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A real-world Markov chain arising in recreational volleyball

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Card shuffling models have provided simple motivating examples for the mathematical theory of mixing times for Markov chains. As a complement, we introduce a more intricate realistic model of a certain observable real-world scheme for mixing human players onto teams. We quantify numerically the effectiveness of this mixing scheme over the seven or eight steps performed in practice. We give a combinatorial proof of the nontrivial fact that the chain is indeed irreducible.

1. Introduction

In introducing Markov chains at some elementary level, the first author always found it difficult to give motivating examples with a real-world story, a plausible probability model, and a fairly rich mathematical structure. Then he realized that he was a regular participant in one such story. A first thought was to write out the model for possible use as an instructional example in an introductory lecture. As often happens, things turned out to be more complicated than first imagined, so it was repurposed as a basis for a challenging undergraduate project to study further aspects of the model. The second author took up the challenge. Some remaining questions that could be used for undergraduate projects are mentioned in Section 6.4.

2. The model

The story concerns recreational volleyball, in a drop-in setting without fixed teams, and where one wants the team compositions to change from game to game, both for socialization and to avoid persistent large differences in team skill levels. Specifically, there are 24 people, and at each stage, there are two ongoing games on two courts, each game between two teams, each team with six players on a half-court. Over the two-hour period there will be seven or eight successive such stages,

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A	F	E	i	vi	v	
B	C	D	ii	iii	iv	
d	c	b	δ	γ	β	start game 0
e	f	a	ϵ	ϕ	α	
F	E	D	iv	iii	ii	
A	B	C	v	vi	i	
b	a	f	β	α	ϕ	end game 0
c	d	e	γ	δ	ϵ	
F	E	D	iv	iii	ii	
f	a	b	A	B	C	
β	α	ϕ	i	vi	v	start game 1
c	d	e	γ	δ	ϵ	

Figure 1. One step of the big chain. As in football, *positions* are relative to the way a team is facing: at start of game 0 both b and B are *front right* in their teams.

everyone always playing. The rule[1] for changing team composition is very simple, exploiting a particular incidental feature of volleyball:

> *At the end of one stage, the players in the back row of each team stay in these positions for the start of the next game, while the front row players move (clockwise in the gym) to the same positions in the next quadrant.*

See Figure 1. The key point is that in volleyball, there are six positions,[2] and players rotate one position each time their team regains the serve, and this happens a *random* number of times during a game. So, relative to initial positions, the three players who finish in the front row will in fact be (to a good approximation) a *uniform* random choice over the six possibilities of three adjacent players, and so we model this as a uniform random choice.

To complete a mathematical model, note that the number of one-position rotations of two opposing teams can differ (because they alternate rotations) by at most 1. So, independently for the two courts, we model the final positions of players in opposing

[1] Actually used in the gym where the first author plays; I don't know how common it is.

[2] By convention numbered 1 to 6, starting in serving position (back right, as facing the net) and ordered counterclockwise. Because players rotate clockwise, this indicates serving order.

teams in a game as rotations by (C_1, C_2) where C_1 is uniform[3] on $\{0, 1, \ldots, 5\}$ and $C_2 = C_1 - 1 + \text{Binomial}\big(2, \frac{1}{2}\big)$ modulo 6, the Binomial term reflecting randomness of the initial serving team and the final serving team.

This specifies a "big" Markov chain on the 24! states (assignment of players to positions). One step of this *big chain* is from the starting positions in one game to the starting positions in the next game, as illustrated in Figure 1. This model is conceptually loosely related to some card shuffling models[4] such as those in [Diaconis 1998; Levin and Peres 2017] in that a "rotation" of team players corresponds to a cut-shuffle of a six-card deck. But unlike playing cards, the volleyball players care about their positions relative to other players for various reasons[5] and this suggests actual observables for study in the model.

3. Results

The central, albeit vague, question is *how effective is this scheme at mixing up the teams*?

In a lecture course, this would provide a real-world example for later discussion of the *mixing times* topic. It seems intuitively obvious that this scheme would mix perfectly in the long run. What does that mean? First observe that the chain is doubly stochastic. To see why, consider the step illustrated in Figure 1. This takes the "start game 0" configuration $\boldsymbol{x}_0$ to a certain "start game 1" configuration $\boldsymbol{x}_1$, via a series of rotations. By reversing that series, one sees that there exists a configuration $\boldsymbol{x}_{-1}$ which, from the same (forwards) series of rotations, takes $\boldsymbol{x}_{-1}$ to $\boldsymbol{x}_0$. This leads to (for fixed $\boldsymbol{x}_0$) a bijection between possible configurations $\boldsymbol{x}_1$ and $\boldsymbol{x}_{-1}$ which preserves transition probabilities; which in turn implies the doubly stochastic property. And that property implies that the uniform distribution on all 24! states is a stationary distribution for the chain.[6]

Basic finite Markov chain theory[7] identifies "mix perfectly in the long run" with *irreducible and aperiodic*, which implies convergence of time-t distributions to a unique stationary distribution, which in our model must be the uniform distribution on all 24! states. Theory also tells us that *irreducible* is equivalent to the property *the directed graph of all possible transitions on the* 24! *states is strongly connected.*

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[3] In a more detailed model, each team wins a random Geometric$\big(\frac{1}{2}\big)$ number of points between rotations, and the game ends when one team reaches 25 points. Then the number of rotations modulo 6 does indeed have distribution close to uniform.

[4] See further discussion in Section 8.

[5] Friendly rivalry between spikers/blockers; more talented setters enable sophisticated fast plays; attractive members of opposite sex; ....

[6] More generally, any card-shuffling scheme in which the shuffling rule depends only on the ranks (positions within deck) and not on the labels of the cards will be doubly stochastic.

[7] In many textbooks such as [Häggström 2002; Privault 2013].

This property is purely combinatorial — the numerical values of the nonzero transition probabilities do not matter. So our first goal is to prove irreducibility and aperiodicity. We give a constructive proof of irreducibility in Section 4, and a proof of aperiodicity in Section 5. Our proofs are rather complicated, and no doubt there exist simpler proofs.

Our second set of results concern numerical calculation or simulation of statistics relating to the realistic short term in this story — seven or eight steps. Some basic observables involve the *friend chain* indicating the relative positions of two players. A variety of numerical results are shown in Section 6. For instance, if your friend does not start on your team, then the probability that you are never on the same team over eight games varies between 0.251 and 0.403 depending on initial relative positions (Table 5). A pedagogic point is that, for numerical calculations, we don't want to work with $24! \times 24!$ transition matrices, but instead exploit symmetry to reduce to question-specific small-state chains. For instance the way a given player moves between games is simple: with chance $\frac{1}{2}$ they stay, with chance $\frac{1}{2}$ they move to the next quadrant. In jargon, this is called the lazy cyclic walk [Levin and Peres 2017].

The bottom line is that, as regards simple observables, this scheme does a reasonable job of mixing up the teams over eight games. However, the central point of sophisticated *mixing time* theory [Levin and Peres 2017] is to go beyond the unquantified "eventually" implied by irreducibility, and instead to quantify when the step-t distribution is close to (in our case) the uniform distribution. The usual quantification involves *variation distance* between distributions on the 24! states, and this is the context of the famous result of [Bayer and Diaconis 1992] for riffle shuffles, informally called "seven shuffles suffice" [Kolata 1990]. Studying variation distance for our big chain, either numerically or via analytic bounds, remains a challenging open problem. We give some preliminary observations in Section 7.

4. The big chain is irreducible

4.1. ***Notation.*** To prove that the "big" chain is irreducible, we will show that it is possible to move from any one given state to any other given state via some sequence of allowable transitions of the chain.

Label the four quadrants (half-courts) A, B, C, D as shown in Figure 2. The change in configuration, from the start of a game to the end of that game, can be represented symbolically in the form

$$A^{x_1} C^{x_2} B^{x_3} D^{x_4},$$

where $0 \le x_i \le 5$ indicates the number of positions (modulo 6) rotated by the team in the relevant quadrant. Figure 3 gives an illustration.

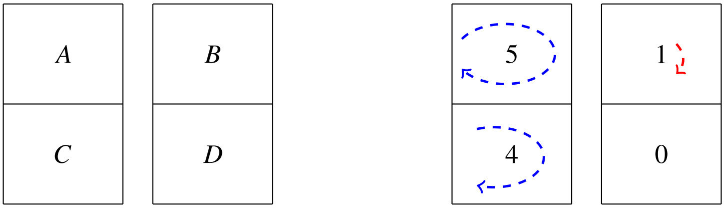

Figure 2. Left: labeling of the four quadrants. Right: rotations involved in step A^5C^4BE.

0	1	2	3	4	5		1	2	8	9	3	4
6	7	8	9	10	11		19	20	14	0	6	7
12	13	14	15	16	17	$\rightarrow$	15	16	17	5	11	10
18	19	20	21	22	23		13	12	18	21	22	23

Figure 3. The effect of step A^5C^4BE.

So the allowable values are $x_1 - x_2 \in \{-1, 0, 1\}$ and $x_3 - x_4 \in \{-1, 0, 1\}$ modulo 6. We then append a symbol E to indicate the final movement (the front row players in each quadrant move to the same positions in the next quadrant). This provides a coding of a step of the chain. For brevity we omit any $x_i = 0$ term and write B instead of B^1. So a typical step is coded in a format like A^5C^4BE. The reader may check that the Figure 1 example is $A^5C^4B^3D^2E$.

A sequence of steps can then be specified by concatenation, so A^5C^4BEEDE represents three steps of the chain, the second (EE) step indicating a game with zero (modulo 6) rotations of each team before the front row switch. We will name certain sequences later as X, F, G, H in describing the construction. The number of steps in a sequence is just the number of E's, when expanded fully. In writing the sequences (such as the definition of X below) we often include spaces for visual clarity but the spaces have no mathematical significance.

One aspect of this notation may be confusing. The sequence $EEEE$ would code the identity move. That means that $BE\ EEEE$ has the same effect as BE. But note that $BEEEE$ is different, and in fact will be a useful device because it has the effect of rotating the players in quadrant B while fixing all other players — see Figure 4. Note also that B^5EEEE is the analogous back-rotation. This syntax issue explains why we sometimes (e.g., in the definition of F below) need to include an initial $EEEE$ in the definition.

4.2. ***High level description 1.*** It is an elementary fact that any permutation of a card deck can be obtained by a sequence of transpositions of two adjacent cards. Indeed the "random adjacent transposition" shuffling scheme is one of the original and

0	1	2	3	4	5		0	1	2	9	3	4
6	7	8	9	10	11	$\to$	6	7	8	10	11	5
12	13	14	15	16	17		12	13	14	15	16	17
18	19	20	21	22	23		18	19	20	21	22	23

Figure 4. The effect of sequence $BEEEE$.

0	1	2	3	4	5		1	0	2	3	4	5
6	7	8	9	10	11	$\to$	6	7	8	9	10	11
12	13	14	15	16	17		12	13	14	15	16	17
18	19	20	21	22	23		18	19	20	21	22	23

(a) Transpose two adjacent players in the same quadrant (sequence F).

0	1	2	3	4	5		2	1	0	3	4	5
6	7	8	9	10	11	$\to$	6	7	8	9	10	11
12	13	14	15	16	17		12	13	14	15	16	17
18	19	20	21	22	23		18	19	20	21	22	23

(b) Transpose two players in the same quadrant with one space in between (G).

0	1	2	3	4	5		8	1	2	3	4	5
6	7	8	9	10	11	$\to$	6	7	0	9	10	11
12	13	14	15	16	17		12	13	14	15	16	17
18	19	20	21	22	23		18	19	20	21	22	23

(c) Transpose two players in the same quadrant with two spaces in between (H).

Figure 5. Transpositions achieved by the specific sequences F, G, H.

most deeply studied examples in the modern theory of mixing times [Aldous 1983; Lacoin 2016; Wilson 2004]. By analogy, we start by showing that any transposition of two players on the same quadrant can be obtained by some sequence of steps. There are three cases, depending on the initial distance between the two players, illustrated in Figure 5, and we will exhibit sequences F, G, H for each case. We will show the first case (adjacent players) in detail.

4.3. ***Sequences that transpose two players.*** We start by introducing a 16-step sequence X defined as

$$X := AE\ \ B^2D^3EEE\ \ A^2C^3E\ \ B^3D^3EEE\ \ A^5E\ \ B^5EEE\ \ AE\ \ B^5EEE\ \ AC^3.$$

$$\begin{array}{|ccc|ccc|}\hline 0 & 1 & 2 & 3 & 4 & 5\\ 6 & 7 & 8 & 9 & 10 & 11\\ \hline 12 & 13 & 14 & 15 & 16 & 17\\ 18 & 19 & 20 & 21 & 22 & 23\\ \hline\end{array} \quad\rightarrow\quad \begin{array}{|ccc|ccc|}\hline 1 & 0 & 2 & 4 & 7 & 8\\ 5 & 3 & 6 & 9 & 10 & 11\\ \hline 12 & 13 & 14 & 15 & 16 & 17\\ 18 & 19 & 20 & 21 & 22 & 23\\ \hline\end{array}$$

Figure 6. The effect of sequence X.

The step-by-step trajectory of sequence X is shown in Figure 7 below, which demonstrates that the effect of X is as shown in Figure 6. The introduction of this X is somewhat magical and hard to explain, but note that for some players it is like a reverse step of the chain.

We can now define the sequence F that transposes the two adjacent players at the left corner of the back row of quadrant A, as shown in Figure 5. Essentially it is just three applications of X. Precisely

$$F := EEEEX\ X\ XEEEE.$$

So F involves 68 steps of the chain. Figure 8 shows the step-by-step trajectory of sequence F.

The other transposition sequences. To transpose the two players at the back row of quadrant A with one space in between, as shown in Figure 5, we use the sequence G defined as

$$G := EEEE\ A^5FA\ F\ A^5FA\ EEEE.$$

This works because the effect of these sequences is to alter the back row as

$$012 \to 021 \to 201 \to 210.$$

Finally, to transpose the players at the upper left corner and at the lower right corner in quadrant A, as shown in Figure 5, we use the sequence H defined as

$$H := FA^5\ FA^5\ FAFAF.$$

The reader may check that this works, and is one place where the initial $EEEE$ in the definition of F is needed.

4.4. ***High level description 2.*** By symmetry, to prove irreducibility it is enough to prove that, from any initial state, one can reach (by some sequence of allowable steps) the reference state shown in Figure 9, where each player i is in position i. Because any permutation of the six players in quadrant A can be derived from a sequence of transpositions, from the existence of transposition sequences (Section 4.3) it suffices to show that we can move, by some sequence of steps, player i to position i, for

1. AE (separate player 2 from 0 and 1)

6	0	1	3	4	5
14	13	12	7	8	2
15	16	17	11	10	9
18	19	20	21	22	23

2–4. B^2D^3EEE (move player 2 to the front right position in A)

6	0	1	8	7	3
2	5	4	21	22	23
12	13	14	15	16	17
18	19	20	9	10	11

5. A^2C^3E (separate players 0 and 1 from 2)

5	2	6	8	7	3
18	19	20	4	1	0
15	16	17	23	22	21
14	13	12	9	10	11

6–8. B^3D^3EEE (move player 0 at the back right of B)

5	2	6	0	1	4
3	7	8	9	10	11
20	19	18	15	16	17
14	13	12	21	22	23

9. A^5E (rotate A to move player 2 at the back right of A)

2	6	8	0	1	4
18	19	20	5	3	7
15	16	17	11	10	9
14	13	12	21	22	23

10–12. B^5EEE (move player 2 to the left of 0)

2	6	8	1	4	7
0	5	3	9	10	11
20	19	18	15	16	17
14	13	12	21	22	23

13. AE

0	2	6	1	4	7
18	19	20	5	3	8
15	16	17	11	10	9
14	13	12	21	22	23

14–16. B^5EEE (move player 1 to the left of 02)

0	2	6	4	7	8
1	5	3	9	10	11
20	19	18	15	16	17
14	13	12	21	22	23

17–20. AC^2EEEEC (same effect as AC^3)

1	0	2	4	7	8
5	3	6	9	10	11
12	13	14	15	16	17
18	19	20	21	22	23

Figure 7. Step-by-step trajectory of sequence *X*.

1. $EEEEX$

1	0	2	4	7	8
5	3	6	9	10	11
12	13	14	15	16	17
18	19	20	21	22	23

2. X

0	1	2	7	3	6
8	4	5	9	10	11
12	13	14	15	16	17
18	19	20	21	22	23

3. $XEEEE$

1	0	2	3	4	5
6	7	8	9	10	11
12	13	14	15	16	17
18	19	20	21	22	23

Figure 8. Step-by-step trajectory of sequence F.

A	0	1	2	B	3	4	5
	6	7	8		9	10	11
C	12	13	14	D	15	16	17
	18	19	20		21	22	23

Figure 9. Reference state.

every player in all the other quadrants B, C, D. We will move players to positions row by row, in the following order:

- Back row of D: (21, 22, 23).
- Back row of C: (18, 19, 20).
- Front row of D: (17, 16, 15).
- Front row of C: (14, 13, 12).
- Back row of B: (5, 4, 3).
- Front row of B: (9, 10, 11).

Each row in turn is *fixed*, in that it remains in place after each subsequent row has been moved to its position.

In the next section we will describe the algorithm in words. A key point is that we will use quadrant A as a kind of temporary stopover for players in transit.

4.5. ***The algorithm.*** As noted in Figure 4 a sequence like $BEEE$ has the effect of rotating a given quadrant by one position, and one can repeat such a sequence. So we can use phrases such as "*rotate* player i to position j" (in the same quadrant). We will call E the *migration* step, so by repeating step E we can use "migrate player i to position j", where player i has the same relative position (e.g., front, right) as j, in the front row. (But remember that all 12 front-row players migrate.) In both cases we of course only do the move if necessary, that is if the player is not already in the desired position.

The algorithm is based on variations of the following Procedure P, where P is one of the quadrants B, C, D, and where the procedure acts to move the required three players to the back row of P.

Procedure P. (1) Label the back row positions of P as a, b, c, left-to-right. (So for $P = D$ we have $(a, b, c) = (21, 22, 23)$.)

(2) Rotate (if in back row) player a to front row. Migrate player a to quadrant A and rotate to front right position. Migrate player a to the front right position in P.

(3) Rotate player a to back right position in P.

(4a) If player b is not in P, repeat actions (2) for player b. This moves player b to the front right position in P.

(4b) Else, rotate P so that player b is in front row, while player a is in back row. Migrate player b to quadrant A. Rotate A to move player b to front right. Rotate P so that player a is returned to back right position in P. Migrate player b to the front right position in P.

(5) Now players (a, b) are in (back right, front right) positions in P: rotate one space to (back center, back right) positions.

(Next we move player c in essentially the same way as player b, Here are the details.)

(6a) If player c is not in P, repeat actions (2) for player c. This moves player c to the front right position in P.

(6b) Else, rotate P so that player c is in front row, while players a, b are in back row. Migrate player c to quadrant A. Rotate A to move player c to front right. Rotate P so that players (a, b) are returned to (back center, back right) positions in P. Migrate player c to the front right position in P.

(7) Rotate players (a, b, c) to (back left, back center, back right) positions in P.

Procedure P moves the required three players to the back row of P. In the algorithm below, once a back row is placed, it is fixed and those players are never moved subsequently. This is because that would require a rotation of P at some subsequent use of (2), which cannot happen because of the *if in back row* condition in (2); these players are already in place.

The algorithm. (1) Apply Procedure D, then Procedure C. (This fixes the back rows of C and D. For the third row (the front row in D) we will use a little trick, to first arrange them on a back row.)

(2) Here, we consider players (17,16,15) who need to be moved to the front row of D. But we label them as (a, b, c) = (5,4,3) and use Procedure B to move them to the back row of B. Then revert to labels (17, 16, 15).

(3) Rotate players (17,16,15) three positions so they become the front row of B.

(4) Migrate one step, so players (17,16,15) become the front row in D. (The previous back rows remain fixed. Next we consider the fourth row, the front row of C. Here another small complication arises; any migration will move the front row of D (fixed above), so we need to ensure that it will not be rotated (we can migrate if necessary), and it is migrated back into place before finishing.)

(5) Here we are considering players (14,13,12). As in (9), label them as $(a, b, c) = (5, 4, 3)$ and use Procedure B to move them to the back row of B. Then revert to labels (14,13,12). This involves a certain number of migrates. When applying Procedure B, we make sure players (17, 16, 15) remain at the front row in some quadrant at all times by not rotating them, and the fixed back court players (18-23) are not moved.

(If we need to rotate a quadrant where players (17,16,15) are in the front row, migrate first until that front row does not contain players (17, 16, 15), b if b is not at the back row of P, and c if b is at the back row of P and c is not, then rotate that quadrant.)

(6) Migrate players (17, 16, 15) to make them the front row in quadrant A. Rotate players (14,13,12) in B three positions so they become the front row in B. Migrate two turns.

(Now all players 12–23 in quadrants C and D have been moved to their reference positions.)

(7) Apply Procedure B. This moves players $(5, 4, 3)$ to the back row of B, as required. It involves some number of migrates, but the players 12–17 assigned to front rows of C and D should remain as front rows of adjacent quadrants, so make sure not to rotate them and migrate them back to their required positions.

(If a rotation is needed, migrate first until that front row does not contain players 12–17, b if b is not at the back row of P, and c if b is at the back row of P and c is not.)

(Now we have fixed all rows of B, C, D except the front row of B. For the front row of B we do a quite different scheme, using the transposition sequences from Section 4.3.)

(8) Here we are considering players $(11, 10, 9)$. Our first goal is to move them to target positions $(2, 1, 0)$ in the back row of A. Any of those players $(11, 10, 9)$ in A can be moved to their target position via transpositions. Remaining players amongst $(11, 10, 9)$ must be in the front row of B. Migrate the front row of B to the front row of A, transpose relevant players to target positions, and migrate back.

(9) Players $(11, 10, 9)$ are at positions $(2, 1, 0)$ in the back row of A. Migrate $(17, 16, 15)$ to the front row of B. Rotate players $(11, 10, 9)$ three positions so they become the front row of A, and migrate to front row of B.

(Now all rows of B, C, D are fixed.)

(10) As noted earlier, any permutation of the six players in quadrant A can be derived from a sequence of transpositions, so we can fix quadrant A by using the transposition sequences F, G, H from Section 4.3.

This completes our proof of irreducibility. Recall that the basic idea was to move players to positions row-by-row. It seems likely that there is some alternate "basic idea" that would lead to a simpler proof, and so we have not tried to optimize the implementation of our basic idea.

5. The big chain is aperiodic

We know that it is possible to move from one state to that same state in four steps — $EEEE$. So to prove that the big chain is aperiodic, it is sufficient to exhibit a sequence of n steps, for some odd n, that move the reference state (Figure 9) to that same state. Such a sequence is shown in Figure 10, There are 163 E's in the unexpanded notation, and 68, 424, and 340 steps from the one F, two G's, and one H, respectively, so there are 995 steps in this sequence.

6. The friend chain

Perhaps the most natural observable to study concerns the positions of two players, say *ego* and *friend*. Naively this would require a 24×23 state chain, but we can exploit some symmetries to make a 26-state chain indicating *relative* positions of the two players. Doing so requires some care; the states are indicated in Figure 11, as explained next.

First note that our process is not invariant under a quarter-turn of the four quadrants,[8] but is invariant under a half-turn, so as in Figure 11, we can assume *ego* is in the left court.

The states of what we will call the *friend chain* indicate relative positions at the start of a game. To describe the state, first record which quadrant *ego* is in (denoted here as initial 1 or 2; that is, quadrants C or A) and then whether *friend*

[8] A friend on the same team in quadrant B might be an opponent in the next game; this cannot happen on quadrant D.

1–13. *AEEEE AEAE CE DEEEE DEDE* (move to a certain other arrangement)

14	7	6	3	4	5
0	21	22	8	2	1
23	15	16	18	11	10
19	20	9	13	12	17

14–32. *EEDEEEE DEEEE DEE CEEEE CEEE D* (fixing back row of D)

14	7	6	3	4	5
8	2	1	10	11	18
17	12	13	0	20	19
9	16	15	21	22	23

33–57. *CEEEE CEEEE CEEE AEEEE AEEE CEEE CEEEE C* (fixing back row of C)

11	10	14	3	4	5
9	16	15	8	2	1
17	12	0	13	7	6
18	19	20	21	22	23

58–67. *EAEEEE AEBEE AE BBBBBE* (fixing (17,16,15) using Procedure B then some migration)

9	12	0	3	4	11
6	7	13	14	10	5
1	2	8	15	16	17
18	19	20	21	22	23

68–87. *BAEEEE AEBEEE AEBE BEEEE BEEEE BEE* (fixing (14,13,12) using Procedure B then some migration)

0	7	6	1	2	8
4	9	3	10	5	11
12	13	14	15	16	17
18	19	20	21	22	23

88–99. *BEEEE BEBEEE AEBEEE* (fixing (5,4,3) using Procedure B)

9	0	7	3	4	5
1	6	10	11	8	2
12	13	14	15	16	17
18	19	20	21	22	23

100–135. *AEEEE AEEEE AGAEEEE AEEEE AEEEE AEEE AEEEE AEEEE AEEEE AE* (fixing (9,10,11) using Steps 15 and 16 of the algorithm)

0	2	8	3	4	5
6	1	7	9	10	11
12	13	14	15	16	17
18	19	20	21	22	23

136–163. *AEEEE AHA AGA AEEEE AEEEE AEEEE AFAEEEE AEEEE AEEEE* (fixing A)

0	1	2	3	4	5
6	1	7	9	10	11
12	13	14	15	16	17
18	19	20	21	22	23

Figure 10. Sample moves.

all 2+	$2T5$	$2T4$	$2T3$	
	$2T6$	$\bullet$	$2T2$	
all 2−	$2O2$	$2O1$	$2O6$	*ego* $\bullet$ in second quadrant
	$2O3$	$2O4$	$2O5$	
all 1+	$1O1$	$1O6$	$1O5$	
	$1O2$	$1O3$	$1O4$	
all 1−	$1T4$	$1T3$	$1T2$	*ego* $\bullet$ in first quadrant
	$1T5$	$1T6$	$\bullet$	

Figure 11. Positions of *friend* relative to *ego* $\bullet$.

is in the same team (denoted by T) or the current opposing team (denoted by O), or on the other court. If on the other court, we only need to note which quadrant (because the position will be randomized during the game), so denote by one of (1+, 1−, 2+, 2−) as illustrated. Finally, writing temporarily ego* for the opponent in the same position as ego, we indicate a friend's position as 1, 2, 3, 4, 5, or 6, counterclockwise from ego or ego*.

This adds up to 26 states, and one can check that the big chain rules define a Markov chain on these states, with transition matrix $\boldsymbol{P}$ as shown in Figures 12 and 13. Its stationary distribution π is induced from the uniform stationary distribution of the big chain: probability $\frac{6}{46}$ for each of the states 1+, 1−, 2+, 2− and probability $\frac{1}{46}$ for each of the 22 remaining states.

6.1. ***Numerics for the friend chain.*** Let us investigate the mixing properties of the friend chain. Standard theory quantifies "closeness to stationarity after n steps" via *variation distance* $d^*(n)$ or *separation distance* $s^*(n)$ from worst-case start, that is, via

$$d^*(n) := \max_i \frac{1}{2} \sum_j |p_{ij}^n - \pi_j|,$$

$$s^*(n) := \max_{i,j} (1 - p_{ij}^n / \pi_j)$$

for the n-step transition matrix $\boldsymbol{P}^n$. Another measure of distance to stationarity is the L^2 or the χ^2 distance. The L^2 distance between P_i and the stationary distribution π after n steps is

$$\| P^n(i, \cdot) - \pi \|_2 = \sqrt{\sum_j \frac{(p_{ij}^n - \pi_j)^2}{\pi_j}}$$

and

$$\| P^n - \pi \|_2 = \max_i \| P^n(i, \cdot) - \pi \|_2.$$

Shown in Table 1 are the numerical values for these distances.

	$1+$	$1-$	$1T2$	$1T3$	$1T4$	$1T5$	$1T6$	$1O1$	$1O2$	$1O3$	$1O4$	$1O5$	$1O6$
$1+$	$\frac{1}{4}$	$\frac{1}{4}$											
$1-$		$\frac{1}{4}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$						
$1T2$			$\frac{1}{3}$						$\frac{1}{6}$				
$1T3$				$\frac{1}{6}$						$\frac{1}{3}$			
$1T4$											$\frac{1}{2}$		
$1T5$						$\frac{1}{6}$						$\frac{1}{3}$	
$1T6$							$\frac{1}{3}$						$\frac{1}{6}$
$1O1$	$\frac{1}{12}$							$\frac{1}{4}$	$\frac{1}{12}$				$\frac{1}{12}$
$1O2$	$\frac{1}{6}$							$\frac{3}{24}$	$\frac{1}{6}$	$\frac{1}{24}$			
$1O3$	$\frac{1}{3}$								$\frac{1}{12}$	$\frac{1}{12}$			
$1O4$	$\frac{5}{12}$									$\frac{1}{24}$		$\frac{1}{24}$	
$1O5$	$\frac{1}{3}$											$\frac{1}{12}$	$\frac{1}{12}$
$1O6$	$\frac{1}{6}$							$\frac{3}{24}$				$\frac{1}{24}$	$\frac{1}{6}$
$2T2$		$\frac{1}{6}$	$\frac{1}{3}$										
$2T3$		$\frac{1}{3}$		$\frac{1}{6}$									
$2T4$		$\frac{1}{2}$											
$2T5$		$\frac{1}{3}$				$\frac{1}{6}$							
$2T6$		$\frac{1}{6}$					$\frac{1}{3}$						
$2O1$	$\frac{1}{12}$	$\frac{5}{12}$											
$2O2$	$\frac{1}{6}$	$\frac{1}{3}$											
$2O3$	$\frac{1}{3}$	$\frac{1}{6}$											
$2O4$	$\frac{5}{12}$	$\frac{1}{12}$											
$2O5$	$\frac{1}{3}$	$\frac{1}{6}$											
$2O6$	$\frac{1}{6}$	$\frac{1}{3}$											
$2+$			$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$		$\frac{1}{36}$	$\frac{2}{36}$
$2-$	$\frac{1}{4}$								$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Figure 12. Transition matrix of the friend chain (first part).

	$2T2$	$2T3$	$2T4$	$2T5$	$2T6$	$2O1$	$2O2$	$2O3$	$2O4$	$2O5$	$2O6$	$2+$	$2-$
$1+$												$\frac{1}{4}$	$\frac{1}{4}$
$1-$						$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$		$\frac{1}{36}$	$\frac{2}{36}$		$\frac{1}{4}$
$1T2$	$\frac{1}{3}$						$\frac{1}{6}$						
$1T3$		$\frac{1}{6}$						$\frac{1}{3}$					
$1T4$									$\frac{1}{2}$				
$1T5$				$\frac{1}{6}$						$\frac{1}{3}$			
$1T6$					$\frac{1}{3}$						$\frac{1}{6}$		
$1O1$	$\frac{1}{24}$				$\frac{1}{24}$							$1\frac{0}{24}$	
$1O2$	$\frac{1}{12}$	$\frac{1}{12}$										$\frac{1}{3}$	
$1O3$	$\frac{1}{24}$	$\frac{1}{6}$	$\frac{1}{8}$									$\frac{1}{6}$	
$1O4$		$\frac{1}{12}$	$\frac{3}{12}$	$\frac{1}{12}$								$\frac{1}{12}$	
$1O5$			$\frac{3}{24}$	$\frac{1}{6}$	$\frac{1}{24}$							$\frac{1}{6}$	
$1O6$				$\frac{1}{12}$	$\frac{1}{12}$							$\frac{1}{3}$	
$2T2$	$\frac{1}{3}$											$\frac{1}{6}$	
$2T3$		$\frac{1}{6}$										$\frac{1}{3}$	
$2T4$												$\frac{1}{2}$	
$2T5$				$\frac{1}{6}$								$\frac{1}{3}$	
$2T6$					$\frac{1}{3}$							$\frac{1}{6}$	
$2O1$	$\frac{1}{24}$				$\frac{1}{24}$	$\frac{1}{4}$	$\frac{1}{12}$				$\frac{1}{12}$		
$2O2$	$\frac{1}{12}$	$\frac{1}{12}$				$\frac{3}{24}$	$\frac{1}{6}$	$\frac{1}{24}$					
$2O3$	$\frac{1}{24}$	$\frac{1}{6}$	$\frac{3}{24}$				$\frac{1}{12}$	$\frac{1}{12}$					
$2O4$		$\frac{1}{12}$	$\frac{3}{12}$	$\frac{1}{12}$				$\frac{1}{24}$		$\frac{1}{24}$			
$2O5$			$\frac{3}{24}$	$\frac{1}{6}$	$\frac{1}{24}$					$\frac{1}{12}$	$\frac{1}{12}$		
$2O6$				$\frac{1}{12}$	$\frac{1}{12}$	$\frac{3}{24}$				$\frac{1}{24}$	$\frac{1}{6}$		
$2+$												$\frac{1}{4}$	$\frac{1}{4}$
$2-$							$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$		$\frac{1}{4}$

Figure 13. Transition matrix of the friend chain (second part).

n	1	2	3	4	5	6	7	8	9
$d^*(n)$	0.957	0.638	0.375	0.263	0.180	0.122	0.083	0.058	0.040
$s^*(n)$	1	1	1	0.933	0.508	0.374	0.297	0.223	0.160
$L^2(n)$	4.690	2.254	1.544	1.05	0.71	0.492	0.339	0.233	0.159

Table 1. Measures of distance to stationarity for the friend chain, after n games.

start	$1+$	$1-$	$1T2$	$1T3$	$1T4$	$1T5$	$1T6$	$1O1$	$1O2$	$1O3$	$1O4$	$1O5$	$1O6$
OT	1.607	1.962	1.803	2.107	2.222	2.107	1.803	3.059	2.894	2.606	2.482	2.606	2.894
ST	1.093	1.515	3.773	2.725	2.314	2.725	3.773	1.421	1.499	1.550	1.523	1.550	1.499

Table 2. Mean number of games (out of eight) in which *friend* is on the opposite team OT (and the same team ST) as *ego*, who starts in the first quadrant.

start	$2T2$	$2T3$	$2T4$	$2T5$	$2T6$	$2O1$	$2O2$	$2O3$	$2O4$	$2O5$	$2O6$	$2+$	$2-$
OT	1.493	1.678	1.700	1.678	1.493	3.059	2.894	2.606	2.482	2.606	2.894	1.962	1.940
ST	3.778	2.698	2.297	2.698	3.778	1.421	1.499	1.550	1.523	1.550	1.499	1.515	1.103

Table 3. Mean number of games (out of eight) in which *friend* is on the opposite team OT (and the same team ST) as *ego*, who starts in the second quadrant.

But these are not observable quantities. More relevant to players is the mean number of games is which *friend* is on the same team, or on the opposing team, as *ego*. This is a simple calculation involving only matrix powers, and shown in Tables 2 and 3 are the mean number of games in which *friend* is on the opponent team and in which *friend* is on the same team.

Notice that there is a symmetry property visible in Tables 2 and 3, i.e., the values under $1T2$ to $1T6$, $1O2$ to $1O6$, $2T2$ to $2T6$, and $2O2$ to $2O6$ are exactly invariant under reversal. Moreover, the values under $1-$ and $2+$ are the same. This shows that if *friend* is playing with or against *ego*, the mean number of games in which *friend* is on the opposite team (or the same team) can be computed depending on the distance from *friend's* initial position to *ego* or *temporary ego's* initial position. In other words, we can reduce the number of states, and the states will depend on *friend's* shortest distance (counterclockwise or clockwise) from *ego*.

A related question is the chance that you *never* play as an opponent (or as teammate) to your friend. Shown in Tables 4 and 5 are the numerical values for “opponent” and “teammate”, respectively, omitting the cases where this is zero (initial opponent, or $1T4$).

start	$1+$	$1-$	$1T2$	$1T3$	$1T5$	$1T6$	
probability	0.098	0.057	0.141	0.026	0.026	0.141	
start	$2T2$	$2T3$	$2T4$	$2T5$	$2T6$	$2+$	$2-$
probability	0.168	0.081	0.082	0.081	0.168	0.057	0.057

Table 4. Probability that *friend* is never an opponent of *ego* over eight games.

start	$1+$	$1-$	$1O1$	$1O2$	$1O3$	$1O4$	$1O5$	$1O6$
probability	0.403	0.292	0.344	0.317	0.271	0.251	0.271	0.317
start	$2O1$	$2O2$	$2O3$	$2O4$	$2O5$	$2O6$	$2+$	$2-$
probability	0.344	0.317	0.271	0.251	0.271	0.317	0.292	0.393

Table 5. Probability that *friend* is never a teammate of *ego* over eight games.

Comparison with random teams. There are several ways one could compare the effect of the mixing scheme we study with the alternate scheme of randomly assigning players to team for every game. For instance, under random mixing, if *friend* starts somewhere which is not the opposite team as *ego*, then in each subsequent game the probability that *friend* is not on the opposite team as *ego* equals $\frac{17}{23}$. So by independence, the probability that *friend* will never be on the opposite team as *ego* over eight games equals $\left(\frac{17}{23}\right)^7 = 0.1205168\ldots \approx 0.121$. We see from Table 4 that this is less than the values under our scheme, if the starting position is only one position away from *ego* or the corresponding position of *ego* on the other half court. In the remaining cases, it is much greater.

Similarly, if *friend* starts somewhere which is not the same team as *ego*, then (under random mixing) in each subsequent game the probability that *friend* is not on the same team as *ego* equals $\frac{18}{23}$. So the probability that *friend* will never be on the same team as *ego* over eight games equals $\left(\frac{18}{23}\right)^7 = 0.1798095\ldots \approx 0.180$. From Table 5 this is always less than the values under our scheme.

6.2. ***Monte Carlo simulations.*** More complicated observables can most easily be addressed via Monte Carlo simulation of the process. For instance:

> What is the probability that *ego* will encounter (as either teammate or opponent) *all* of the other 23 players during an eight-game sequence?

By Monte Carlo, the probability ≈ 0.595 if *ego* starts at the first half court, or ≈ 0.675 if *ego* starts in the second half court. (Intuitively, these figures differ because in the former case one has more overlap between opponents in the first and second games.) Over ten games, these probabilities increase to 0.814 and 0.857. These simulations were done with one million trials, so we can take confidence intervals to be ± 0.001.

6.3. *Solving linear systems.* Aside from computing powers of the transition matrix or using Monte Carlo simulations to answer numerical questions, a textbook method for calculation is via solving linear systems of equations. We will illustrate by two examples which can be done by hand — more complicated examples could be done numerically.

Example. *For each of the various positions for players on the same team, what is the expected number of games until they are no longer on the same team?*

Recall the Figure 11 notation. By symmetry, we can take the first player to be *ego* in the first quadrant, and the other player to be one of $1T2$, $1T3$, $1T4$, or *ego* in the second quadrant, and the other player to be one of $2T2$, $2T3$, $2T4$. For one of those initial other player positions i, let t_i be the expected number of games it takes until the two players are on different teams. Consider $i = 1T2$. There are three equally likely outcomes of the first match. Either neither of those two players migrates, in which case the configuration remains $1T2$. Or both players migrate, in which case the configuration becomes $2T2$. Or exactly one player migrates, in which case they are now on different teams. Repeating this analysis for each i leads to the following linear system of equations:

$$\begin{aligned} t_{1T2} &= 1 + \tfrac{1}{3}t_{1T2} + \tfrac{1}{3}t_{2T2}, & t_{2T2} &= 1 + \tfrac{1}{3}t_{1T2} + \tfrac{1}{3}t_{2T2}, \\ t_{1T3} &= 1 + \tfrac{1}{6}t_{1T3} + \tfrac{1}{6}t_{2T3}, & t_{2T3} &= 1 + \tfrac{1}{6}t_{1T3} + \tfrac{1}{6}t_{2T3}, \\ t_{1T4} &= 1, & t_{2T4} &= 1. \end{aligned}$$

By symmetry we have $t_{1T2} = t_{2T2},\ t_{1T3} = t_{2T3} = t_{1T5} = t_{2T5},\ t_{1T4} = t_{2T4}$. Solving theses equations leads to

$$t_{1T2} = t_{2T2} = 3, \quad t_{1T3} = t_{2T3} = t_{1T5} = t_{2T5} = \tfrac{3}{2}, \quad t_{1T4} = t_{2T4} = 1.$$

In fact this example could be solved without writing down equations, by first observing that in each case the number of games required has a Geometric(p) distribution, with $p = \frac{1}{3}, \frac{2}{3}, 1$ according as the initial distance between players being 1, 2, 3.

Example. *Starting the friend Markov chain from one of the states* $1\pm$ *or* $2\pm$, *what is the probability that* ***ego*** *and* ***friend*** *will play on the same team before they play on opposing teams?*

Here we do need to write down the equations. Let p_i be the probability that *ego* and *friend* will play on the same team before they play on opposing teams if *friend* starts on state i, where i is one of $1\pm$ and $2\pm$. While they are on opposite courts, there are 4 equally likely possibilities for the players to migrate or not. From the transition matrix of the friend chain, we obtain the linear system

$$\begin{aligned} p_{1+} &= \tfrac{1}{4}p_{1+} + \tfrac{1}{4}p_{1-} + \tfrac{1}{4}p_{2+} + \tfrac{1}{4}p_{2-}, \\ p_{1-} &= \tfrac{1}{4} + \tfrac{1}{4}p_{1-} + \tfrac{1}{4}p_{2-}, \\ p_{2+} &= \tfrac{1}{4} + \tfrac{1}{4}p_{2+} + \tfrac{1}{4}p_{2-}, \\ p_{2-} &= \tfrac{1}{4}p_{1+} + \tfrac{1}{4}p_{2-}. \end{aligned}$$

The solution is

$$p_{1+} = \tfrac{3}{11}, \quad p_{1-} = p_{2+} = \tfrac{4}{11}, \quad p_{2-} = \tfrac{1}{11}.$$

Hence, if *friend* starts on one of the states in $1+$, $1-$, $2+$, and $2-$, the probabilities that *ego* and *friend* will play on the same team before they play on opposing teams are $\frac{3}{11}$, $\frac{4}{11}$, $\frac{4}{11}$, and $\frac{1}{11}$, respectively.

6.4. ***Suggestions for course projects.*** As mentioned in the introduction, one could use this topic as a source of projects to accompany a course in Markov chains with some emphasis on computation. Here are some suggestions.

(1) Repeat the analysis for the simpler model is which there are only 12 players and one court. At the end of each match, the players in one front row swap positions with the players in the other front row.

(2) Similar to the two examples in Section 6.3 above, calculate the expected amount of time until *ego* and *friend* are on the same team (or opposite teams) for each possible initial configuration.

(3) (suggested by a referee) Calculate the *fundamental matrix* of the friend chain. The fundamental matrix determines the mean time to go from one given initial state to another given target state. Moreover, in the context of comparison with random teams (the number of games you play with or against your friend in the Markov scheme, versus the number in the purely random model), the fundamental matrix determines the $n \to \infty$ limit of the expected difference between the two schemes.

(4) A more challenging theory project is to improve our bounds on the mixing time. The construction in Section 4 implicitly gives a (very large) upper bound, and the next section gives lower bounds, but these are surely far from optimal.

7. Mixing time for the big chain

At a research level there has been extensive study of mixing times for many different card-shuffling models, usually in the asymptotic setting (as the size n of card deck goes to ∞). Our model is rather specific to the $n = 24$ case, so we have not sought to embed it into some family allowing large n.

n	1	2	3	4	5	6	7	8	9
$d(n)$	0.5	0.25	0.25	0.125	0.125	0.0625	0.0625	0.0313	0.0313

Table 6. Variation distance for the lazy cyclic walk.

One can get lower bounds on mixing time by considering specific functions of the chain, and the variation distances in Table 1 for the friend chain are a lower bound for the distances in the big chain. A first step [Aldous 1983] in studying some "local moves" shuffles such as random adjacent transpositions was to obtain lower bounds by studying motion of initially adjacent cards. In our model, the "friends" maximal-start variation distance in Table 1 is indeed[9] from the case of initial adjacent players.

Recall that the progress of ego around the four quadrants is just the lazy cyclic walk, for which variation distance to stationarity in recorded in Table 6.

These values are slightly less than the values in Table 1 for the friend chain — both are priori lower bounds for the variation distance for the big chain.

To find a better lower bound for the mixing time for the big chain, we can combine the two aforementioned ideas and form a 52-state (big friend) chain, where *ego* can be in any of the four quadrants. In this new chain, we label the positions with respect to *ego* similar to the friend chain (see Figures 11 and 14). We first record which quadrant *ego* is in (denoted as 1, 2, 3 or 4) and then whether friend is in the same team (denoted by T) or the current opposing team (denoted by O), or on the other court. If on the other court, we only note which quadrant, so denote by one of $+$ or $-$. Finally, writing temporarily ego* for the opponent in the same position as ego, we indicate a friend's position as 1, 2, 3, 4, 5, or 6, counterclockwise from ego or ego*. Its stationary distribution π is induced from the uniform stationary distribution of the big friend chain: probability $\frac{6}{92}$ for each of the states $1+$, $1-$, $2+$, $2-$, $3+$, $3-$, $4+$, and $4-$, and probability $\frac{1}{92}$ for each of the 44 remaining states.

The transition matrix for this big friend chain is easy to obtain from the transition matrix of the friend chain. Consider the transition matrix of the friend chain in Figures 12 and 13. We can view it as a matrix composed of smaller matrices as in Table 7, where T_{11}, T_{12}, T_{21}, and T_{22} are 13×13 matrices. The states used in the transition matrix for the big friend chain are arranged as follows: $1+$, $1-$, $1T2$ to $1T6$, $1O1$ to $1O6$, $2T2$ to $2T6$, $2O1$ to $2O6$, $2+$, $2-$, $3+$, $3-$, $3T2$ to $3T6$, $3O1$ to $3O6$, $4T2$ to $4T6$, $4O1$ to $4O6$, $4+$, and $4-$, and transition matrix for the big friend chain is shown in Table 8.

We can now investigate the mixing properties of the big friend chain. Shown in Table 9 are the distances from stationarity for the big chain.

[9]Except for opening games.

all 4−	$4O5$	$4O4$	$4O3$	
	$4O6$	$4O1$	$4O2$	*ego* • in fourth quadrant
all 4+	$4T2$	•	$4T6$	
	$4T3$	$4T4$	$4T5$	

all 3−	•	$3T6$	$3T5$	
	$3T2$	$3T3$	$3T4$	*ego* • in third quadrant
all 3+	$3O4$	$3O3$	$3O2$	
	$3O5$	$3O6$	$3O1$	

Figure 14. Positions of *friend* relative to *ego* •.

	1	2
1	T_{11}	T_{12}
2	T_{21}	T_{22}

Table 7. Transition matrix for the friend chain.

	1	2	3	4
1	T_{11}	T_{12}	0	0
2	0	T_{22}	T_{21}	0
3	0	0	T_{11}	T_{12}
4	T_{21}	0	0	T_{22}

Table 8. Transition matrix for the big friend chain.

n	1	2	3	4	5	6	7	8	9
$d^*(n)$	0.978	0.713	0.520	0.340	0.242	0.168	0.125	0.085	0.058
$s^*(n)$	1	1	1	1	0.827	0.681	0.461	0.391	0.272
$L^2(n)$	6.708	2.977	1.868	1.228	0.827	0.563	0.387	0.266	0.183

Table 9. Measures of distance to stationarity for the big friend chain, after n games.

Notice that the values we got for $d^*(n)$ in the big friend chain are larger than that of the friend chain, improving the lower bound of the big friend chain.

8. Final remarks

We have mentioned analogies with card shuffling several times, because rotations of team players correspond to a cut-shuffle of a six-card deck. Our model is equivalent to a certain (not very easily implemented physically) random shuffle of a 24-card deck via first breaking into four subdecks. Persi Diaconis (personal communication) remarks that casinos and some fantasy games involve shuffling decks much larger than the usual 52-card deck, and this is often done via some scheme involving breaking into subdecks, shuffling each in some way, and recombining in some way. Such schemes (thereby loosely analogous to our model) have generally not been studied in mathematical probability, an exception being the casino shelf shuffling machines studied by Diaconis, Fulman and Holmes [Diaconis et al. 2013].

We have interpreted the underlying question

how effective is this scheme at mixing up the teams?

in terms of mixing times, that is, implicitly by comparison with the alternative of randomly assigning players to teams for each game. An opposite alternative would be some analog of "design of statistical experiments" schemes, deterministically assigning players to teams in each round in such a way that relative positions of two players were as uniformly spread as possible. At a practical level, our scheme is much easier and faster to implement than either alternative. Moreover its implementation is *robust* to small variation in number of players, which is common in informal settings: an extra player will rotate off court, or a five-player team always has three players deemed front row.

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