Pattern avoiding permutations and Brownian excursion

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Combinatorial Stochastic Processes
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231-avoiding permutations

Definition

A permutation σ is said to be **231**-avoiding if there does not exist $i < j < k$ such that $\sigma(k) < \sigma(i) < \sigma(j)$.

- $\sigma_1 = 3754621$ is NOT **231**-avoiding.
- $\sigma_2 = 2154367$ is **231**-avoiding.

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- Knuth ('69): The number of **231**-avoiding permutations of size n is

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

231-avoiding permutations

- Miner, Pak 2013 – *The shape of random pattern avoiding permutations.*
- Janson, Nakamura, Zeilberger 2013 – *On the asymptotic statistics of the number of occurrences of multiple permutation patterns.*
- Janson 2014 – *Patterns in random permutations avoiding the pattern* **132**.
- Madras, Pehlivan 2014 – *Structure of Random* **312***-avoiding permutations.*

Fixed Points

Theorem (Montmort 1708)

Let σ_n be a uniformly random permutation of $\{1, 2, \ldots, n\}$. The number of fixed point of σ_n converges in distribution to a Poisson(1) random variable as $n \to \infty$.

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Theorem (Miner-Pak '13)

Let σ_n be a uniformly random **231***-avoiding permutation of $\{1, 2, \ldots, n\}$. The expected number of fixed points of σ_n is asymptotic to $\frac{\Gamma(1/4)}{2\sqrt{\pi}} n^{1/4}$ as $n \to \infty$.*

Fixed Points

Theorem (Hoffman-R-Slivken '14)

- *Let* σ_n *be a uniformly random* **231***-avoiding permutation of* $\{1, 2, \ldots, n\}$.
- *Let* $\mathrm{Fix}_n(t) = \#\{i \in \{1, 2, \ldots, [t]\} : \sigma_n(i) = i\}$.

Then

$$\left(\frac{1}{n^{1/4}}\mathrm{Fix}_n(nt)\right)_{t\in[0,1]} \xrightarrow{d} \left(\frac{1}{2^{7/4}\sqrt{\pi}}\int_0^t \frac{1}{\mathbb{e}_u^{3/2}}du\right)_{t\in[0,1]},$$

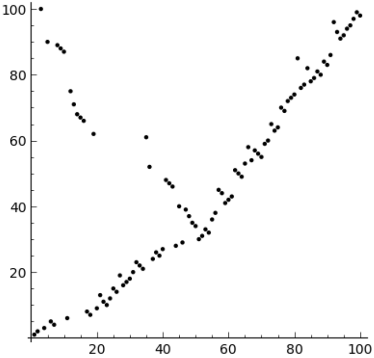
where $(\mathbb{e}_t)_{t\in[0,1]}$ *is standard Brownian excursion.*

Theorem (Hoffman-R-Slivken '14)

If n is large and σ_n is a uniformly random **231***-avoiding permutation of $\{1, 2, \ldots, n\}$ then, appropriately rescaled,*

$$(i - \sigma_n(i))_{1 \leq i \leq n}$$

almost looks like a Brownian excursion.



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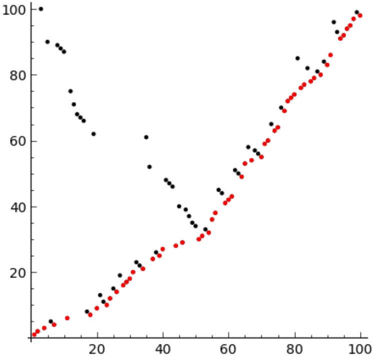

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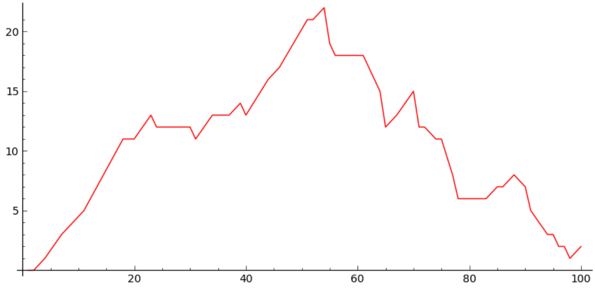

Figure: $i - \sigma_n(i)$ for "good" values of i.

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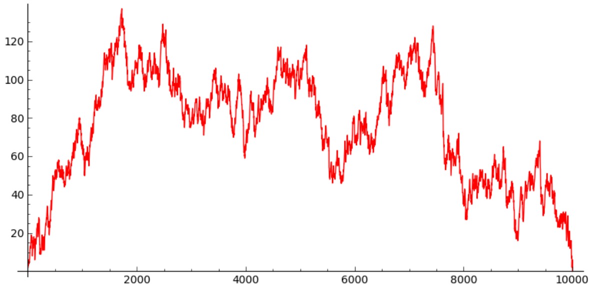
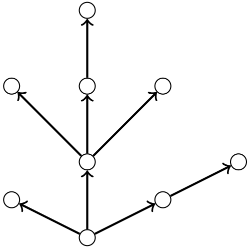


Figure: An example for $n = 10000$

$\mathbf{231}$-avoiding permutations

A bijection between trees with $n+1$ vertices and $\mathbf{231}$-avoiding permutations of $\{1, 2, \ldots, n\}$.



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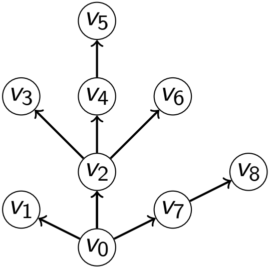

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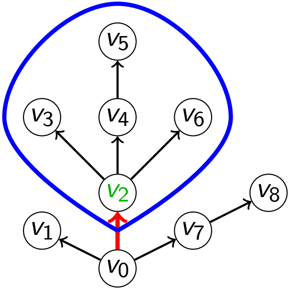
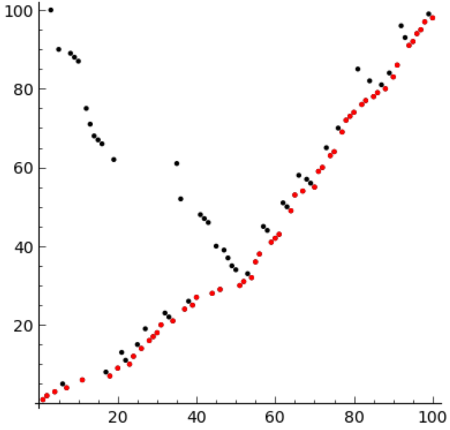


$$\sigma_{\mathbf{t}}(2) = 2 + 5 - 1 = 6$$

$$\sigma_{\mathbf{t}}(i) = i + |\mathbf{t}_{v_i}| - \mathrm{ht}(v_i)$$

The Good Points



$$i - \sigma_{\mathbf{t}}(i) = \mathrm{ht}(v_i) - |\mathbf{t}_{v_i}|$$