The Metric Coalescent Process

joint with David Aldous

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Two Related Processes

Two stochastic processes:

1. The Compulsive Gambler
   1. Finite agent based model,
   2. Finite Markov Information Exchange (FMIE) framework.
2. Metric Coalescent
   1. Measure-valued Markov process,
   2. Defined for any metric space (S, d).

FMIE Processes

General Setup: interacting particle systems reinterpreted as stochastic social dynamics.

1. n agents; each in some state $X_i(t) \in \mathfrak{S}$ for each time $t \geq 0$
2. Each pair of agents (i, j) meet at the times of a Poisson process of rate ν_{ij}
3. At meeting times t between pairs of agents (i, j), the states transition

$$(X_i(t-), X_j(t-)) \to (X_i(t), X_j(t))$$

according to some deterministic or random rule

$$F : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{S} \times \mathfrak{S}.$$

FMIE Processes

Some familiar (and less familiar) examples:

1. Stochastic epidemic models; SIR model, etc.
2. Density dependent Markov chains (For ex. Kurtz 1978)
3. Averaging process, take $\mathfrak{S} = \mathbb{R}$ as money. Upon meeting two agents average their money. (Aldous-Lanoue 2012).

$$F(a, b) = \left(\frac{a+b}{2}, \frac{a+b}{2} \right)$$

4. The iPod Model, an FMIE variant of the Voter Model (Aldous-Lanoue 2013)

The goal is to study how the (non-asymptotic) behaviour depends on the finite meeting rates ν_{ij}. Analogous to the study of mixing time for finite Markov chains.

Compulsive Gambler Process

Simple FMIE process with agents' state space $\mathfrak{S} = \mathbb{R}_{\geq 0}$, interpreted as money. When agents i and j meet they play a fair, winner take all game. I.e. the transition function is

$$F(a,b) = \begin{cases} (a+b,0) & \text{with prob. } \frac{a}{a+b} \\ (0,a+b) & \text{with prob. } \frac{b}{a+b} \end{cases}$$

In the finite agent setting, we assume the total initial wealth is normalized

$$\sum_{i \in \text{Agents}} X_i(0) = 1.$$

Importantly this allows us to view the state of the process as a probability measure of the set of agents.

Compulsive Gambler Process

This model first studied in the setting of d-regular graphs and Galton-Watson Trees by Aldous-Salez. Some results on proportion of agents "still alive" at a time $t > 0$, in particular $t = \infty$ [ALS14].

The rest of today's talk will focus on a very particular variant of the CG, one with dependent rates ν_{ij}.

Extending the CG Process

We can reformulate the CG as a measure-valued Markov process in terms of:

1. A metric space (S, d),
2. A rate function $\phi(x)\colon\ \mathbb{R}_{>0} \to \mathbb{R}_{>0}$,

The Metric Coalescent (MC) is then a continuous time $P_{\text{fs}}(S)$-valued Markov process, generalizing the CG as follows. For any $\mu \in P_{\text{fs}}(S)$:

1. The atoms $s_i, 1 \leq i \leq \#\mu$ of μ are identified as the agents,
2. The masses $\mu(s_i)$ as their respective current wealth,
3. The meeting rates between agents i and j given by ϕ and the metric as

$$\nu_{ij} = \phi(d(s_i, s_j))$$

A Visualization

A simulation of the Metric Coalescent process on $S = [0, 1]^2$ started from finitely supported approximations of the uniform measure:

1. Link

Developed by Weijian Han.

Main Theorem

Let (S, d) be a locally compact, separable metric space and let the rate function $\phi(x)$ satisfy

$$\lim_{x \downarrow 0} \phi(x) = \infty.$$

Our main result for the Metric Coalescent is as follows [Lan14]:

Main Theorem

There exists a unique, cadlag, Feller continuous $P(S)$-valued Markov process $\mu_t, t \geq 0$ defined from any initial measure $\mu_0 \in P(S)$ s.t. if μ_0 is compactly supported:

1. $\mu_t \in P_{\text{fs}}(S)$ for all $t > 0$, almost surely;
2. For each $t_0 > 0$, the process $(\mu_t, t \geq t_0)$ is distributed as the Metric Coalescent started at μ_{t_0};

Proof Idea: Naive Approach

The "naive" proof idea for constructing $\mu_t, t \geq 0$ for a generic $\mu \in P(S)$ is to approximate μ with a sequence of finitely supported $\mu^i \in P_{fs}(S)$ for $i \geq 1$. Then for $t \geq 0$ define (the random measure) μ_t as the weak limit

$$\mu_t = \lim_i \mu_t^i.$$

Feller continuity in the Main Theorem retroactively implies that this sequence of random measures does converge, however – even ignoring the coupling issues here – this approach isn't so fruitful in proving the Main Theorem.

Some progress is made in [Lan14] following this idea using moment methods.

Proof Idea: Exchangeable Coalescents

Key Idea: replace the symmetric "random winners at meeting times" dynamics between agents with "deterministic winners according to a size-biased initial ranking". This allows us to view the MC as an exchangeable partition process and enables a wide variety of tools. Among these used:

1. A comparison to Kingman's Coalescent,
2. Two separate applications of de Finetti's theorem,
3. An explicit formula for moments of

$$\int f \, d\mu_t$$

for $f : S \to \mathbb{R}$.

Further Directions

Two directions for further research:

1. **Coming Down From Infinity:** We know that for compactly supported μ_0 initial measures, μ_t is finitely supported for all positive times $t > 0$. It is easy to construct non-compactly supported μ_0 for which this isn't true. What more can be said?
2. **Time Reversal:** A classical result on Kingman's Coalescent is its duality under a time reversal to a conditioned Yule process. Viewing the MC as a "geometrization" of KC, can something similar be said?

References

Thanks for listening!

For further information on these two processes and a complete reference list.

[ALS14] D. Aldous, D. Lanoue, and J. Salez, *The Compulsive Gambler Process*, ArXiv e-prints (2014).

[Lan14] D. Lanoue, *The Metric Coalescent*, ArXiv e-prints (2014).