



To stay discovered: On tournament mean score sequences and the Bradley–Terry model

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In memory of Larry Shepp, 1936 – 2013

Abstract

On being told that a piece of work he thought was his discovery had duplicated an earlier mathematician's work, Larry Shepp once replied "Yes, but when *I* discovered it, it *stayed* discovered." In this spirit we give discussion and probabilistic proofs of two related known results (Moon 1963, Joe 1988) on random tournaments which seem surprisingly unknown to modern probabilists. In particular, our proof of Moon's theorem on mean score sequences seems more constructive than previous proofs. This provides a comparatively concrete introduction to a longstanding mystery, the lack of a canonical construction for a joint distribution in the representation theorem for convex order.

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1. Introduction

1.1. An analogy

As an analogy for the two theorems to be discussed, we recall some very well-known facts.
First note:

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(a) A probability measure $\mu = \text{dist}(X_1, \dots, X_n)$ on \mathbb{R}^n with each $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 < \infty$ has a covariance matrix $\Gamma_{ij} = \mathbb{E}X_i X_j < \infty$.

(b) If Γ is a symmetric $n \times n$ matrix such that $f(\mathbf{x}) = f(x_1, \dots, x_n) \propto \exp(-\mathbf{x}^T \Gamma^{-1} \mathbf{x}/2)$ is a well-defined probability distribution, then its covariance matrix is Γ .

What is true, but not obvious *a priori*, is that essentially³ every covariance matrix (as defined in (a)) can be used to define the Gaussian distribution in (b). This is the Fact 1 for our analogy. The reader will likely perceive this fact as a consequence of the multivariate CLT, although there is an arguably conceptually simpler proof (see Section 3.3). Fact 2 is the familiar characterization of covariance matrices as symmetric positive semi-definite matrices.

Our Theorems 1 and 2 may be regarded as analogs of Facts 1 and 2 in a different context.

1.2. Statements of the two theorems

In everyday language *tournament* usually means a single-elimination tournament. In graph theory an n -team *tournament* means the set of win–lose results of league play in which each pair of teams plays once. So there are exactly $2^{\binom{n}{2}}$ tournaments. Now consider a completely arbitrary probability distribution μ on the set of $2^{\binom{n}{2}}$ tournaments. For $1 \leq i \leq n$ write x_i for the expectation of the number of wins by team i . Call such a sequence $\mathbf{x} = (x_i, 1 \leq i \leq n)$ a *mean score sequence*. In this context, an analog (as mathematically tractable, for instance) of the Gaussian distribution is the following *Bradley–Terry* model. Take real parameters $(\lambda_i, 1 \leq i \leq n)$ and let match results be independent with

$$p_{ij} := \mathbb{P}(i \text{ beats } j) = L(\lambda_i - \lambda_j) \tag{1}$$

for the logistic function

$$L(u) := \frac{e^u}{1 + e^u}, \quad -\infty < u < \infty.$$

The set \mathcal{X}_n^{B-T} of mean score sequences arising from the Bradley–Terry model must be a subset of the set \mathcal{X}_n of all mean score sequences. But it turns out that, as suggested by the Gaussian analogy, we get essentially *all* mean score sequences from a Bradley–Terry model.

Theorem 1 (Joe [8]). *The closure in \mathbb{R}^n of the set \mathcal{X}_n^{B-T} is \mathcal{X}_n .*

We need *closure* to get the extreme cases of the mean score sequence where for some k , $x_i = i - 1$ for all $i \geq k$.

In fact Theorem 1 is not explicitly stated and proved in [8], but it is a straightforward consequence of results there. We give both a simple heuristic argument and a careful proof in Section 3.

As background to the second theorem, recall that the following definition and representation (see, e.g., [15], section 2.A) of *convex order* \leq are useful in several areas of probability.

Definition. For finite mean probability measures on \mathbb{R} , $\mu \leq \nu$ means

$$\int \phi d\mu \leq \int \phi d\nu \quad \text{for all convex } \phi \text{ such that the integrals exist.} \tag{2}$$

³ We need *essentially* to handle the degenerate case: precisely, the set of matrices defined by (a) is the closure of the set defined by (b).

Representation (Strassen [16]). $\mu \preceq \nu$ if and only if there exists a joint distribution (X, Y) with marginal distributions μ and ν such that

$$X = \mathbb{E}(Y|X). \tag{3}$$

In the special case where μ and ν are uniform on n -element multisets \mathbf{x} and \mathbf{y} , written by convention in increasing⁴ order as $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$, the convex order $\mu \preceq \nu$ is equivalent to the notion that \mathbf{x} is majorized by \mathbf{y} , also written as $\mathbf{x} \preceq \mathbf{y}$, and usually defined by

$$\sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i, \quad 1 \leq k \leq n, \quad \text{with equality for } k = n. \tag{4}$$

The breadth of utility of this notion of majorization, mostly outside probability, is demonstrated in the book [11]. The relation $\mathbf{x} \preceq \mathbf{y}$ can be thought of intuitively as “ \mathbf{x} is less spread out than \mathbf{y} ,” as discussed in [11].

The second theorem is our analog of the characterization of covariance matrices as symmetric positive semi-definite matrices. This connection is discussed further after (8) below.

Theorem 2 (Moon [12]). *An increasing sequence of real numbers $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is a mean score sequence of some random tournament if and only if $\mathbf{x} \preceq (0, 1, \dots, n - 1)$.*

We give proofs of these theorems in Sections 2 and 3, and then discuss previous proofs and the broader context in Section 4. From that broader context, we feel that the most interesting part of these results is how should one prove the “if” part of Theorem 2, so we start by giving two proofs of that.

2. Proofs of Theorem 2

2.1. First proof of Theorem 2 (“if” part)

Given $\mathbf{x} \preceq (0, 1, \dots, n - 1)$, the representation (3) holds with X uniform on $\{x_1, \dots, x_n\}$ and Y uniform on $(0, 1, \dots, n - 1)$, and then $\mu_i(\cdot) := \mathbb{P}(\cdot|X = x_i)$ define probability distributions $\mu_i(\cdot)$, $1 \leq i \leq n$, on integers $\{0, 1, \dots, n - 1\}$ such that

- (i) $\mu_i(\cdot)$ has mean x_i ,
- (ii) $\sum_i \mu_i(j) = 1$, all $0 \leq j \leq n - 1$.

The proof is easily understood in terms of a model for football⁵ – note this is separate from the initial “tournament” model.

When teams i and j play, their goal scores are independent with distributions μ_i and μ_j . Define p_{ij} as the mean number of points earned by i against j , when a win earns 1 point and a tie earns 1/2 point.

To complete the proof we need only check

- (iii) $\sum_{j \neq i} p_{ij} = x_i$, $1 \leq i \leq n$;

in other words, check

$$\sum_{j \neq i} \chi(\mu_i, \mu_j) = x_i, \quad 1 \leq i \leq n, \tag{5}$$

⁴ Increasing means non-decreasing.

⁵ Soccer (U.S.).

where for independent random variables X and \hat{X} with distributions ν and $\hat{\nu}$ we define

$$\chi(\nu, \hat{\nu}) = \mathbb{P}(X > \hat{X}) + \frac{1}{2}\mathbb{P}(X = \hat{X}).$$

Note that χ is linear in each argument and that, for the uniform distribution λ on $\{0, 1, \dots, n - 1\}$ we have $\chi(\delta_k, \lambda) = (k + \frac{1}{2})/n$. So by linearity and (i), $\chi(\mu_i, \lambda) = (x_i + \frac{1}{2})/n$. By (ii) we may write $\lambda = \frac{1}{n}(\mu_i + \sum_{j \neq i} \mu_j)$ and then by linearity

$$x_i + \frac{1}{2} = \chi(\mu_i, \mu_i) + \sum_{j \neq i} \chi(\mu_i, \mu_j).$$

But $\chi(\mu_i, \mu_i) = \frac{1}{2}$ by symmetry, verifying (5).

2.2. Second proof of Theorem 2 (“if” part)

A permutation $\pi : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ defines a special kind of “season results,” in which team i loses to team j if and only if $\pi(i) < \pi(j)$, and therefore team i wins exactly $\pi(i) - 1$ games. By considering this special type of “totally ordered” season result, the converse will follow from the lemma below, because the mean score sequence for the corresponding “random total order” is $(q(i) - 1, 1 \leq i \leq n)$.

Lemma 3. *Let $q : \{1, 2, \dots, n\} \rightarrow [1, n]$ be a function such that $q(1 + U_n) \leq 1 + U_n$, where U_n has the uniform distribution on $\{0, 1, \dots, n - 1\}$. Then there exists a probability distribution over permutations π such that*

$$\mathbb{E}\pi(i) = q(i), 1 \leq i \leq n.$$

Proof. Write I_n and J_n for random variables with the uniform distribution on $\{1, 2, \dots, n\}$. The relation $q(I_n) \leq J_n$ is equivalent, using the representation (3), to saying that we can construct a joint distribution for (I_n, J_n) such that

$$\mathbb{E}(J_n | I_n = i) = q(i), 1 \leq i \leq n.$$

Now the matrix with entries

$$p_{ij} := n\mathbb{P}(I_n = i, J_n = j)$$

is doubly stochastic, so by Birkhoff’s theorem it is a mixture of permutation matrices. In other words, there is a probability distribution over permutations π such that $p_{ij} = \mathbb{P}(\pi(i) = j)$. But this just says

$$\mathbb{P}(\pi(i) = j) = \mathbb{P}(J_n = j | I_n = i),$$

and so

$$\mathbb{E}\pi(i) = \mathbb{E}(J_n | I_n = i) = q(i). \quad \square$$

2.3. Proof of Theorem 2 (“only if” part)

First consider the deterministic case, so there is an integer-valued sequence $0 \leq x_1^* \leq x_2^* \leq \dots \leq x_n^*$ of wins. Fix a convex function ϕ .

Suppose $x_1^* \geq 1$. Change the results of the games won by team 1, one game at a time, to make them a loss for team 1. At each such step x_1^* decreases by 1 and some x_i^* increases

by 1, and by convexity the value of $\sum_i \phi(x_i^*)$ can only increase. Continue until reaching a configuration with $x_1^* = 0$. Now suppose $x_2^* \geq 2$. Again change the results of the games won by team 2 against teams $i > 2$, one game at a time, to make them a loss for team 2. Again this can only increase $\sum_i \phi(x_i^*)$. Continue until reaching a configuration with $x_1^* = 0$ and $x_2^* = 1$. Eventually we reach the configuration with $x_i^* = i - 1, 1 \leq i \leq n$. So the original configuration satisfies $\sum_i \phi(x_i^*) \leq \sum_i \phi(i - 1)$, establishing (2) in the deterministic case.

In the random case write X_i for the random number of wins by team i . So $x_i = \mathbb{E}X_i$, and by Jensen’s inequality $\phi(x_i) \leq \mathbb{E}\phi(X_i)$. So

$$\sum_i \phi(x_i) \leq \sum_i \mathbb{E}\phi(X_i) = \mathbb{E} \left[\sum_i \phi(X_i) \right]$$

and the quantity in brackets is bounded by $\sum_i \phi(i - 1)$, by the deterministic case. This establishes (2) in general.

3. The Bradley–Terry model

3.1. The max-entropy heuristic

There is a one sentence explanation of [Theorem 1](#):

If \mathbf{x} is a mean score sequence, then by definition there exists a matrix with non-negative off-diagonal entries $p_{ij} = 1 - p_{ji}$ satisfying the constraints $\sum_{j \neq i} p_{ij} = x_i, 1 \leq i \leq n$; and the max-entropy such matrix is of the Bradley–Terry form (1).

To elaborate, the problem

$$\text{maximize } - \sum_{i \neq j} p_{ij} \log p_{ij} \text{ subject to } \sum_{j \neq i} p_{ij} = x_i, 1 \leq i \leq n,$$

is solved, in classical applied mathematics, by introducing Lagrange multipliers $\lambda_i, 1 \leq i \leq n$, and solving

$$\text{maximize } - \sum_{i \neq j} p_{ij} \log p_{ij} + \sum_i \lambda_i (\sum_{j \neq i} p_{ij} - x_i) \text{ over } \lambda_i, 1 \leq i \leq n.$$

By setting $\frac{d}{dp_{ij}}(\cdot) = 0$, the solution satisfies

$$-\log p_{ij} + \log(1 - p_{ij}) + \lambda_i - \lambda_j = 0,$$

implying that the matrix (p_{ij}) is indeed of the Bradley–Terry form (1).

We learned this max-entropy argument from Joe [8]. How much detail needs to be added to make a completely rigorous proof is a matter of taste; we give a rather fussy argument next.

3.2. Strong stochastic transitivity

In statistical modeling contexts such as [14] one says that p_{ij} on tournaments of size n has the *strong stochastic transitivity* (SST) property if (after relabeling $[n]$ if necessary)

$$p_{ij} \text{ is increasing in } i, \text{ for any fixed } j. \tag{6}$$

Note that this condition implies that the matrix P with entries p_{ij} is increasing down columns and decreasing along rows.

Lemma 4 (Joe [8], Theorems 2.3 and 2.7). Let $\mathbf{x} \preceq (0, 1, \dots, n - 1)$. Suppose that ψ is strictly convex. If p_{ij}^* minimizes $\sum_{i \neq j} \psi(p_{ij})$ over p_{ij} on tournaments of size n with mean score sequences \mathbf{x} , then p_{ij}^* has SST.

We remark that in fact the above follows by a special case of Theorem 2.7 in [8]. Here we need only the case of functions $\psi(P) = \sum_{i \neq j} \psi(p_{ij})$ for strictly convex ψ , whereas in [8] this result is shown to hold for strictly Schur convex functions ψ .

In [8] and earlier combinatorial literature the SST property is instead defined to mean

$$\min\{p_{ij}, p_{jk}\} \geq 1/2 \Rightarrow p_{ik} \geq \max\{p_{ij}, p_{jk}\}. \tag{7}$$

Unfortunately, despite this same terminology in use in the literature, the notions (6) and (7) are not equivalent. For example, consider $n = 3$ with all $p_{ij} = 1/2$ except $p_{13} = 1 - p_{31} < 1/2$. Such a distribution has (6) but not (7). It is however true that (7) implies (6). This is likely well-known, but we cannot find a published proof (it does not appear in the old survey [6]) so we have included a proof in the Appendix.

Proof of Theorem 1. We show that if $\mathbf{x} \prec (0, 1, \dots, n - 1)$, in the sense that $\sum_{i=1}^k x_i > \binom{k}{2}$ for $1 \leq k < n$ and $\sum_{i=1}^n x_i = \binom{n}{2}$, then $p_{ij} = L(\lambda_i - \lambda_j)$ for some $(\lambda_i, 1 \leq i \leq n)$. The theorem follows.

To this end, consider minimizing $\sum_{i \neq j} p_{ij} \log p_{ij}$, subject to all $p_{ij} \leq 1$, $p_{ij} + p_{ji} = 1$ and $x_i = \sum_{j \neq i} p_{ij}$. Since $\mathbf{x} \preceq (0, 1, \dots, n - 1)$, a solution p_{ij}^* exists. We claim that all $p_{ij}^* \in (0, 1)$. This concludes the proof, noting that the stationary point of

$$\sum_{i \neq j} p_{ij} \log p_{ij} + \sum_{i=1}^{n-1} \lambda_i (x_i - \sum_{j \neq i} p_{ij})$$

corresponds to some $p_{ij}^* = L(\lambda_i^* - \lambda_j^*)$ with all $x_i = \sum_{j \neq i} L(\lambda_i^* - \lambda_j^*)$.

To establish the claim, suppose towards a contradiction that some $p_{ij}^* = 1$. Since $\psi(x) = x \log x$ is strictly convex, p_{ij}^* has SST in the sense of (6) by Lemma 4. Let u be the minimal index such that $p_{uj}^* = 1$ for some j . Let v be the maximal index such that $p_{uv}^* = 1$. Note that since $\mathbf{x} \prec (0, 1, \dots, n - 1)$ we have $v < u - 1$, as else, since p_{ij}^* has SST, we would find that $p_{ij}^* = 0$ for all $i \leq u - 1$ and $j \geq u$, and so $\sum_{i=1}^{u-1} x_i = \binom{u-1}{2}$.

Next, consider q_{ij} obtained from p_{ij}^* by decreasing $p_{uv}^* = 1$ by ε and increasing $p_{u,v+1}^*$ and $p_{v+1,v}^*$ by ε . That is, let

- (i) $q_{vu} = 1 - q_{uv} = \varepsilon$,
- (ii) $q_{u,v+1} = 1 - q_{v+1,u} = p_{u,v+1}^* + \varepsilon$,
- (iii) $q_{v+1,v} = 1 - q_{v,v+1} = p_{v+1,v}^* + \varepsilon$,

and $q_{ij} = p_{ij}^*$ for all other i, j . Since $v < u - 1$, and so $v + 1 \neq u$, all

$$\sum_{j \neq i} q_{ij} = \sum_{j \neq i} p_{ij}^* = x_i.$$

Moreover, by the choice of u and v , and since $v + 1 < u$, note that $p_{u,v+1}^* < 1$ and $p_{v+1,v}^* < 1$. Therefore all $q_{ij} \leq 1$, for all small $\varepsilon > 0$. Hence q_{ij} is a probability distribution with mean score sequence \mathbf{x} .

Finally, observe that, for $\alpha, \beta \in (0, 1)$, differentiating

$$\ell(\varepsilon) + \ell(\alpha + \varepsilon) + \ell(\beta + \varepsilon)$$

with respect to ε , where $\ell(x) = x \log x + (1 - x) \log(1 - x)$, we obtain

$$\log \left(\frac{\varepsilon}{1 - \varepsilon} \frac{\alpha + \varepsilon}{1 - \alpha - \varepsilon} \frac{\beta + \varepsilon}{1 - \beta - \varepsilon} \right) < 0$$

for all small $\varepsilon > 0$. It follows that

$$\sum_{i \neq j} q_{ij} \log q_{ij} < \sum_{i \neq j} p_{ij}^* \log p_{ij}^*$$

for all small $\varepsilon > 0$, contradicting the minimality of p_{ij}^* . \square

3.3. Comments on max-entropy

The Section 1.1 analogy, that (a) is the closure of the set defined by (b), can also be proved by max-entropy instead of the multivariate CLT. And in the context of this paper, we could call p_{ij} Cauchy if $p_{ij} = C(\alpha_i - \alpha_j)$, where $C(x) = 2^{-1} + \pi^{-1} \arctan x$ is the CDF of a standard Cauchy. The same argument above works, replacing the entropy function $u \log u$ by the function $-(2\pi)^{-1} \log \sin(u\pi)$, to show that the set of mean score sequences from Cauchy matrices is dense in the set of all mean score sequences. In fact, this is only one example of a more general principle; see the discussion on p. 920 in Joe [8] following Theorem 2.7.

4. Discussion

One purpose of this paper is simply to juxtapose Theorems 1 and 2, which come from rather different research communities. As noted below, Moon’s theorem has many proofs using different textbook theorems — and indeed could be used as a running example in a first undergraduate course in discrete mathematics. As probabilists we wanted to find “probabilistic” arguments, and our two “if” proofs in Section 2 are (we believe) new and probabilistic. In particular the “football story” in Section 2.1 seems memorable. The relevant discrete mathematics literature starts with Landau’s theorem [10], the deterministic analog of Moon’s theorem characterizing score sequences of non-random tournament outcomes. Moon’s original proof [12], and a more general version (their Theorem 3.9) in Moon and Pullman [13], used network flow feasibility properties, and later Bang and Sharp [2] used Hall’s theorem on systems of distinct representatives, Cruse [5] used linear programming methods, and Thornblad [17] derived it from the deterministic case [10] via a rather lengthy argument. To us it is more natural to explicitly exploit the Strassen representation, as our two “if” proofs do in different ways. The “only if” part could alternatively be proved as an easy consequence of Landau’s theorem and [11] Proposition 12.D.1. Our proof (by “Robin Hood moves” [11]) avoids Landau’s theorem.

As mentioned before, convex order and the general Strassen representation are useful tools in several areas of probability theory. Because an arbitrary distribution on \mathbb{R} can be approximated by uniform distributions on n -element sets, the general case is conceptually very similar to the majorization case $\mathbf{x} \preceq \mathbf{y}$ defined at (4). This setting is treated at great length in the book [11]. Instead of the “coupling of random variables” picture natural to modern probabilists, the representation is stated there in the equivalent form

$$\text{if } \mathbf{x} \preceq \mathbf{y} \text{ then } \mathbf{x} = \mathbf{A}\mathbf{y} \text{ for some doubly stochastic matrix } \mathbf{A}. \tag{8}$$

There are constructive proofs of this (e.g., [11], Theorem 2.B.2). Combined with our “football” proof we obtain a completely constructive proof of Moon’s theorem. To our knowledge, previous proofs cannot be made constructive so easily.

Further to the Section 1.1 analogy between Theorem 2 and the characterization of covariance matrices as symmetric positive semi-definite matrices (with those associated with Gaussian densities being positive definite), we note that by Chao and Wong [4] there is in fact a matrix A as in (8) that is positive semi-definite (and they give an explicit formula). Moreover, by Brualdi, Hwang and Pyo [3], for all \mathbf{x} arising from a Bradley–Terry model (i.e., by the proof of Theorem 1, all “non-degenerate” \mathbf{x} such that $\sum_{i=1}^k x_i > \binom{k}{2}$ for $k < n$) there is a positive definite A .

To us, the most interesting part of the bigger picture surrounding convex order is that there is apparently no “canonical” choice of joint distribution in (3), (8): proofs may be constructive but they involve rather arbitrary choices and the resulting joint distributions are not easily described. Recent literature on *peacocks* [7] studies continuous-parameter processes increasing in convex order, via many different constructions, and ideas from that literature might be relevant in our context.

We encountered this field while exploring the Bradley–Terry model as a basic mathematical toy model for sports results — see [1], [9] for references to the extensive literature in that field. Abstractly there is a map G from the set of $-\infty < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n < \infty$, centered by requiring $\sum_i \lambda_i = 0$, to $\mathbf{x} \in \mathbb{R}^n$ defined by

$$x_i = \sum_{j \neq i} L(\lambda_i - \lambda_j).$$

This map has some range \mathcal{X}_n^{B-T} , but it is hard to see directly from that definition what is the range of G . Theorem 1 answers that question. However the inverse function G^{-1} giving $(\lambda_1, \dots, \lambda_n)$ in terms of $\mathbf{x} \in \mathcal{X}_n$ remains obscure; we do not have an explicit formula. So we cannot say anything about how the win-probabilities $p_{ij} = L(\lambda_i - \lambda_j)$ depend on \mathbf{x} . In particular, we do not know whether these win-probabilities p_{ij} can always be obtained within the “football model” by some choice of distributions μ_i (equivalently of matrix A at (8)).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Proof that (7) implies (6)

Suppose we have (7). Consider the graph G with directed edges $j \rightarrow i$ for $p_{ij} \geq 1/2$. By induction, all induced $H \subset G$ have a vertex i_* such that $j \rightarrow i_*$ for all $j \neq i_* \in H$. Indeed, suppose this holds for all such subgraphs of size $m < n$. Let H be an induced subgraph of size $m + 1$, and consider $H' = H - v$ for some $v \in H$. Let i'_* be such that $j \rightarrow i'_*$ for all $j \neq i'_* \in H'$. If $v \rightarrow i'_*$ in H , then let $i_* = i'_*$. Otherwise, if $v \not\rightarrow i'_*$ (and so $i'_* \rightarrow v$) in H , then (7) implies that all other $j \rightarrow v$ are in H (as else, since $j \rightarrow i'_*$, the presence of some $v \rightarrow j$ in H and (7) would imply that $v \rightarrow i'_*$ in H , contrary to our assumption), and so let $i_* = v$. This completes the induction. The case $H = G$ gives i_* such that $p_{i_*,j} \geq 1/2$ for all j . Therefore, we can recursively relabel $[n]$ so that $p_{ij} \geq 1/2$ for all pairs $i > j$. Then assumption (7) directly implies the SST property in the sense of (6).

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