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The compulsive gambler process

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Abstract

In the *compulsive gambler* process there is a finite set of agents who meet pairwise at random times (i and j meet at times of a rate-ν_{ij} Poisson process) and, upon meeting, play an instantaneous fair game in which one wins the other's money. We introduce this process and describe some of its basic properties. Some properties are rather obvious (martingale structure; comparison with Kingman coalescent) while others are more subtle (an "exchangeable over the money elements" property, and a construction reminiscent of the Donnelly-Kurtz look-down construction). Several directions for possible future research are described. One – where agents meet neighbors in a sparse graph – is studied here, and another – a continuous-space extension called the *metric coalescent* – is studied in Lanoue (2014).

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1 Introduction

The style of models known to probabilists as Interacting Particle Systems (IPS) [13] have found use in many fields across the mathematical and social sciences. Often the underlying conceptual picture is of a social network, where individual "agents" meet pairwise and update their "state" (opinion, activity etc) in a way depending on their previous states. This picture motivates a precise general setup we call *Finite Markov Information Exchange (FMIE) processes* [1]. Consider a set $\mathbf{Agents}$ of n agents and a nonnegative array (ν_{ij}), indexed by unordered pairs $\{i, j\}$, which is irreducible (i.e. the graph of edges corresponding to strictly positive entries is connected). Assume

- Each unordered pair i, j of agents with $\nu_{ij} > 0$ meets at the times of a rate-ν_{ij} Poisson process, independent for different pairs.

Call this collection of Poisson processes the *meeting process*; the array (ν_{ij}) specifies the *meeting model*. A specific FMIE is a specific rule (deterministic or random) for updating states, so this encompasses most of the familiar IPS models such as the voter model and

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contact process. But our emphasis differs from the classical emphasis of IPS in several ways; the states are typically numerical rather than categorical, the number n of agents is finite (though we consider $n \to \infty$ asymptotics) and we focus on obtaining rough results for general meeting rates rather than sharp results for very specific meeting rates.

One specific FMIE model is the *averaging process* [2] in which agents initially have different amounts of money; whenever two agents meet, they share their combined money equally. In this paper we introduce and study a conceptually opposite model, the *compulsive gambler process*. In the “standard" form of the model, agents each start with one unit money. When two agents meet, if they each have non-zero money (say amounts a and b) then they instantly play a fair game in which one agent acquires the combined amount $a + b$ (so with probabilities $a/(a + b)$ and $b/(a + b)$) respectively). In the “normalized" form of the model the initial fortunes are non-negative real numbers, and we scale so that the total money equals 1.

This is an invented model for which we do not claim realism[1], but we do claim some mathematical interest as an intermediary between IPS theory and coalescent theory.

1.1 Elementary observations

First consider a fixed meeting model on n agents. Write $\mathbf{Z}(t) = (Z_i(t), i \in \mathbf{Agents})$ for the time-t configuration of the standard compulsive gambler process; agent i has $Z_i(t)$ units of money. The following assertions are true, and mostly obvious. We will give proofs, where necessary, and some crude quantifications in section 2 (Lemmas 2.2 and 2.8).

(i) $\mathbf{Z}(t)$ is a finite-state continuous-time Markov chain which, at some a.s. finite random time T, reaches some absorbing configuration $\mathbf{Z}^*$ in which there is some random non-empty set $\mathcal{T}$ of agents who are *solvent*, i.e. have non-zero money.

(ii) If $\nu_{ij} > 0$ for all $j \neq i$ then $|\mathcal{T}| = 1$ a.s., and we call T the *fixation time*. Furthermore, because each $(Z_i(t), 0 \leq t < \infty)$ is a martingale we have $\mathbb{P}(\mathcal{T} = \{i\}) = \mathbb{P}(Z_i^* = n) = 1/n$ for each agent i.

(iii) If $\nu_{ij} = 0$ for some $j \neq i$ then $\mathbb{P}(|\mathcal{T}| = 1)$ is strictly between 0 and 1.

These facts suggest more quantitative questions to ask, in the setting of a sequence $(\nu_{ij}^{(n)})$ of meeting models with $n \to \infty$. How does $T^{(n)}$ behave? In case (iii), how do $|\mathcal{T}^{(n)}|$ and the distribution $\mathbb{P}(Z_i^* \in \cdot | i \in \mathcal{T}^{(n)})$ of a typical “final fortune" behave? In either case we can ask how the process of the number of solvent agents

$$N(t) := |\{i : Z_i(t) > 0\}|$$

behaves over $0 \leq t \leq T$. If the meeting model has some spatial structure then what can we say about the spatial structure of the set of solvent agents at time t?

1.2 Techniques and results

It turns out that a surprising variety of techniques can be exploited in the study of the compulsive gambler process. Amongst these techniques, to be described in section 2, the most natural are martingale results (Lemma 2.1) and elementary bounds obtained by comparison with the Kingman coalescent (e.g. Lemma 2.2). Less obvious is Lemma 2.4: instead of making the random choices of game-winners at the meeting times, we can insert initial randomness and then have a deterministic rule for game-winners. In the “standard" case that construction has a symmetry property (Lemma 2.6): the

[1] Any perceived analogy between averaging/compulsive gambler models and socialism/capitalism is entirely the reader’s responsibility.

deterministic rule is based on a uniformly random labeling of initial currency notes as $1, \ldots, n$, and conditional on the configuration of fortunes at time t, the allocation of note-labels to agents is uniformly random. This last method parallels methods used in the study of *exchangeable coalescents* [4, 5, 6], though the precise relation is not clear to us and we have not explicitly used results from that theory.

Our main purpose is to lay the groundwork for future research by describing explicitly these techniques (section 2). In this paper we pursue analysis in only one direction, by studying the setting where the meeting model is that agents meet neighbors in a sparse graph (section 3). We give bounds on the density of the ultimately solvent agents on regular graphs, and then study the case of trees. There is a general recursion on finite trees, and that methodology extends to certain infinite trees (regular and Galton-Watson) on which some expect calculations and bounds are obtained. In particular one can calculate the asymptotic density (3.14) of solvent agents under the sparse Erdős-Rényi meeting model, a result which can alternatively be seen in terms of the *short-time* behavior of the Kingman coalescent (section 3.3).

Several other, perhaps more interesting, directions of current or future research are outlined in the final section 4,

2 Four basic techniques

We now abbreviate “compulsive gambler" to CG. Fix a meeting model (ν_{ij}) on a set $\mathbf{Agents}$ of n agents. In developing these basic techniques we will generally work in the normalized setting (see remark below).

Write $\mathbf{X}(t) = (X_i(t), i \in \mathbf{Agents})$ for the time-t configuration of the normalized CG process; agent i has $X_i(t)$ units of money, and the state space is $\{\mathbf{x} = (x_i) : x_i \geq 0 \ \forall i, \sum_i x_i = 1\}$. The CG process is specified by its transition rates. For each ordered distinct pair (j,k) with $\min(x_j, x_k) > 0$,

$$\mathbf{x} \to \mathbf{x}^{(j,k)} \text{ at rate } \nu_{jk} \tfrac{x_j}{x_j + x_k}; \quad \text{ where} \tag{2.1}$$

$$\mathbf{x}_i^{(j,k)} = x_i, i \neq j,k; \quad \mathbf{x}_j^{(j,k)} = x_j + x_k; \quad \mathbf{x}_k^{(j,k)} = 0.$$

There is an initial state $\mathbf{x}(0) = (x_i(0))$, which we will sometimes need to assume has full support, that is

$$x_i(0) > 0 \ \forall i.$$

We remark that, given a result for the normalized CG process $\mathbf{X}(t)$, by taking initial state $\mathbf{x}_i(0) = 1/n \forall i$ and rescaling we can deduce the analogous result for the standard CG process $\mathbf{Z}(t)$ with $Z_i(0) = 1 \forall i$. In fact we can usually formulate results so that they remain true with $\mathbf{Z}$ substituted for $\mathbf{X}$.

Write

$$N(t) := |\{i : X_i(t) > 0\}|$$

for the number of *solvent* agents at time t. Note that when $\nu_{ij} > 0 \ \forall j \neq i$ it is clear there is a finite fixation time

$$T = \min\{t : N(t) = 1\} < \infty \text{ a.s.}$$

2.1 Martingale properties

We first record some notation for the elementary stochastic calculus of integrable bounded variation processes. Such a process (Y_t) has a Doob-Meyer decomposition $Y_t = M_t + A_t$, where (M_t) is a martingale and (A_t) is predictable, which can be written in differential notation as $dY_t = dM_t + dA_t$. To avoid introducing new symbols, we write $\mathbb{E}(dY_t|\mathcal{F}_t)$ for dA_t.

Lemma 2.1. *For the normalized CG process* $(\mathbf{X}(t))$*, for any meeting rates:*
(i) $(X_i(t), 0 \le t < \infty)$ *is a martingale.*
(ii) For $j \neq i$*,* $(X_i(t)X_j(t), 0 \le t < \infty)$ *is a supermartingale.*
(iii) For $f : \mathbf{Agents} \to \mathbb{R}$ *write* $M_f(t) = \sum_i f(i) X_i(t)$*. Given a metric* d *on* $\mathbf{Agents}$ *write*

$$L_f := \max_{j \neq i} \tfrac{|f(j)-f(i)|}{d(i,j)}, \quad \nu^* := \max_{j \neq i} \nu_{ij} d^2(i,j).$$

Then $(M_f(t), 0 \le t < \infty)$ *is a martingale, and*

$$\mathbb{E} M_f^2(t) - M_f^2(0) \le \tfrac{1}{2} \nu^* L_f^2 t. \tag{2.2}$$

(iv) Let $\psi : \mathbf{Agents} \times \mathbf{Agents} \to [0, \infty)$ *be such that* $\psi(i,j) \equiv \psi(j,i)$ *and* $\psi(i,i) \equiv 0$*. Define*

$$\Psi(\mathbf{x}) = \sum_{\{i,j\}} x_i x_j \psi(i,j), \quad \Psi_\nu(\mathbf{x}) = \sum_{\{i,j\}} \nu_{ij} x_i x_j \psi(i,j).$$

Then the process

$$\Psi(\mathbf{X}(t)) + \int_0^t \Psi_\nu(\mathbf{X}(s)) \, ds$$

is a martingale.

Proof. (i) and (ii) are straightforward. Inequality (iii) can be deduced from the explicit formula for $\mathbb{E} M_f^2(t)$ given later in Lemma 2.7, or directly was follows. $M_f(\cdot)$ is a martingale, and we calculate

$$\mathbb{E}(dM_f^2(t) | \mathcal{F}(t)) = dt \sum_{\{i,j\}} \nu_{ij} X_i(t) X_j(t) (f(j) - f(i))^2,$$

the sum being over unordered pairs. From (ii) we have $\mathbb{E} X_i(t) X_j(t) \le x_i x_j$, where $\mathbf{x} = (x_i)$ is the initial configuaration, so taking expectation

$$\begin{aligned}
\mathbb{E}(dM_f^2(t)) &\le dt \ \times \tfrac{1}{2} \sum_i \sum_{j:j \neq i} (\nu^* d^{-2}(i,j) \times (L_f d(i,j))^2 \times x_i x_j) \\
&= dt \ \times \tfrac{1}{2} \nu^* L_f^2 \sum_i \sum_{j:j \neq i} x_i x_j \\
&\le dt \ \times \tfrac{1}{2} \nu^* L_f^2.
\end{aligned}$$

This establishes (iii).

For (iv), consider a configuration $\mathbf{x}$ and a pair $\{i,j\}$. We can write $\Psi(\mathbf{x})$ in the form

$$\Psi(\mathbf{x}) = a + b(i) x_i + b(j) x_j + x_i x_j \psi(i,j)$$

where a does not depend on x_i or x_j, and $b(i) = \sum_{k \neq i,j} x_k \psi(i,k)$, $b(j) = \sum_{k \neq i,j} x_k \psi(j,k)$. If i and j meet then $\Psi(\mathbf{x})$ updates to a new value $\Psi^+(\mathbf{x})$ as follows:

if i wins (chance $\frac{x_i}{x_i + x_j}$) then $\Psi^+(\mathbf{x}) = a + b(i)(x_i + x_j)$

if j wins (chance $\frac{x_j}{x_i + x_j}$) then $\Psi^+(\mathbf{x}) = a + b(j)(x_i + x_j)$.

So, if i and j meet, then the expectation of the increment $\Psi^+(\mathbf{x}) - \Psi(\mathbf{x})$ equals $-x_i x_j \psi(i,j)$. This implies

$$\mathbb{E}(d\Psi(\mathbf{X}(t)) | \mathcal{F}(t)) = - \sum_{\{i,j\}} \nu_{ij} X_i(t) X_j(t) \psi(i,j) \, dt = -\Psi_\nu(\mathbf{X}(t)) \, dt$$

which is assertion (iv).

□

By scaling Lemma 2.1 applies unchanged to the standard CG process $(\mathbf{Z}(t))$, except that a scaling factor n^2 arises on the right side of (2.2).

2.2 The Kingman coalescent

In the particular case

$$\nu_{ij} = 1, \quad j \neq i$$

of the meeting model, the CG process (assuming the initial state $\mathbf{x}(0)$ has full support) is essentially the well-studied *Kingman coalescent* [4]. In this case the process $(N(t), 0 \leq t < \infty)$ is the "pure death" Markov chain, started at n, with transition rates $q_{m,m-1} = \binom{m}{2}$, from which it immediately follows that the fixation time

$$T := \min\{t : N(t) = 1\} \tag{2.3}$$

is a.s. finite with expectation

$$\mathbb{E}T = \sum_{m=2}^{n} 1/\binom{m}{2} = 2(1 - n^{-1}). \tag{2.4}$$

Here is a simple application.

Lemma 2.2. *Consider the normalized CG process with a meeting model for which $\nu_* := \min_{j \neq i} \nu_{ij} > 0$, and with arbitrary initial state $\mathbf{x}(0)$.*
(i) The fixation time T satisfies $\mathbb{E}T \leq 2/\nu_*$;
(ii) $\mathbb{P}(N(t) > r) \leq \frac{2}{r\nu_* t}, \ r \geq 2$;
(iii) $\mathbb{E}N(t) \leq \frac{C}{t\nu_*}, \ 0 < t < \infty$ for some numerical constant $C < \infty$;
(iv) Let L be the agent who has acquired all the money at time T. Then $\mathbb{P}(L = i) = x_i(0), \quad \forall i \in \mathbf{Agents}$.

Proof. Although the process $(N(t))$ is typically not Markov, when $N(t) = m$ the conditional intensity of a transition $m \to m-1$ is at least $\nu_* \binom{m}{2}$, so (i) follows by comparison with the Kingman chain result (2.4). Similarly, write $T_{(r)} = \min\{t : N(t) \leq r\}$ and $T_{(r)}^{\text{King}}$ for the corresponding quantity for the Kingman chain. Then

$$\begin{aligned} \mathbb{P}(N(t) > r) &= \mathbb{P}(T_{(r)} > t) \\ &\leq t^{-1}\mathbb{E}T_{(r)} \\ &\leq \nu_*^{-1} t^{-1} \mathbb{E}T_{(r)}^{\text{King}} \text{ by comparison with the Kingman chain.} \end{aligned}$$

And

$$\mathbb{E}T_{(r)}^{\text{King}} = \sum_{m=r+1}^{n} 1/\binom{m}{2} \leq 2/r.$$

So (ii) follows by comparison. A similar argument, calculating $\mathrm{var}(T_{(r)}^{\text{King}})$ and using Chebyshev's inequality, establishes (iii). Assertion (iv) follows from the martingale property (Lemma 2.1(i)) of $(X_i(t))$, applying the optional sampling theorem at time T. $\square$

By comparison with the Kingman chain result (2.4) we also get the lower bound

$$2(1 - n^{-1})\tfrac{1}{\nu^*} \leq \mathbb{E}T \tag{2.5}$$

for $\nu^* := \max_{j \neq i} \nu_{ij}$ on an n-agent space; this holds for the normalized CG process whose initial state has full support. But the following result will often be stronger in the standard case.

Proposition 2.3. *For the standard CG process on an n-element space $\mathbf{Agents}$ with $\nu_* := \min_{j \neq i} \nu_{ij} > 0$, the fixation time T satisfies*

$$\frac{1}{\nu^*} \leq \frac{1}{\binom{n}{2}} \sum_{\{i,j\}} \frac{1}{\nu_{ij}} \leq \mathbb{E}T.$$

Proof. Applying Lemma 2.1(iv) with $\psi(i,j) := 1_{(j\neq i)}/\nu_{ij}$ we see that

$$\Psi(\mathbf{Z}(t)) + \int_0^t \Psi_\nu(\mathbf{Z}(s))\, ds$$

is a martingale, for

$$\Psi(\mathbf{z}) = \sum_{\{i,j\}} \frac{z_i z_j}{\nu_{ij}}, \quad \Psi_\nu(\mathbf{z}) = \sum_{\{i,j\}} z_i z_j.$$

By the optional sampling theorem

$$\mathbb{E}\Psi(\mathbf{Z}(0)) - \mathbb{E}\Psi(\mathbf{Z}(T)) = \mathbb{E}\int_0^T \Psi_\nu(\mathbf{Z}(s))\, ds. \tag{2.6}$$

But $\Psi(\mathbf{Z}(0)) = \sum_{\{i,j\}} \frac{1}{\nu_{ij}}$ and $\Psi(\mathbf{Z}(T)) = 0$ and

$$\sum_{\{i,j\}} z_i z_j = \tfrac{1}{2}(n^2 - \sum_i z_i^2) \leq \tfrac{1}{2}(n^2 - n) = \binom{n}{2}.$$

So (2.6) implies

$$\sum_{\{i,j\}} \frac{1}{\nu_{ij}} \leq \binom{n}{2} \mathbb{E}T$$

which is the inequality $\frac{1}{\binom{n}{2}} \sum_{\{i,j\}} \frac{1}{\nu_{ij}} \leq \mathbb{E}T$. The leftmost inequality is immediate. □

2.3 The augmented process

Given a probability distribution $\pi = (\pi_i)$ on the set $\mathbf{Agents}$ of n agents with each $\pi_i > 0$, take independent random variables η_i with Exponential(π_i) distributions. Define a random ordering $\prec$ on $\mathbf{Agents}$ by

$$i \prec j \text{ if } \eta_i < \eta_j. \tag{2.7}$$

This is one of several equivalent definitions of the *size-biased random ordering* [10] associated with π. For instance, defining a random bijection $F : \{1, \ldots, n\} \to \mathbf{Agents}$ by

$$\begin{aligned}
\mathbb{P}(F(1) = i) &= \pi_i \\
\mathbb{P}(F(2) = j | F(1) = i) &= \pi_j/(1 - \pi_i),\ j \neq i \\
\mathbb{P}(F(3) = k | F(1) = i, F(2) = j) &= \pi_k/(1 - \pi_i - \pi_j),\ \{i,j,k\} \text{ distinct} \\
&\cdots
\end{aligned}$$

and so on, then the size-biased random ordering could be defined as

$$i \prec j \text{ if } F^{-1}(i) < F^{-1}(j). \tag{2.8}$$

We want to consider the normalized CG process with some initial configuration $\mathbf{x}(0) = (x_i(0))$. Take the size-biased random ordering $\prec$ on $\mathbf{Agents}$ associated with the probability distribution $\mathbf{x}(0)$. Conditional on the realization of $\prec$, we can define a variation of the CG process in which, when two agents i, j with non-zero money meet, the winner is always the agent who comes earlier in $\prec$ (if $i \prec j$ then i is the winner). In other words, the transition rates (2.1) become

$$\mathbf{x} \to \mathbf{x}^{(j,k)} \text{ at rate } \nu_{jk} \text{ if } \min(x_j, x_k) > 0 \text{ and } j \prec k. \tag{2.9}$$

Call this the *augmented process* $(\mathbf{X}(t), \prec)$ with initial state $(\mathbf{x}(0), \prec)$. Note that the random order $\prec$ does not change with time. The notation is justified by the lemma below, which says that the first component $(\mathbf{X}(t))$ of the augmented process evolves as the normalized CG process.

Lemma 2.4. *In the augmented process* $((\mathbf{X}(t), \prec), 0 \le t < \infty)$ *with initial state* $(\mathbf{x}(0), \prec)$*, the component* $(\mathbf{X}(t), 0 \le t < \infty)$ *evolves as the normalized CG process with initial configuration* $\mathbf{x}(0)$*.*

Proof. Recall the elementary facts that, for independent Exponential r.v.'s η_1, η_2 with rates λ_1, λ_2,

$$\mathbb{P}(\eta_1 < \eta_2) = \tfrac{\lambda_1}{\lambda_1+\lambda_2} \tag{2.10}$$

$$\text{the conditional dist. of } \eta_1 \text{ given } \eta_1 < \eta_2 \text{ is Exponential}(\lambda_1 + \lambda_2). \tag{2.11}$$

Implement the augmented process using the order $\prec$ at (2.7) given by independent Exponential($x_i(0)$) r.v.'s $(\eta_i, i \in \mathbf{Agents})$. Write $\mathcal{F}(t) = \sigma(\mathbf{X}(s), 0 \le s \le t)$, and note this does not include the random order $\prec$. We claim that for each t

> conditionally on $\mathcal{F}(t)$, the r.v.'s $(\eta_i : \ i \in \mathbf{Agents}, X_i(t) > 0)$ are independent Exponentials with rates $X_i(t)$.

It is enough to check this remains true inductively over meetings. If agents i and j meet at t with non-zero fortunes $X_i(t-)$ and $X_j(t-)$, then by the evolution rule for the augmented process

> on the event $\{\eta_i < \eta_j\}$ we have $X_i(t) = X_i(t-) + X_j(t-)$ and $X_j(t) = 0$

and similarly on the complementary event. By inductive hypothesis η_i and η_j are independent Exponentials of rates $X_i(t-)$ and $X_j(t-)$; fact (2.11) then says that η_i has Exponential $X_i(t)$ distribution and the induction goes through.

Having established the claim, consider again what happens when agents i and j meet at t with non-zero fortunes $X_i(t-)$ and $X_j(t-)$. The probability that the update is to $X_i(t) = X_i(t-) + X_j(t-)$ and $X_j(t) = 0$ (and similarly for the complementary event) is the probability of the event $\{\eta_i < \eta_j\}$, which by the claim and (2.10) equals $X_i(t-)/(X_i(t-) + X_j(t-))$. But this is the dynamics of the CG process. □

See [12] for uses of this result in the context of the metric coalescent.

2.4 The token process

For a *standard* CG process $(\mathbf{Z}(t))$ we can define the augmented process by scaling from the "uniform" ($x_i(0) \equiv 1/n$) case of the normalized CG process. But there is a more concrete and informative expansion of this notion, which we will describe here. First, here is a story which might help visualize what is going on. (In talks we ask several audience members to each place an actual currency note on the table, so we can demonstrate the story.) Real-world currency notes have serial numbers; imagine each agent starting with one note with a random serial number, so that the ranking (smallest to largest) of the n notes is uniformly random. When two agents with non-zero money meet, we specify that the agent who wins the game is determined as the agent who possesses, in their collection at that time, the smallest-ranked note. The winner adds the loser's notes to his pile of notes.

In the story, each agent has a set of notes, but what is relevant is not the precise serial numbers but the relative rankings of each of the n serial numbers. In the formalization below, $(S_i(t))$ is the set of rankings of all the notes owned by agent i at time t.

Note this story is consistent with the "uniform" case of Lemma 2.4, which is essentially the context where we record only the relative orders of each agent's *smallest-ranked* note. But in contrast to Lemma 2.4, what we do next is useful only in the "standard" context.

To formalize the story above, given meeting rates $(\nu_{ij}, i, j \in \mathbf{Agents})$ we first take a uniformly random bijection $F : \{1, \ldots, n\} \to \mathbf{Agents}$. Visualize *tokens* $1, \ldots, n$ being randomly dealt to the agents. Define a process $\mathbf{S}(t) = (S_i(t), i \in \mathbf{Agents})$ to have initial configuration

$$S_i(0) = \{F^{-1}(i)\},\ i \in \mathbf{Agents}$$

and transition rates (copying (2.9))

$$\mathbf{S} \to \mathbf{S}^{(j,k)} \text{ at rate } \nu_{jk} \text{ if } S_j \text{ and } S_k \text{ non-empty and } \min S_j < \min S_k \tag{2.12}$$

where

$$S_j^{(j,k)} = S_j \cup S_k,\ S_k^{(j,k)} = \emptyset,\ S_i^{(j,k)} = S_i \text{ for } i \neq j, k.$$

So $S_i(t)$ is just the set of tokens held by agent i at time t, and at a meeting the game is always won by the owner of the smallest (lowest-ranked) token. Call $(\mathbf{S}(t), 0 \leq t < \infty)$ the *token process*, and write

$$\mathbf{Z}(t) = (|S_i(t)|,\ i \in \mathbf{Agents}); \quad \mathcal{F}(t) = \sigma(\mathbf{X}(s),\ 0 \leq s \leq t).$$

As discussed above, Lemma 2.4 implies

Corollary 2.5. *In the token process* $(\mathbf{S}(t), 0 \leq t < \infty)$*, the process* $\mathbf{Z}(t) := (|S_i(t)|,\ i \in \mathbf{Agents})$ *evolves as the standard CG process.*

Corollary 2.5 plays a key role in the development of the metric coalescent in [12]. And we will see in sections 2.6 and 3 how it enables us to use simple intuitive arguments in our discrete setting. Corollary 2.5 is reminiscent of the Donnelly-Kurtz look-down construction [9] but we do not see a precise connection.

Lemma 2.6 below says: if we just see the number of tokens that each agent has, then the assignment of tokens to agents is uniformly random over possible assignments.

Lemma 2.6. *In the token process, for each* t*, the conditional distribution of* $(S_i(t), i \in \mathbf{Agents})$ *given* $\mathcal{F}(t)$ *is uniformly random over all partitions* $(B_i, i \in \mathbf{Agents})$ *of* $\{1, \ldots, n\}$ *with* $|B_i| = X_i(t)\ \forall i$*.*

Here (B_i) is a labeled set partition and $|B_i|$ is the cardinality of block B_i.

Proof. As in the proof of Lemma 2.4, it is enough to check that the assertion remains true inductively over meetings. Given that $S_{j_1}(t)$ and $S_{j_2}(t)$ are non-empty, the event of a meeting of (j_1, j_2) in $(t, t + dt)$ is independent of $(S_{j_1}(t), S_{j_2}(t))$. Such a meeting causes either S_{j_1} or S_{j_2} to become $S_{j_1}(t) \cup S_{j_2}(t)$ and the other to become empty. Now checking that the induction goes through reduces to checking the following elementary fact about merging components of uniformly random partitions, which we leave to the reader.

Take $(n_i, i \in I)$ with each $n_i \geq 1$ and $\sum_i n_i = n$. Take two elements j_1, j_2 of I, write j_0 for a new symbol and let $I' := (I \setminus \{j_1, j_2\}) \cup \{j_0\}$ and $n_{j_0} = n_{j_1} + n_{j_2}$. Take a uniformly random partition $(B_i, i \in I)$ of $\{1, \ldots, n\}$ into components with $|B_i| = n_i\ \forall i \in I$. Construct a random partition $(B_i', i \in I')$ by setting $B_{j_0} = B_{j_1} \cup B_{j_2}$ and $B_i' = B_i$ for other i. Then
(i) $(B_i', i \in I')$ is a uniformly random partition of $\{1, \ldots, n\}$ into components with $|B_i'| = n_i\ \forall i \in I'$;
(ii) The event "the minimum element of B_{j_1} is smaller than the minimum element of B_{j_2}" is independent of the random partition $(B_i', i \in I')$.

In applying this fact in our setting, the point is that the information revealed by the change in $\mathbf{Z}(\cdot)$ at the meeting is precisely the identity of j_1, j_2 and whether the event in (ii) occurs, but conditioning on these does not destroy uniformity. $\square$

Lemma 2.6 can be extended to the general normalized process – a more detailed treatment is given in [12] section 2.4.

2.5 Moment calculations

Lemma 2.6 allows us to do various calculations with a standard CG process $(\mathbf{Z}(t))$, such as the second-moment calculations below. Recall that (by the martingale property) we know $\mathbb{E}Z_i(t) \equiv 1$.

Lemma 2.7. *For a standard CG process,*

$$\mathbb{E}[Z_i(t)Z_j(t)] = \exp(-\nu_{ij}t), \ j \neq i. \tag{2.13}$$

$$\mathbb{E}[Z_i(t)(Z_i(t)-1)] = \sum_{j:j\neq i}(1-\exp(-\nu_{ij}t)). \tag{2.14}$$

So for $f : \mathbf{Agents} \to \mathbb{R}$ *we have*

$$\mathbb{E}\left(\sum_i f_i Z_i(t)\right)^2 = \sum_i f_i^2 \left[1 + \sum_{j:j\neq i}(1-\exp(-\nu_{ij}t))\right] + \sum_i \sum_{j:j\neq i} f_i f_j \exp(-\nu_{ij}t)). \tag{2.15}$$

Proof. Lemma 2.6 gives

$$\mathbb{P}(1 \in S_i(t), 2 \in S_j(t)|\mathcal{F}(t)) = \frac{Z_i(t)Z_j(t)}{n(n-1)}$$

and taking expectation

$$\mathbb{P}(1 \in S_i(t), 2 \in S_j(t)) = \frac{1}{n(n-1)}\mathbb{E}[Z_i(t)Z_j(t)].$$

From the dynamics of the token process, the event $\{1 \in S_i(t), 2 \in S_j(t)\}$ happens if and only if $F(1) = i, F(2) = j$ and $\tau_{ij} > t$, where τ_{ij} is the first meeting time of i and j. So

$$\mathbb{P}(1 \in S_i(t), 2 \in S_j(t)) = \frac{1}{n(n-1)}\ \mathbb{P}(\tau_{ij} > t).$$

These last two identities give (2.13). One can deduce (2.14) from (2.13) and the "martingale" fact $\mathbb{E}|S_i(t)| = 1$, but let us see how it follows by the same kind of argument as above. Lemma 2.6 gives

$$\mathbb{P}(\{1,2\} \subseteq S_i(t)|\mathcal{F}(t)) = \frac{Z_i(t)(Z_i(t)-1)}{n(n-1)}$$

and taking expectation

$$\mathbb{P}(\{1,2\} \subseteq S_i(t)) = \frac{1}{n(n-1)}\mathbb{E}[Z_i(t)(Z_i(t)-1)].$$

From the dynamics of the token process, the event $\{\{1,2\} \subseteq S_i(t)\}$ happens if and only if $F(1) = i$ and $F(2) =$ some j for which $\tau_{ij} \leq t$, and so

$$\mathbb{P}(\{1,2\} \subseteq S_i(t)) = \frac{1}{n(n-1)}\ \sum_{j:j\neq i}\mathbb{P}(\tau_{ij} \leq t).$$

These last two identities give (2.14). Finally, (2.15) follows from (2.13) and (2.14) by expanding the square. $\square$

2.6 Elementary properties of the standard CG process

Here we prove the remaining "mostly obvious" assertions about the standard CG process from section 1.1, and some minor extensions. Fix a meeting model (ν_{ij}) on n agents. Write G for the graph whose edges are the pairs (i,j) with $\nu_{ij} > 0$. Recall G is connected by assumption. An *anticlique* (or *independent set*) in G is a set A of vertices such that there is no edge with both end-vertices in A. There is a finite set of configurations $\mathbf{z}$ that can be reached by the standard CG process. Such a configuration is absorbing if and only if $\{i : z_i \geq 1\}$ is an anticlique. The process must reach some absorbing configuration at some a.s. finite time T, because $N(t)$ (the number of solvent agents) decreases by one every time the configuration changes. Write $\mathcal{T}$ for the random set of agents with non-zero money at T.

Lemma 2.8. *For the standard CG process:*
(i) $\mathbb{E}T \leq (n-1)/\delta$*, where* $\delta := \min\{\nu_{ij} : \ \nu_{ij} > 0\}$*.*
(ii) For each pair $\{i,j\}$ *with* $\nu_{ij} = 0$ *we have* $\mathbb{P}(i \text{ and } j \in \mathcal{T}) \geq \frac{2}{n(n-1)}$*.*
(iii) $\mathbb{P}(i \in \mathcal{T}) \geq \frac{1}{1+d(i)}$ *and so* $\mathbb{E}|\mathcal{T}| \geq \sum_i \frac{1}{1+d(i)}$*, where* $d(i)$ *is the degree of vertex* i *in* G*.*
(iv) $\mathbb{P}(|\mathcal{T}| = 1) > 0$*.*

Note that, if $\nu_{ij} = 0$ for some pair $\{i,j\}$, then (ii) implies $\mathbb{P}(|\mathcal{T}| = 1) < 1$.

Proof. If $N(t) = m$ and the configuration is not an anticlique then the conditional intensity of a transition $m \to m-1$ is at least δ, implying (i) by comparison with the pure death process with constant transition rate δ. For (ii) consider the token process from section 2.4. If $\nu_{ij} = 0$ then with probability $1/\binom{n}{2}$ agents i and j have tokens 1 and 2; if so, then neither can lose a game, so both must end in $\mathcal{T}$. Similarly for (iii), with probability $1/(1+d(i))$ agent i's token is smaller than all its $d(i)$ neighbors' tokens; if so, then agent i cannot lose a game, implying $i \in \mathcal{T}$. So $\mathbb{P}(i \in \mathcal{T}) \geq 1/(1+d(i))$, which is (iii). For (iv), consider a spanning tree for G. We can order its edges as

$$e_1 = (\ell_1, v_1), \ e_2 = (\ell_2, v_2), \ \ldots \ e_{n-1} = (\ell_{n-1}, v_{n-1})$$

in such a way that each ℓ_i is a leaf of the subtree in which edges $e_1, \ldots, e_{i-1}$ have been deleted. With non-zero probability, the first $n-1$ meetings in the meeting process are over the edges $e_1, \ldots, e_{n-1}$ in that order; and with non-zero probability, the game involving (ℓ_i, v_i) is won by v_i for each i. If this happens then v_{n-1} ends up with all the money. □

3 The sparse graph setting

Throughout section 3 we study the standard CG process $(\mathbf{Z}(t))$. Consider a connected finite graph G with n vertices and which is r-regular, for $r \geq 3$ (so if r is odd then n must be even). Take the set $\mathbf{Agents}$ as the vertices of G, and the meeting rates as

$$\begin{aligned} \nu_{ij} &= 1 \text{ if } (i,j) \text{ is an edge} \\ &= 0 \text{ if not.} \end{aligned} \tag{3.1}$$

As observed in section 1.1, the standard CG process must terminate in a random configuration $\mathbf{Z}^*$ with some random set $\mathcal{T}$ of solvent agents. We study the density of solvent agents:

$$\rho(G) := n^{-1}\mathbb{E}|\mathcal{T}|.$$

What are the possible values of $\rho(G)$, in terms of n and r? Consider first the lower bound.

Lemma 3.1. *(i)* $\rho(G) \geq \frac{1}{r+1}$*.*
(ii) If n *is a multiple of* r *then there exists a graph* G *such that*

$$\rho(G) \leq \tfrac{1}{r}\left(1 + \tfrac{2\kappa_r}{r(r-1)}\right)$$

where κ_r*, defined by (3.2) below, is such that* $\kappa_r \uparrow \kappa_\infty < \infty$ *as* $r \uparrow \infty$*.*

Proof. Assertion (i) repeats Lemma 2.8(iii). For (ii), consider the graph G constructed as follows. Take n/r disjoint graphs $C_1, \ldots, C_{n/r}$, each being the complete graph on r vertices with one edge (a_i, b_i) removed. Then add edges $(b_1, a_2), (b_2, a_3), \ldots, (b_{n/r}, a_1)$ to make G.

For each $1 \le i \le n/r$, the only possible way for $\mathcal{T}$ to contain more than one vertex of C_i is for $\mathcal{T}$ to contain the two vertices $\{a_i, b_i\}$ (because any other pair of agents in C_i will meet). So

$$\begin{aligned} \mathbb{E}|\mathcal{T} \cap C_i| &\le 1 \times \mathbb{P}(|\mathcal{T} \cap C_i| \le 1) + 2 \times \mathbb{P}(|\mathcal{T} \cap C_i| = 2) \\ &= 1 + q, \text{ where } q := \mathbb{P}(\{a_i, b_i\} \subseteq \mathcal{T}) \end{aligned}$$

and then

$$\rho(G) := n^{-1}\mathbb{E}|\mathcal{T}| \le n^{-1}(n/r)(1+q) = \tfrac{1}{r}(1+q).$$

To study q we use the token process representation from section 2.4. Observe first that if the smallest token amongst C_i is initially possessed by some agent c_i which is not a_i or b_i, then agent c_i can never lose a bet, because each agent that c_i meets has either zero money or has its original token as its smallest token. Moreover in this case neither a_i nor b_i can be in $\mathcal{T}$, because each meets c_i. So the only way that the event $\{a_i, b_i\} \subseteq \mathcal{T}$ can occur is if one of $\{a_i, b_i\}$ has the smallest original token amongst agents C_i. For $2 \le m \le r$ consider the event

$D_m := \{$ a_i has the smallest, and b_i has the m'th smallest, original token amongst agents $C_i\}$.

Because the tokens are initially uniformly randomly distributed,

$$q = 2 \sum_{m=2}^{r} \mathbb{P}(D_m \text{ and } \{a_i, b_i\} \subseteq \mathcal{T}) = \tfrac{2}{r(r-1)} \sum_{m=2}^{r} \mathbb{P}(\{a_i, b_i\} \subseteq \mathcal{T} | D_m).$$

Now we need to study what happens on the event D_m in order to bound the (conditional) probability of $\{a_i, b_i\} \subseteq \mathcal{T}$. First note that the possibility of a loss by a_i or b_i to its neighbor outside C_i can only lower this probability, so we can ignore this possibility. And the possibility of a win by a_i or b_i against its neighbor outside C_i makes no difference. So we need only consider meetings within C_i. But then we need only consider meetings within the set C_i^m of the agents with the m smallest tokens, because these cannot lose to the other agents. And then we only need to consider such meetings between solvent agents, and each such meeting will reduce the number of solvent agents by 1. In order that $b_i \in \mathcal{T}$ it is necessary that no such meeting involves b_i. Inductively on j such meetings having occurred without involving b_i, there are $m-j-1$ other solvent agents in C_i^m, so the conditional probability that the next such meeting also does not involve b_i equals $\frac{\binom{m-j-1}{2}}{\binom{m-j}{2}-1}$, where the "-1" arises because the edge (a_i, b_i) is not present. So now we have shown

$$q \le \tfrac{2}{r(r-1)} \sum_{m=2}^{r} \sigma_m, \text{ where } \sigma_m := \prod_{j=0}^{m-3} \frac{\binom{m-j-1}{2}}{\binom{m-j}{2} - 1}.$$

But σ_m decreases as order m^{-2}, establishing the bound in (ii) for

$$\kappa_r = \sum_{m=2}^{r} \sigma_m. \tag{3.2}$$

□

Finding somewhat tight upper bounds complementary to those in Lemma 3.1 seems more difficult. In the following sections we study the standard CG process on trees, in particular on the infinite r-ary tree $\mathbb{T}_r$, where the random set $\mathcal{T}$ of solvent agents in the $t \to \infty$ limit has some density

$$\rho(\mathbb{T}_r) := \mathbb{P}(i \in \mathcal{T}).$$

It is well-known [8] that there exist, for fixed $r \geq 3$, sequences $(G_{n,r}, n \geq n_0(r))$ of r-regular n-vertex connected graphs (derived e.g. from typical realizations of random r-regular graphs) which converge to $\mathbb{T}_r$ in the sense of local weak convergence (Benjamini-Schramm convergence). That is, the distribution of the restriction of $\mathbb{T}_r$ to vertices within any fixed graph distance from a uniformly random root vertex converges to that restriction of rooted $\mathbb{T}_r$. It is intuitively clear (we outline the argument in section 3.4) that for such a sequence we will have

$$\lim_n \rho(G_{n,r}) = \rho(\mathbb{T}_r). \tag{3.3}$$

By analyzing the CG process on $\mathbb{T}_r$ we will show (Corollary 3.5) that $\rho(\mathbb{T}_r) \sim 2/r$ as $r \to \infty$. Granted that result, we can summarize Lemma 3.1 and the discussion above as follows.

Proposition 3.2. *Define*

$$\begin{aligned} a^*(r) &= \sup_{(G_{n,r})} \limsup_n \rho(G_{n,r}) \\ a_*(r) &= \inf_{(G_{n,r})} \liminf_n \rho(G_{n,r}) \end{aligned}$$

the sup *and* inf *over sequences* $(G_{n,r}, n \geq n_0(r))$ *of* r*-regular* n*-vertex connected graphs. Then*

$$\begin{aligned} a_*(r) &\sim \frac{1}{r} \text{ as } r \to \infty \\ a^*(r) &\geq \frac{2 - o(1)}{r} \text{ as } r \to \infty. \end{aligned}$$

We conjecture (based solely on vague intuition) that in fact $a^*(r) \sim 2/r$ as $r \to \infty$, in other words that locally tree-like graphs are asymptotically extremal for this problem.

3.1 Finite trees

Consider the standard CG process on a finite tree $\mathbb{T}$, with the constant meeting rates (3.1) over edges. We establish a recursion, Lemma 3.3, for the distribution of $Z_{(\mathbb{T},o)}(t)$, the fortune of agent o at time t. The CG process uses only (some of) the first meeting times τ_e across edges, which are independent with Exponential(1) distribution; by a deterministic time-change we can suppose instead the distribution is Uniform(0,1), simplifying calculations below.

For $0 \leq t, z \leq 1$, set

$$\phi_{(\mathbb{T},o)}(z,t) := 1 - \mathbb{E}\left[z^{Z_{(\mathbb{T},o)}(t)}\right].$$

For a neighbor i of o (written $i \sim o$), we let $\mathbb{T}_i$ denote the subtree of $\mathbb{T}$ (as viewed from root o) consisting of i and all its descendants.

Lemma 3.3. *For* $0 \leq z, t \leq 1$*,*

$$\phi_{(\mathbb{T},o)}(z,t) = \int_z^1 \prod_{i \sim o} \left(1 - \int_0^t \phi_{(\mathbb{T}_i,i)}(\xi,u)du\right) d\xi.$$

Proof. Let $Y(t)$ denote the fortune of agent o at time t in the modified process where o systematically wins every game she plays. Clearly,

$$Y(t) \stackrel{d}{=} 1 + \sum_{i \sim o} Z_{(\mathbb{T}_i, i)}(\tau_{oi}) \mathbf{1}_{\{\tau_{oi} \le t\}}, \tag{3.4}$$

with the terms in the sum being independent. The original process can be coupled with the modified process in the natural way, such that they coincide as long as o has not lost a game (in the original process). Hence, almost surely under this coupling,

$$Z_{(\mathbb{T},o)}(t) = Y(t) \text{ or } Z_{(\mathbb{T},o)}(t) = 0.$$

From the “fair game" structure of the CG process we know $\mathbb{E}\left[Z_{(\mathbb{T},o)}(t)\right] = 1$. But conditioning on the times of meetings does not alter the “fair game" structure; and because the times of meetings determine $Y(t)$, we have $\mathbb{E}\left[Z_{(\mathbb{T},o)}(t) \mid Y(t)\right] = 1$. So the conditional distribution of $Z_{(\mathbb{T},o)}(t)$ given $Y(t)$ must be

$$Z_{(\mathbb{T},o)}(t) = \left\{ \begin{array}{lll} Y(t) & \text{w.p} & 1/Y(t) \\ 0 & \text{w.p} & 1 - 1/Y(t). \end{array} \right. .$$

Hence,

$$\phi_{(\mathbb{T},o)}(z,t) = \mathbb{E}\left[\mathbb{E}[1 - z^{Z_{(\mathbb{T},o)}(t)} | Y]\,\right] = \mathbb{E}\left[\frac{1 - z^{Y(t)}}{Y(t)}\right] = \int_z^1 \mathbb{E}\left[\xi^{Y(t)-1}\right] d\xi. \tag{3.5}$$

Now (3.4) gives

$$\mathbb{E}\left[\xi^{Y(t)-1}\right] = \prod_{i \sim o} \left(1 - \int_0^t \phi_{(\mathbb{T}_i,i)}(\xi, u) du\right)$$

and substituting into (3.5) completes the proof. □

3.2 Infinite trees

The infinite d-ary tree. We next consider the case where $(\mathbb{T}, o)$ is the infinite $d-$ary tree rooted at o, that is each vertex has $d \ge 1$ children. Writing

$$\phi_d(z,t) = 1 - \mathbb{E}\left[z^{Z_{(T,o)}(t)}\right] \tag{3.6}$$

Lemma 3.3 gives the functional identity

$$\phi_d(z,t) \quad = \quad \int_z^1 \left(1 - \int_0^t \phi_d(\xi, u) du\right)^d d\xi. \tag{3.7}$$

We could not determine $\phi_d(z,t)$ explicitly, but we will establish the following bounds, which are sufficient to obtain asymptotics as $d \to \infty$.

Lemma 3.4. *Setting $\varepsilon_d = \frac{2}{d} \log\left(1 + \frac{d}{2}\right) < 1$, we have for $0 \le z, t \le 1$ and $d \ge 1$,*

$$\frac{2(1-z)(1-\varepsilon_d)}{2(1-\varepsilon_d) + d(1-z)t} \le \phi_d(z,t) \le \frac{2(1-z)}{2 + d(1-z)t}.$$

In particular, $\mathbb{P}(Z_{(\mathbb{T},o)}(1) \ne 0) = \phi_d(0,1) \sim 2/d$ as $d \to \infty$.

Proof. First note that, by (3.7),

$$-\frac{\partial \phi_d}{\partial t}(z,t) \quad = \quad d \int_z^1 \phi_d(\xi, t) \left(1 - \int_0^t \phi_d(\xi, u) du\right)^{d-1} d\xi,$$

whereas, using again (3.7) and the fact that $\phi_d(1,t) \equiv 0$,

$$\begin{aligned}(\phi_d(z,t))^2 &= -2\int_z^1 \phi_d(\xi,t)\frac{\partial \phi_d}{\partial z}(\xi,t)d\xi \\ &= 2\int_z^1 \phi_d(\xi,t)\left(1-\int_0^t \phi_d(\xi,u)du\right)^d d\xi.\end{aligned}$$

Combining these two identities, we obtain

$$\frac{\partial(1/\phi_d)}{\partial t}(z,t) = \frac{d}{2} \times \frac{\int_z^1 \phi_d(\xi,t)\left(1-\int_0^t \phi_d(\xi,u)du\right)^{d-1} d\xi}{\int_z^1 \phi_d(\xi,t)\left(1-\int_0^t \phi_d(\xi,u)du\right)^{d} d\xi}. \tag{3.8}$$

Since ϕ_d is $[0,1]-$valued, we see that

$$\frac{\partial(1/\phi_d)}{\partial t}(z,t) \geq \frac{d}{2}.$$

Integrating with respect to t gives

$$\frac{1}{\phi_d(z,t)} \geq \frac{1}{\phi_d(z,0)} + \frac{td}{2} = \frac{1}{1-z} + \frac{td}{2}$$

which rearranges to the claimed upper bound. From this upper bound, it follows that for $0 \leq z,t \leq 1$,

$$\int_0^t \phi_d(z,u)du \leq \varepsilon_d, \tag{3.9}$$

which we may plug into (3.8) to obtain:

$$\frac{\partial(1/\phi_d)}{\partial t}(z,t) \leq \frac{d}{2} \times \frac{1}{1-\varepsilon_d}.$$

Integrating with respect to t gives

$$\frac{1}{\phi_d(z,t)} \leq \frac{1}{\phi_d(z,0)} + \frac{td}{2(1-\varepsilon_d)} = \frac{1}{1-z} + \frac{td}{2(1-\varepsilon_d)}$$

which rearranges to the claimed lower bound. □

The $r-$regular tree. The $r-$regular infinite tree consists of r copies of a $(r-1)-$ary tree, connected to a root. Letting $\phi_r^*(z,t)$ denote the corresponding function, as at (3.6), the general recursion from Lemma 3.3 gives

$$\phi_r^*(z,t) = \int_z^1 \left(1-\int_0^t \phi_{r-1}(\xi,u)du\right)^r d\xi.$$

Comparing with (3.7) and using (3.9), it follows that

$$(1-\varepsilon_{r-1})\phi_{r-1} \leq \phi_r^* \leq \phi_{r-1},$$

so that ϕ_r^* satisfies the same $r \to \infty$ asymptotics as does ϕ_{r-1}. In particular,

Corollary 3.5. *On the $r-$regular infinite tree $\mathbb{T}_r$, the probability $\rho(\mathbb{T}_r) = \phi_r^*(0,1)$ that a given agent finishes with non-zero money is $2/r + o(1/r)$ as $r \to \infty$.*

Galton-Watson trees. When the rooted tree (T, o) is a random Galton-Watson tree with degree distribution $\{\pi_n : n \geq 0\}$, the general recursion from Lemma 3.3 immediately leads to a recursive distributional equation for the annealed generating function

$$\phi(z, t) := 1 - \mathbb{E}\left[z^{Z_{(T,o)}(t)}\right],$$

where expectation is now taken with respect to both the randomness of the rooted tree and the randomness of the CG process. Letting $F_\pi(x) = \sum_n \pi_n x^n$ denote the degree generating function of the Galton-Watson tree, we readily obtain:

$$\phi(z, t) \quad = \quad \int_z^1 F_\pi\left(1 - \int_0^t \phi(\xi, u)du\right) d\xi. \tag{3.10}$$

Extracting useful information from this equation for a general distribution $\{\pi_n : n \geq 0\}$ remains an open problem.

3.3 The Poisson-Galton-Watson tree, the sparse Erdős-Rényi graph and the short-time behavior of the Kingman coalescent

In the case where $\{\pi_n : n \geq 0\}$ is the Poisson distribution with mean $c \geq 0$ we have $F_\pi(z) = e^{cz-c}$) and equation (3.10) specializes to

$$\phi(z, t) = \int_z^1 \exp\left(-c \int_0^t \phi(\xi, u)\right) d\xi. \tag{3.11}$$

Differentiating with respect to z and t yields the partial differential equation

$$\frac{\partial^2 \phi}{\partial z \partial t} = -c\phi \frac{\partial \phi}{\partial z}.$$

In other words, the bivariate function $\frac{\partial \phi}{\partial t} + \frac{c}{2}\phi^2$ is constant in the variable z. Taking $z = 1$ in (3.11) shows that the constant is in fact 0, so in fact $\frac{\partial \phi}{\partial t} + \frac{c}{2}\phi^2 = 0$. This implies

$$\frac{\partial(1/\phi)}{\partial t} = \frac{-\frac{\partial \phi}{\partial t}}{\phi^2} = \frac{c}{2}.$$

Since $\phi(z, 0) = 1 - z$ we deduce that

$$\frac{1}{\phi(z, t)} = \frac{1}{1 - z} + \frac{ct}{2}.$$

Rearranging,

$$\phi(z, t) \quad = \quad \frac{2(1 - z)}{2 + c(1 - z)t}.$$

Identifying this generating function, we find that the fortune $Z(t)$ of the agent at the root has distribution specified by

$$\mathbb{P}(Z(t) > 0) = \tfrac{2}{2+ct} \tag{3.12}$$

the conditional distribution of $Z(t)$ given $Z(t) > 0$ is Geometric($\frac{2}{2+ct}$). (3.13)

Because the Poisson-Galton-Watson tree $\mathbb{T}_c^*$ is the local weak limit of the sparse Erdős-Rényi random graph $\mathcal{G}(n, c/n)$ it seems intuitively clear (see section 3.4) that

$$\mathbb{E}\rho(\mathcal{G}(n, c/n)) \to \mathbb{E}\rho(\mathbb{T}_c^*) = \mathbb{P}(Z(1) > 0) = \tfrac{2}{2+c}. \tag{3.14}$$

Let us outline an interesting alternative explanation of why (3.12, 3.13) arise here. Under our time-change (first meetings occur at Uniform$(0,1)$ random times) consider the process of all first meetings (whether or not agents are solvent) on $\mathcal{G}(n,c/n)$. This process arises from a two-stage construction: for each edge e of the complete graph on n vertices, first select e with probability c/n, then (if selected) assign the Uniform random meeting time. But for large n this is tantamount to saying that, for each edge e of the complete graph, the first meeting occurs at rate c/n over $0 < t < 1$. Now with this description of the meeting process, the process of fortunes of solvent agents in the standard CG process on $\mathcal{G}(n,c/n)$ becomes

> each pair of fortunes is, at rate c/n, replaced by one fortune, the sum of the pair of fortunes

Here we are considering the collection of fortunes as unordered – not retaining the identity of agents.

Now consider the Kingman coalescent, in the spirit of more general stochastic coalescence models [3, 4, 5, 6], as a process of coalescing partitions of $\{1,2,\ldots,n\}$, being the special case in which each pair of blocks merges at constant rate. But take this rate to be $1/n$ instead of 1. The $n \to \infty$ limit distribution of block sizes at time τ (for uniformly random choice of block) in this short-time limit regime is known to be Geometric($\frac{2}{2+\tau}$) – see Construction 2 and equation (25) in [3] for an intuitive explanation in terms of a process of coalescing intervals on $\mathbb{Z}$, or [11] for an analytic derivation. The evolution of the process of block sizes in this Kingman coalescent is

> each pair of blocks is, at rate $1/n$, replaced by one block, whose size in the sum of the two block sizes.

So up to the factor c, in the $n \to \infty$ limits the process of fortunes of solvent agents in the standard CG process on $\mathcal{G}(n,c/n)$ is the same as the process of block sizes in this Kingman coalescent. In particular the distribution in (3.13), for the fortune at time t of a uniformly random solvent agent in the standard CG process on $\mathcal{G}(n,c/n)$, is the same as the distribution of a uniformly random block in the Kingman coalescent at time $\tau = ct$, which is Geometric($\frac{2}{2+ct}$).

3.4 Local weak convergence

It seems plausible that, in some considerable generality, when we have Benjamini-Schramm convergence of a sequence $(\mathcal{G}_n)$ of finite graphs to a rooted infinite graph $\mathcal{G}_\infty$, then one can define uniquely the standard CG process on $\mathcal{G}_\infty$, and this process is the limit (in the natural analogous “local weak convergence" sense) of the standard CG processes on $\mathcal{G}_n$. We have asserted this as “intuitively clear" in the particular cases (3.3,3.14) where the limit graph is the infinite r-ary tree $\mathbb{T}_r$ or the Poisson-Galton-Watson tree $\mathbb{T}_c^*$. In fact a sufficient condition is as follows. Write $N(d)$ for the number of length-d paths from the root in $\mathcal{G}_\infty$. The sufficient condition (clearly satisfied in our two cases) is

$$\mathbb{E}N(d) = o((d-2)!) \text{ as } d \to \infty. \tag{3.15}$$

By the token process representation the probability that the root acquires a token along a given path of length d is at most $1/d!$, so condition (3.15) ensures that the behavior of the CG process at the root of $\mathcal{G}_\infty$ is determined “locally" by the initial tokens in an a.s. finite region. More details of this argument can be found at [14].

4 Directions for future research

Here are some other directions of current or future research.

The metric coalescent. This concerns a continuous-space extension. Take a suitable space S, write $\mathcal{P}(S)$ for the space of probability measures μ on S and write $\mathcal{P}_{\text{fs}}(S) \subset \mathcal{P}(S)$ for the subspace of finite support probability measures. Consider a symmetric function $\nu : S^2 \to \mathbb{R}_{\geq 0}$. For any $\mu \in \mathcal{P}_{\text{fs}}(S)$, we can consider the normalized compulsive gambler process for which the set $\mathbf{Agents}$ is the support $\{s_1, \ldots, s_n\}$ of μ, the meeting rates are the $\nu(s_i, s_j)$, and the initial distribution of money is μ; and moreover we can regard the states of the process as elements of $\mathcal{P}_{\text{fs}}(S)$. So we can reconsider the normalized compulsive gambler process as a Markov process, specified by the function ν, whose state space is (all of) $\mathcal{P}_{\text{fs}}(S)$. Then we can ask, inspired by the Kingman coalescent and its extensions [5], whether it makes sense to imagine this process starting with a general (in particular, non-atomic) initial state $\mu_0 \in \mathcal{P}(S)$. This topic is studied in detail in [12] in the context of a complete separable locally compact metric space (S, d) and meeting rates of the form

$$\nu(s, s') = \phi(d(s, s'))$$

for some continuous function $\phi(\cdot) > 0$. The main result of [12] is that, under the condition $\lim_{x \downarrow 0} \phi(x) = \infty$, the standardized compulsive gambler process on $\mathcal{P}_{\text{fs}}(S)$ extends to a Feller process (the *metric coalescent*) on all of $\mathcal{P}(S)$. In particular, for an initial $\mu_0 \in \mathcal{P}(S)$ with compact support, the metric coalescent process $(\mu_t, 0 \leq t < \infty)$ has finite support at each $t_0 > 0$, evolves as the compulsive gambler process over $[t_0, \infty)$ and satisfies the initial condition

$$\mathbb{P}(\lim_{t \downarrow 0} \mu_t = \mu_0 \text{ in } \mathcal{P}(S)) = 1.$$

A key ingredient in the proof is Corollary 2.5 of this paper. In informal language, Corollary 2.5 says that for the standard compulsive gambler process, instead of determining the game winners at the meeting times, we can do so via initial randomization, as follows. Initially each agent has a currency note with a random serial number; when two solvent agents meet, each has a collection of notes, but now the winner is always the owner of the lowest-ranked note, ranking by serial number. In other words we start with a uniformly random ordering $s_1, \ldots, s_n$ of $\mathbf{Agents}$, ranked by serial number of note. In the continuous-space setting we can do the same construction but starting with i.i.d. (μ_0) random samples $s_1, \ldots, s_n$. For each n we now have a $\mathcal{P}_{\text{fs}}(S)$-valued process $(\mu_t^{(n)}, 0 \leq t < \infty)$. But as n varies these processes have a natural coupling and the metric coalescent can be constructed as the a.s. $n \to \infty$ limit process.

Infinite discrete space. For a countable infinite set $\mathbf{Agents}$ where we do not (as we did at (3.15)) require meeting rates to be constant or zero, an alternative sufficient condition for the standard compulsive gambler process $(\mathbf{Z}(t))$ to be well-defined is

$$\nu^* := \sup_i \sum_{j: j \neq i} \nu_{ij} < \infty. \tag{4.1}$$

In section 3 we studied the case of the r-regular tree. For another direction, consider the case where $\mathbf{Agents} = \mathbb{Z}^d$ and the meeting rates are

$$\nu_{ij} = ||j - i||^{-\alpha} \tag{4.2}$$

for some $\alpha > d$, implying (4.1). Consider the mean density of solvent agents at time t

$$\rho(t) := \mathbb{P}(Z_i(t) \neq 0)$$

and the conditional distribution $Z^*(t)$ of $Z_i(t)$ given $Z_i(t) \neq 0$, for which $\mathbb{E}Z^*(t) = 1/\rho(t)$ because $\mathbb{E}Z_i(t) \equiv 1$. Heuristic arguments, based on supposing the positions of solvent agents do not become “clustered", suggest that

$$\rho(t) \asymp t^{-\beta} \text{ for } \beta = \tfrac{d}{\alpha}.$$

It is then plausible that

$$\rho(t)Z^*(t) \to_d Z^*, \quad \text{for some } Z^* \text{ such that } \mathbb{E}Z^* = 1$$

and then that the process has a scaling limit, the limit being a process whose states are (locally finite support) measures on $\mathbb{R}^d$.

Long-range meeting on the torus. Consider the d-dimensional discrete torus $\mathbb{Z}_m^d$ and take meeting rates as at (4.2). By the heuristics above, for $\alpha > d$ we expect the fixation time T to scale as $\rho^{-1}(m^{-d}) = m^\alpha$. In this setting it makes sense to consider also the case $\alpha < d$. In that case an agent will tend to meet distant agents rather than nearby ones – loosely speaking, there is a "phase transition" in the behavior of the process at $\alpha = d$. (Let us mention in passing that a detailed study of phase transitions in the first-passage percolation model with rates (4.2) has been given in [7]). However by comparison with the Kingman coalescent in the case $\alpha < d$ we still expect T to scale as m^α, reflecting the minimum meeting rate of $m^{-\alpha}$. And somewhat surprisingly we can establish the order of magnitude of T in both cases without any detailed analysis: Proposition 2.3 easily implies that in the torus model there exist constants $c_{d,\alpha}$ and $C_{d,\alpha}$ such that

$$c_{d,\alpha} m^\alpha \le \mathbb{E}T \le C_{d,\alpha} m^\alpha. \tag{4.3}$$

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