METEOR PROCESS

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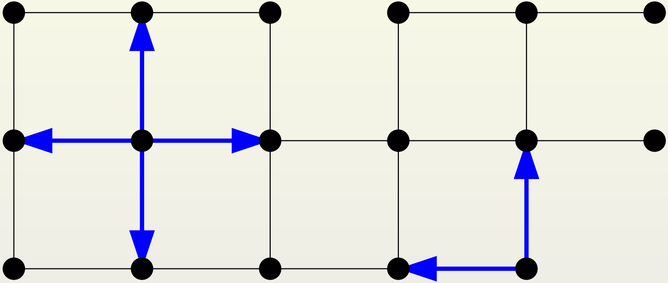
Collaborators and preprints

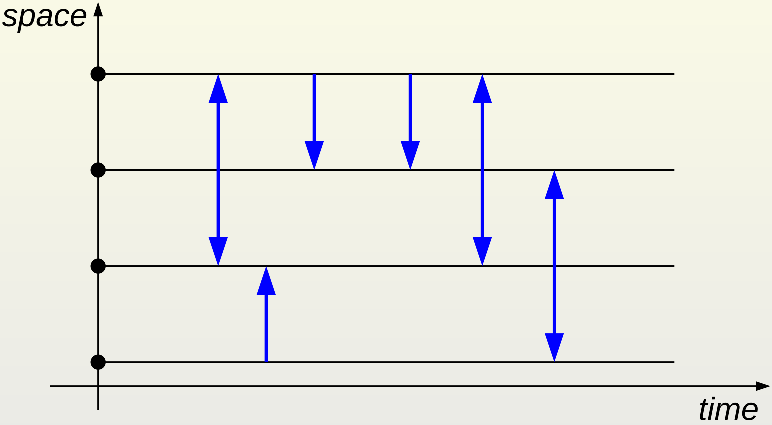
Joint work with Sara Billey, Soumik Pal and Bruce E. Sagan.

Math Arxiv:
http://arxiv.org/abs/1308.2183
http://arxiv.org/abs/1312.6865

Mass redistribution



Mass redistribution (2)



Related models

Chan and Prałat (2012)
Crane and Lalley (2013)
Ferrari and Fontes (1998)
Fey-den Boer, Meester, Quant and Redig (2008)
Howitt and Warren (2009)

Model

G - simple connected graph (no loops, no multiple edges)
V - vertex set
M_t^x - mass at $x \in V$ at time t
Assumption: $M_0^x \in [0, \infty)$ for all $x \in V$
$\mathcal{M}_t = \{M_t^x, x \in V\}$
N_t^x - Poisson process at $x \in V$
The Poisson processes are assumed to be independent.
The "meteor hit" (mass redistribution event) occurs at a vertex when the corresponding Poisson process jumps.

THEOREM

If the graph has a bounded degree then the meteor process is well defined for all $t \geq 0$.

Example. Suppose that G is a triangle. The following are possible mass process transitions.
$(1, 2, 0) \to (0, 5/2, 1/2)$

General graph - stationary distribution

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$(1, \pi/2, 2 - \pi/2) \to (1 + \pi/4, 0, 2 - \pi/4)$

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$(1, \pi/2, 2 - \pi/2) \to (1 + \pi/4, 0, 2 - \pi/4)$
The state space is stratified.

THEOREM

Suppose that the graph is finite. The stationary distribution for the process $\mathcal{M}_t$ exists and is unique. The process $\mathcal{M}_t$ converges to the stationary distribution exponentially fast.

Proof (sketch). Consider two mass processes $\mathcal{M}_t$ and $\widetilde{\mathcal{M}}_t$ on the same graph, with different initial distributions but the same meteor hits.

Stationary distribution - existence and uniqueness

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$t \to \sum_{x \in V} \left| \mathcal{M}_t^x - \widetilde{\mathcal{M}}_t^x \right|$ is non-increasing.

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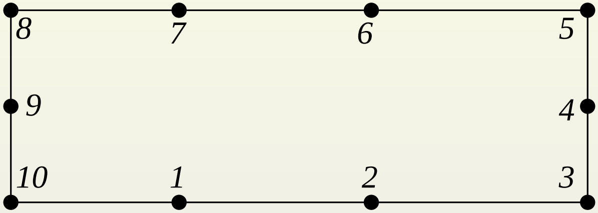
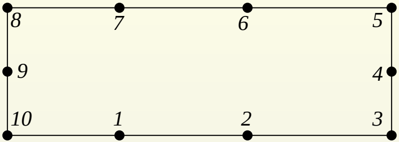
Hairer, Mattingly and Scheutzow (2011)

Circular graphs



C_k - circular graph with k vertices
Q_k - stationary distribution for $\mathcal{M}_t$ on C_k

Circular graphs - moments of mass at a vertex



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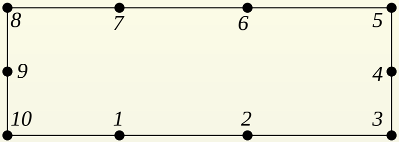

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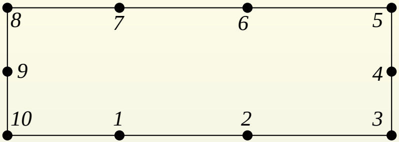

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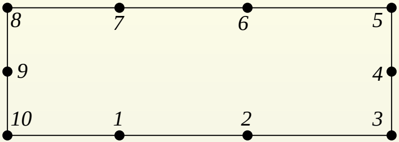

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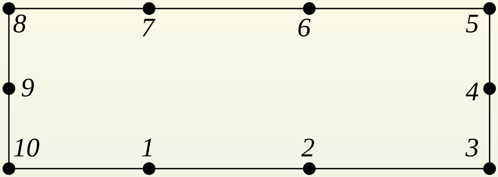
Circular graphs - correlation and independence



$$\lim_{k\to\infty} \mathrm{Cov}_{Q_k}(M_0^x, M_0^y) = 0, \qquad x \not\leftrightarrow y$$

If $x \not\leftrightarrow y$ then M_0^x and M_0^y do not appear to be asymptotically independent under Q_k's.
$(M_0^x)^2$ and M_0^y seem to be asymptotically correlated under Q_k, if $x \not\leftrightarrow y$.

From circular graphs to $\mathbb{Z}$

C_k - circular graph with k vertices
Q_k - stationary distribution for $\mathcal{M}_t$ on C_k

THEOREM

For every fixed n, the distributions of $(M_0^1, M_0^2, \dots, M_0^n)$ under Q_k converge to a limit Q_∞ as $k \to \infty$.

The theorem yields existence of a stationary distribution Q_∞ for the meteor process on $\mathbb{Z}$.

Similar results hold for meteor processes on C_k^d and $\mathbb{Z}^d$.

Moments of mass at a vertex in $\mathbb{Z}^d$

THEOREM

$$E_{Q_\infty} M_0^x = 1, \qquad x \in V$$

$$\operatorname{Var}_{Q_\infty} M_0^x = 1, \qquad x \in V$$

$$\operatorname{Cov}_{Q_\infty}(M_0^x, M_0^y) = -\frac{1}{2d}, \qquad x \leftrightarrow y$$

$$\operatorname{Cov}_{Q_\infty}(M_0^x, M_0^y) = 0, \qquad x \not\leftrightarrow y$$

Mass fluctuations in intervals of $\mathbb{Z}$

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For every n,

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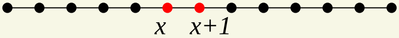
Flow across the boundary



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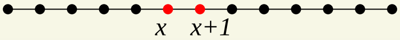


F_t^x - net flow from x to $x+1$ between times 0 and t

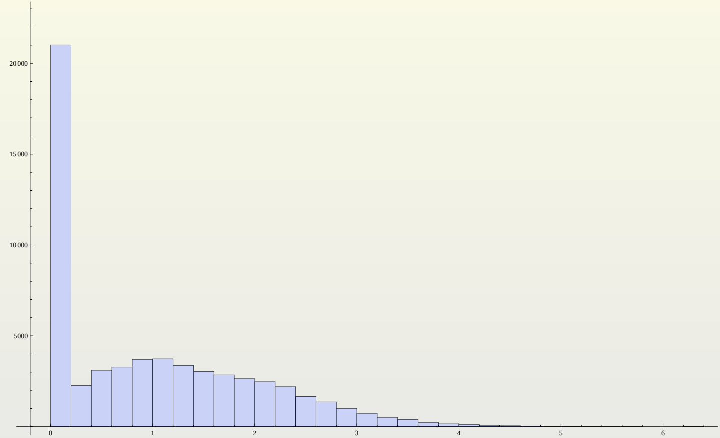
THEOREM

Under Q_∞, for all $x \in \mathbb{Z}$ and $t \geq 0$,

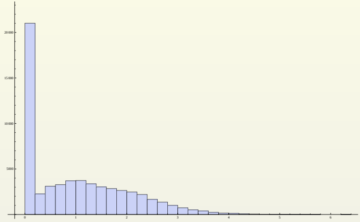
$$\mathrm{Var}\, F_t^x \leq 4.$$

Mass distribution at a vertex of $\mathbb{Z}$

A simulation of M_0^x under Q_∞.



Mass distribution at a vertex of $\mathbb{Z}$ (2)



$P_{Q_\infty}(M_0^x = 0) = 1/3$

$E_{Q_\infty} M_0^x = 1, \qquad \mathrm{Var}_{Q_\infty} M_0^x = 1$

Is Q_∞ a mixture of a gamma distribution and an atom at 0? No.

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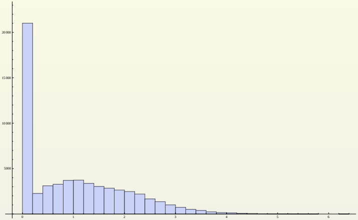

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Is Q_∞ a mixture of a gamma distribution and an atom at 0? No.

One can find an exact and rigorous value for $E_{Q_k}(M_0^x)^n$ for every n and k.
We cannot find asymptotic formulas when $k \to \infty$.

Support of the stationary distribution

Assume that $|V| = k$, and $\sum_{x \in V} M_0^x = k$.

Let $\mathcal{S}$ be the simplex consisting of all $\{S_x, x \in V\}$ with $S_x \geq 0$ for all $x \in V$ and $\sum_{x \in V} S_x = k$.

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Let $\mathcal{S}^*$ be the set of $\{S_x, x \in V\}$ with $S_x = 0$ for at least one $x \in V$.

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THEOREM

The (closed) support of the stationary distribution for $\mathcal{M}_t$ is equal to $\mathcal{S}^*$.

DEFINITION

Suppose that $\mathcal{M}_0$ is given and $k = \sum_{v \in V} M_0^v$.
For each $j \geq 1$, let $\{Y_n^j, n \geq 0\}$ be a discrete time symmetric random walk on G with the initial distribution $P(Y_0^j = x) = M_0^x / k$ for $x \in V$. We assume that conditional on $\mathcal{M}_0$, processes $\{Y_n^j, n \geq 0\}$, $j \geq 1$, are independent.

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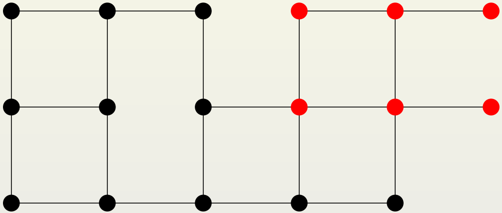
Recall Poisson processes N^v and assume that they are independent of $\{Y_n^j, n \geq 0\}$, $j \geq 1$. For every $j \geq 1$, we define a continuous time Markov process $\{Z_t^j, t \geq 0\}$ by requiring that the embedded discrete Markov chain for Z^j is Y^j and Z^j jumps at a time t if and only if N^v jumps at time t, where $v = Z_{t-}^j$.

WIMPs and convergence rate

The rate of convergence to equilibrium for $\mathcal{M}_t$ cannot be faster than that for a simple random walk. Justification: Consider expected occupation measures.

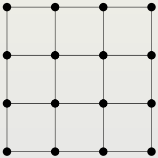


Rate of convergence on tori

THEOREM

Consider the meteor process on a graph $G = C_n^d$ (the product of d copies of the cycle C_n). Consider any distributions (possibly random) of mass $\mathcal{M}_0$ and $\widetilde{\mathcal{M}}_0$, and suppose that $\sum_x M_0^x = \sum_x \widetilde{M}_0^x = |V| = n^d$, a.s. There exist constants c_1, c_2 and c_3, not depending on G, such that if $n \geq 1 \vee c_1 \sqrt{d \log d}$ and $t \geq c_2 d n^2$ then one can define a coupling of mass processes $\mathcal{M}_t$ and $\widetilde{\mathcal{M}}_t$ on a common probability space so that,

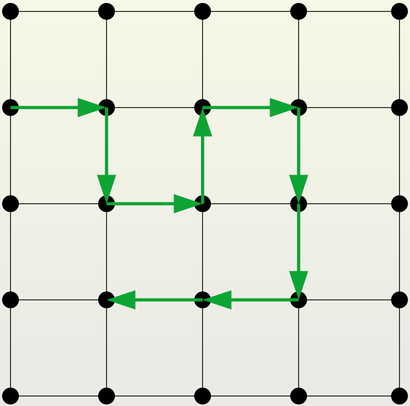
$$E\left(\sum_{x \in V} |M_t^x - \widetilde{M}_t^x|\right) \leq \exp(-c_3 t/(dn^2))|V|.$$



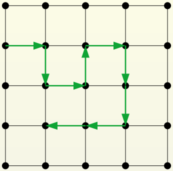
Earthworm

Earthworm = simple random walk

Redistribution events occur at the sites visited by earthworm



Earthworms equidistribute soil



THEOREM

Fix $d \geq 1$ and let $\mathbf{M}_t^n$ be the empirical measure process for the earthworm process on the graph $G = C_n^d$. Assume that $M_0^v = 1/n^d$ for $v \in V$ (hence, $\sum_{v \in V} M_0^v = 1$).

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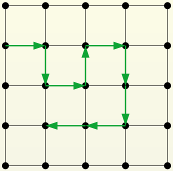

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(i) For every n, the random measures $\mathbf{M}_t^n$ converge weakly to a random measure $\mathbf{M}_\infty^n$, when $t \to \infty$.

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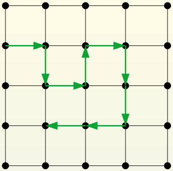

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(i) For every n, the random measures $\mathbf{M}_t^n$ converge weakly to a random measure $\mathbf{M}_\infty^n$, when $t \to \infty$.

(ii) For $R \subset \mathbb{R}^d$ and $a \in \mathbb{R}$, let $aR = \{x \in \mathbb{R}^d : x = ay \text{ for some } y \in R\}$ and $\widehat{\mathbf{M}}_\infty^n(R) = \mathbf{M}_\infty^n(nR)$. When $n \to \infty$, the random measures $\widehat{\mathbf{M}}_\infty^n$ converge weakly to the random measure equal to, a.s., the uniform probability measure on $[0, 1]^d$.

Craters in circular graphs

$G = C_k$

There is a crater at x at time t if $M_t^x = 0$.

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Craters in circular graphs

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Under the stationary distribution, the distribution of craters in C_k is the same as the distribution of peaks in a random (uniform) permutation of size k.

Peaks in random permutations

It is possible to find a formula for the probability of a given peak set in a random permutation.

THEOREM

$$P(\text{crater at } 1) = 1/3,$$

$$P(\text{crater at 1 followed by exactly } n \text{ non-craters}) = \frac{n(n+3)2^{n+1}}{(n+4)!},$$

$$P(\text{no craters at } 1, 2, \ldots, n) = \frac{2^{n+1}}{(n+2)!}.$$

THEOREM

$$
\begin{aligned}
&P(\text{crater at 1 followed by } i \text{ non-craters, then a crater,}\\
&\quad \text{then exactly } j \text{ non-craters})\\
&= \frac{2^{i+j}}{(i+j+5)!}\left[(i+j+4)\left(j\binom{i+j+1}{i-1} + (j+1)\binom{i+j+1}{i}\right.\right.\\
&\left.\left. + (i+1)\binom{i+j+1}{i+1} + i\binom{i+j+1}{i+2} - 2(i+j+1)\right) + ij\binom{i+j+4}{i+2}\right].
\end{aligned}
$$

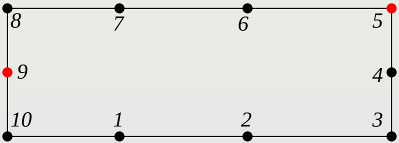
Craters repel each other

THEOREM

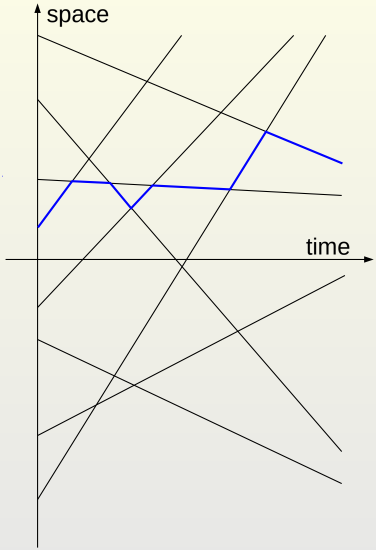
Consider the meteor process on a circular graph C_k in the stationary regime. Let $\mathcal{G}$ be the family of adjacent craters, i.e., $(i,j) \in \mathcal{G}$ if an only if there are craters at i and j and there are no craters between i and j. For $r > 1$, let

$$A_r^1 = \left\{ \frac{\max_{(i,j)\in\mathcal{G}_1} |i-j|}{\min_{(i,j)\in\mathcal{G}_1} |i-j|} \leq 1 + r \right\}.$$

Let H_n^1 be the event that there are exactly n craters at time 0. For every $n \geq 2$, $p < 1$ and $r > 1$ there exists $k_1 < \infty$ such that for all $k \geq k_1$, $P(A_r^1 \mid H_n^1) > p$.



Systems of non-crossing paths



Non-crossing continuous path models:
Harris (1965)
Spitzer (1968) - shown
Dürr, Goldstein and Lebowitz (1985)
Tagged particle in exclusion process:
Arratia (1983)

Free path scaling: $dX \approx (dt)^{\alpha}$
Non-crossing path scaling:
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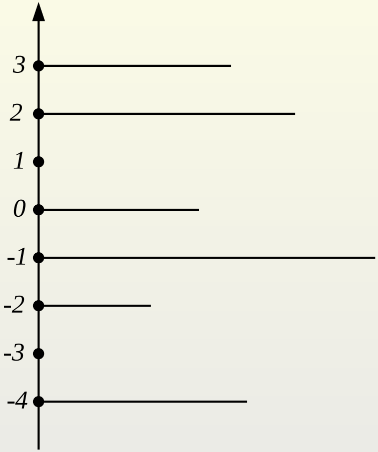
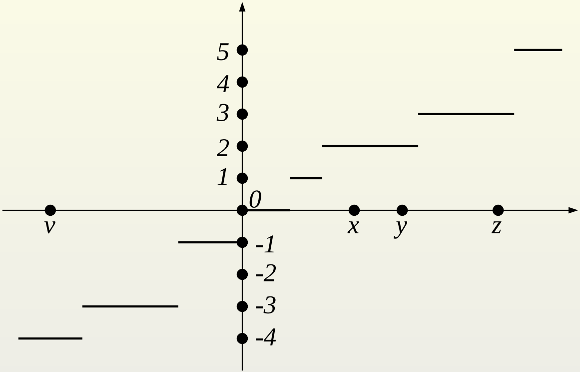
Meteor process on $\mathbb{Z}$

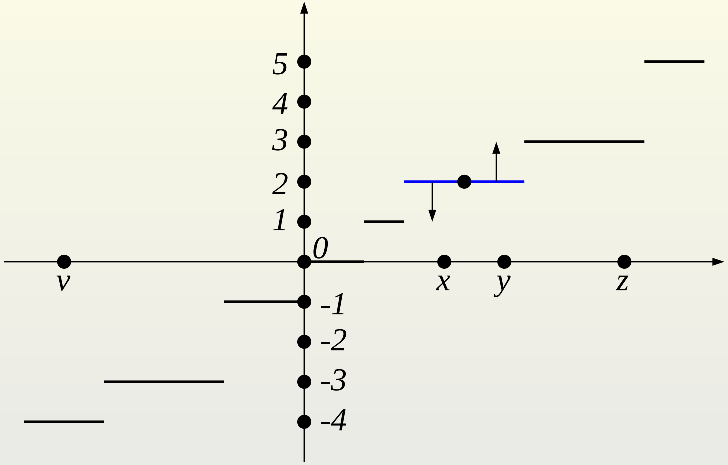


There is no time scale in this picture. The horizontal axis represents mass.

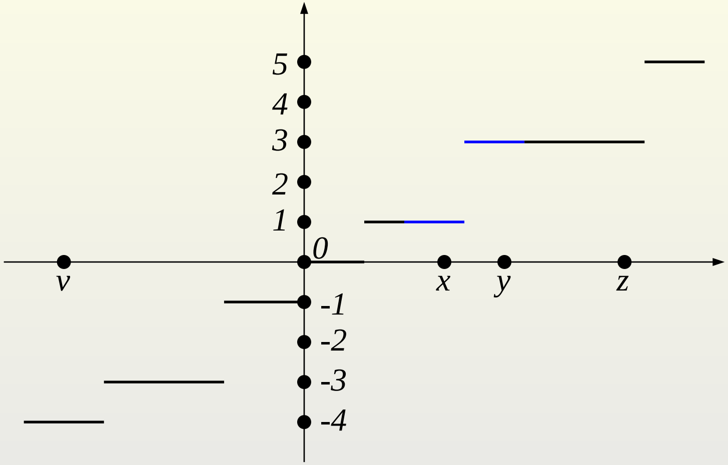


The horizontal axis represents mass. The state of the meteor process at time t is represented as an RCLL function $H_t^{\cdot}$. For example, $H_t^x = H_t^y = 2$, $H_t^v = -4$ and $H_t^z = 3$.

Jump of meteor process



Jump of meteor process (2)



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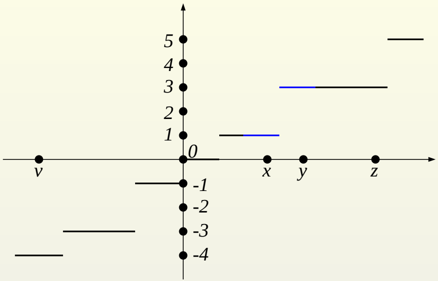

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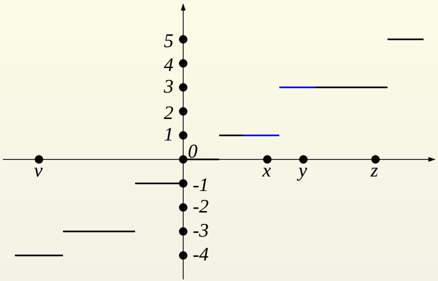

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THEOREM

Suppose that that the meteor process is in the stationary distribution Q. Then for every $\alpha < 2$ there exists $c < \infty$ such that for every $x \in \mathbb{Z}$ and $t \geq 0$,

$$E|H_t^x - H_0^x|^\alpha \leq c.$$

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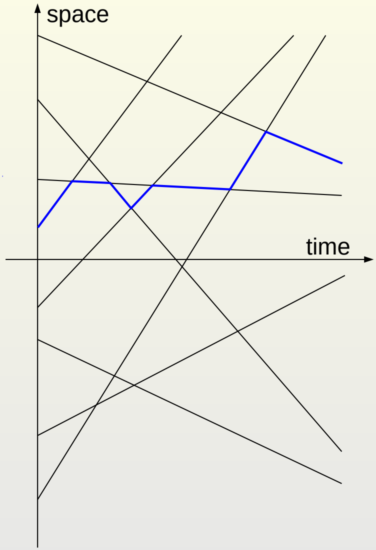

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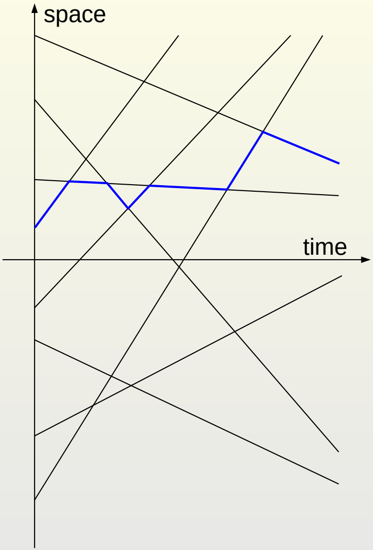

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Harris (1965)
Spitzer (1968) - shown
Dürr, Goldstein and Lebowitz (1985)
Tagged particle in exclusion process:
Arratia (1983)

Free path scaling: $dX \approx (dt)^{\alpha}$
Non-crossing path scaling:
$dX \approx (dt)^{\alpha/2}$

Meteor model
Free path scaling: $dX \approx (dt)^{1/2}$
Non-crossing path scaling:
$dX \approx (dt)^{0}$