Exchangeable graph-valued Markov processes

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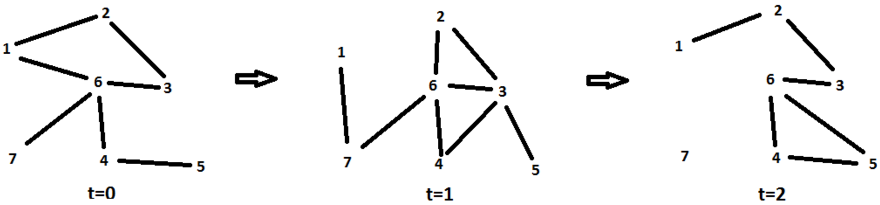
Graph-valued Markov process

- $\mathcal{G}_{\mathbb{N}}$: graphs with vertex set $\mathbb{N} = \{1, 2, \ldots\}$.
- $\mathcal{G}_n$: graphs with vertex set $[n] := \{1, \ldots, n\}$.
- *adjacency matrix/array*: $G = (G_{ij})_{i,j\geq 1} \in \mathcal{G}_{\mathbb{N}}$.
- *relabeling*: $\sigma : \mathbb{N} \to \mathbb{N}$ (permutation), $G \mapsto G^{\sigma} := (G_{\sigma(i)\sigma(j)})_{i,j\geq 1}$.
- *restriction*: $\mathcal{G}_{\mathbb{N}} \to \mathcal{G}_n$, $(G_{ij})_{i,j\geq 1} \mapsto G|_{[n]} := (G_{ij})_{1\leq i,j\leq n}$.

$\Gamma = (\Gamma_t)_{t\geq 0}$ *is a Markov process on* $\mathcal{G}_{\mathbb{N}}$ *satisfying*

- exchangeability*: for all* $\sigma : \mathbb{N} \to \mathbb{N}$, $\Gamma^{\sigma} := (\Gamma_t^{\sigma})_{t\geq 0}$ *is a version of* Γ*.*
- (Markovian) consistency*:* $\Gamma^{[n]} := (\Gamma_t|_{[n]})_{t\geq 0}$ *is a Markov chain on* $\mathcal{G}_n$*, for every* $n = 1, 2, \ldots$.



Weakly exchangeable arrays

An exchangeable graph Γ is a weakly exchangeable $\{0,1\}$-valued array $\Gamma = (\Gamma_{ij})_{i,j\geq 1}$.

Aldous–Hoover theorem: $\Gamma =_{\mathcal{L}} \Gamma^* = (\Gamma^*_{ij})_{i,j\geq 1}$ with

$$\Gamma^*_{ij} = f(\alpha, \xi_i, \xi_j, \eta_{\{i,j\}}), \quad i,j \geq 1,$$

where $f(\cdot, b, c, \cdot) = f(\cdot, c, b, \cdot)$ and $\alpha, (\xi_i)_{i\geq 1}, (\eta_{\{i,j\}})_{1\leq i<j}$ are i.i.d. Uniform[0,1].

(I) overall effect: α

(II) vertex effect: $\{\xi_i\}_{i\geq 1}$

(III) edge effect: $\{\eta_{\{i,j\}}\}_{1\leq i<j}$

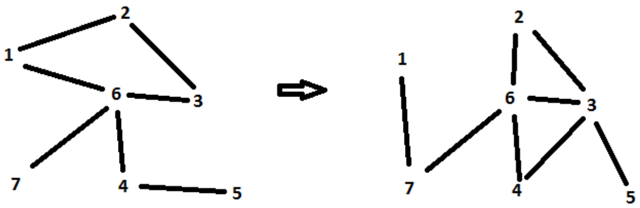
Theorem

$\Gamma = (\Gamma_t)_{t \geq 0}$ *an exchangeable, consistent Markov process on* $\mathcal{G}_{\mathbb{N}}$*. Then there are three types of discontinuity:*

(I) **global jump***: a positive fraction of all edges changes status;*

(II) **single-vertex jump***: a positive fraction of edges incident to a single vertex change, everything else stays the same;*

(III) **single-edge flip***: a single edge changes status, everything else stays the same.*



Characterization of discontinuities

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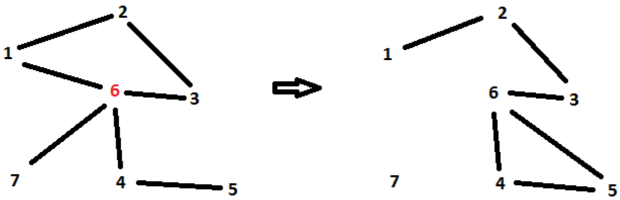

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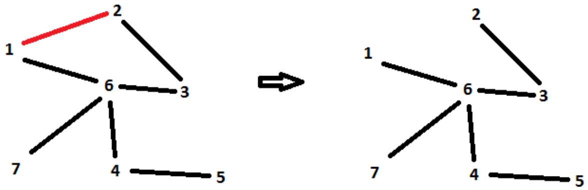

Characterization of discontinuities

Theorem

$\Gamma = (\Gamma_t)_{t\geq 0}$ an exchangeable, consistent Markov process on $\mathcal{G}_{\mathbb{N}}$. The jump measure decomposes into three parts:

(I) **unique σ-finite measure on $\{0,1\}\times\{0,1\}$-valued arrays**: *random function $W : [0,1]^4 \times \{0,1\} \to \{0,1\}$ (weakly exchangeable array) so that $\Gamma_{t-} \mapsto \Gamma_t$ with*

$$\Gamma_t(i,j) = W(\alpha, \xi_i, \xi_j, \eta_{\{i,j\}}, \Gamma_{t-}(i,j)),$$

where $\{\alpha; (\xi_i); (\eta_{\{i,j\}})\}$ are i.i.d. Uniform[0,1].

(II) **unique σ-finite measure on 2×2 stochastic matrices**: *there is a unique $i = 1,2,\ldots$ for which $(\Gamma_{t-}(i,1), \Gamma_{t-}(i,2),\ldots)$ jumps according to a 2×2 stochastic matrix S.*

(III) **unique constants $\mathbf{c}_{01}, \mathbf{c}_{10} \geq 0$**: *determine jump rates of each edge.*

Comments:

- Compare to the Lévy-Itô characterization of exchangeable coalescent processes (binary coagulation, multiple collisions).
  - (I) **binary coagulation**: two blocks merge, everything else stays the same (“continuous jumps”);
  - (II) **multiple collisions**: multiple blocks merge simultaneously (“discrete jumps”).
- There is an associated projection of Γ into the space of graph limits.