Genesis of Gamma Bursts in Neural Local Field Potentials

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(See-1)/(Sii+1) = 0.45455
Envelope extracted via Hilbert transform
Model of interacting populations of excitatory and inhibitory neurons

\[ V_E(t), V_I(t) \text{ are potentials (or conductances, firing rates).} \]

\[ \tau_E \frac{dV_E(t)}{dt} = (-V_E(t) + S_{EE} V_E(t) - S_{EI} V_I(t)) dt + \sigma_E dW_E(t) \]

\[ \tau_I \frac{dV_I(t)}{dt} = (-V_I(t) - S_{II} V_I(t) + S_{IE} V_E(t)) dt + \sigma_I dW_I(t) \]

i.e.,

\[ dV = -A V dt + B dw \]

where

\[ A = \begin{pmatrix} (1-S_{EE})/\tau_E & S_{EI}/\tau_E \\ -S_{IE}/\tau_I & (1+S_{II})/\tau_I \end{pmatrix} \]

\[ B = \begin{pmatrix} \sigma_E/\tau_E & 0 \\ 0 & \sigma_I/\tau_I \end{pmatrix} \]

We want: the deterministic model has damped oscillations to (0,0).
Model: \[ dV = -AV \, dt + B \, dW \]

Damped oscillations of deterministic model implies
- \[ A \] has eigenvalues \(-\lambda \pm \imath \omega\)
  \[ 0 < \lambda \ll \omega \]

\[ \lambda = \frac{1}{2} \left( \frac{1 - S_{EE}}{\zeta_E} + \frac{1 + S_{II}}{\zeta_I} \right) \]

\[ \omega = \left( \frac{S_{EI} S_{IE}}{\zeta_E \zeta_I} - \frac{1}{4} \left( \frac{1 - S_{EE}}{\zeta_E} - \frac{1 + S_{II}}{\zeta_I} \right)^2 \right)^{1/2} \]

The power spectral density of \( V_E(t) \) is

\[ \text{PSD}_E(f) = \frac{A_{12}^2 \sigma_E^2 / \zeta_I^2 + (A_{12}^2 + f^2) \sigma_E^2 / \zeta_E^2}{(\omega^2 + A_{12}^2 + f^2)^2 + 4(\lambda f)^2} \]

For large \( f \) this is \( \sim 1/f^2 \); for small \( f \) this is \( \sim \text{constant} \).

This PSD looks like the PSD of an Ornstein-Uhlenbeck process, except that \( \rightarrow \)
Finding Gamma Bursts in Sample Paths of $V(t)$.

Gamma bursts occur as $V(t)$ evolves in time. The PSD cannot see them. However, sample paths of $V(t)$ have additional structure.

Using [Baxendale & G. J. Math Bio 2011] we see that as $N/w$ becomes small,

$$V(t) = (V_E(t), V_I(t)) \approx \frac{\sigma}{\lambda} Q R_{-wt} S(\lambda t),$$

where

$$S(t) = (S_1(t), S_2(t)),$$

independent O.U. processes.

$$R_s = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix},$$

$Q$ is a matrix $\exists Q^{-1}(-A)Q = \begin{pmatrix} -\omega & w \\ -\omega & -\lambda \end{pmatrix}$

e.g. $Q = \begin{pmatrix} -\omega & -A_{11} \\ 0 & -A_{21} \end{pmatrix}$,

$$\sigma = \left( \frac{1}{2} \text{Tr} \left( Q^{-1}BB^T(Q')^T \right) \right)^{1/2}$$
The approximation:

\[ dV(t) = -AV(t) \, dt + B \, dW_t \]

Let \( Y(t) = Q \, V(t) \):

\[ dY(t) = \begin{pmatrix} -A & W \\ -W & -A \end{pmatrix} Y(t) \, dt + Q^t B \, dW_t. \]

Let \( Y(t) = R_{-\omega t} Z(t) \):

\[ dZ(t) = -A Z(t) \, dt + R_{\omega t} Q^t B \, dW(t) \]

Let \( U(t) = \frac{\sqrt{d}}{\sigma} Z(t/\alpha) \):

\[ dU(t) = -U(t) \, dt + R_{\omega t/\alpha} \frac{Q^t B}{\sigma} \, dW(t) \]

Limit \( \alpha/\omega \to 0 \):

\[ dS(t) = -S(t) + dW(t) \]

Unwind:

\[ V(t) \approx \frac{\sigma}{\alpha} Q \, R_{-\omega t} S(\alpha t) \]
Gamma bursts are related to amplitude

\[ S(t) = |S(t)| \left( \cos \phi(t), \sin \phi(t) \right) \]

where

\[ \phi(t) = \arctan \frac{S_1(t)}{S_2(t)}. \]

The approximation

\[ V(t) \approx \frac{\alpha}{\sqrt{2}} QR_{-wt} S(2t) \]

needs

\[ V(t) \approx \frac{\alpha}{\sqrt{2}} |S(2t)| \left( \cos(\omega t + \phi(2t)), \sin(\omega t + \phi(2t)) \right) \]

The matrix \( Q \) produces a constant phase difference between \( V_E(t) \) and \( V_I(t) \) of

\[ \frac{2\omega}{A_{11} - A_{22}}. \]

The ratio of their magnitudes is

\[ \left( -\frac{A_{21}}{A_{12}} \right)^{1/2} \]

Gamma bursts are excursions of \( V_E(t), V_I(t) \) away from zero.
The processes
\[ Z(t) = |S(t)| \]
and
\[ \phi(t) = \arctan S_1(t)/S_2(t) \]
satisfy the system of stochastic differential equations
\[ dZ(t) = \left( \frac{1}{Z(t)} - Z(t) \right) dt + dW(t) \]
\[ d\phi(t) = \frac{1}{Z(t)} d\theta(t), \]
where \( W(t) \) is Brownian motion
and \( \theta(t) \) is Brownian motion run on a unit circle.

In \( V(t) = \frac{\theta}{n} Q|S(2t)| \cos(-\omega t + \phi(t)) \sin(-\omega t + \phi(t)) \)
they run slowly, at rate \( \lambda \), relative to \( \omega \).

Phase-slip of \( V(t) \) relative to the rotation \( R(t) \) moves quickly (peaks of "instantaneous frequency")
when \( Z(t) \) is near zero, i.e. between excursions of the O.U. process \( S(t) \).