure.

Structure of the MMC

In order to have a MMC, the coupling should satisfy the following:

- There is a *deterministic* system of mirrors $\{M(t)\}_{t \geq 0}$ which can *evolve in time* such that, for each t, Y_t is obtained by reflecting X_t in $M(t)$.
- Under suitable regularity assumptions, the moving mirror can be parametrized in a *smooth way*.
- These lead to (implicit) functional equations on the drift, via stochastic calculus.

Rigidity Theorems for MMC

Theorem

(B'-Kendall)

If there exist $x_0, y_0 \in \mathbb{R}^d$ *and* $r > 0$ *such that there exists a Markovian maximal coupling of the diffusion processes* X *and* Y *starting from* x *and* y *for every* $x \in B(x_0, r)$ *and* $y \in B(y_0, r)$*, then there exist a real scalar* λ*, a skew-symmetric matrix* T *and a vector* $c \in \mathbb{R}^d$ *such that*

$$\mathbf{b}(x) = \lambda x + Tx + c$$

for all $x \in \mathbb{R}^d$*.*

Stronger version for one dimension

Theorem

(B'-Kendall)

There exists a Markovian maximal coupling of one dimensional diffusions X and Y starting from x_0 and y_0 respectively if and only if the drift b is either linear or $b(x) = -b(x_0 + y_0 - x)$ for all $x \in \mathbb{R}$.

Remark: This determines *all one dimensional diffusions* (with general diffusion coefficient) for which MMC holds, via scale functions.

Conclusion and Remarks

- There exists a complete characterization for time-nonhomogeneous drifts.
- Towards general multidimensional diffusions / diffusions on manifolds, work in progress.
- When MMC does not exist, we can look at efficient couplings (coupling rate of *same order* as T.V. distance). Some work in this direction has been done for *Kolmogorov Diffusions*.

Thank You!