Better bounds for the worst case of the stretch-length tradeoff in geometric networks

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In this write-up, the worst case study of the stretch-length tradeoff in geometric networks is considered. The original problem is raised by Prof. Aldous in his paper[1]. The notations and definitions here will be consistent with the paper.

For geometric spanner networks, how well the network provides short routes is often measured by the *stretch* or *spanning ratio* of the network, which is defined as

$$S := \max_{v \neq w} \frac{r(v,w)}{d(v,w)} \geq 1$$

where $d(v,w)$ denotes the Euclidean distance and $r(v,w)$ denotes the route length.

The underlying setting is a configuration of n cities at arbitrary positions $\mathbf{z}_n = (z_1, \cdots, z_n)$ in a square of area n. For a network $\mathscr{N}$ which connects all these cities, write $S(\mathscr{N})$ for its stretch and write

$$L(\mathscr{N}) = \frac{1}{n} \times (\text{network length of } \mathscr{N})$$

for normalised network length. Define

$$\psi_n(\mathbf{z}_n, s) := \inf\{L(\mathscr{N}) : S(\mathscr{N}) \leq s\}$$

the infimum over all networks connecting the cities $\mathbf{z}_n$. In the paper, the worst case and the average case are studied. The worst-case is defined as

$$\Psi^{worst}(s) = \lim_{n\to\infty} \sup_{\mathbf{z}_n} \psi_n(\mathbf{z}_n, s)$$

As shown in the paper, such limit exists but is is very difficult to find a way to get a formula for that. For this reason, estimation for this quantity is more practical. Prof. Aldous derived upper bounds on $\Psi^{worst}(s)$ for some special values from specific constructions:

$$\Psi^{worst}(2) \leq 4, \quad \Psi^{worst}(\frac{3}{2}) \leq 4\sqrt{2}, \quad \Psi^{worst}(\sqrt{2}) \leq 4\sqrt{3}$$

In this write-up, I figure out some new ways to construct networks and these constructions give a better upper bound for $\Psi^{worst}(s)$. The main result is the following:

Theorem 1.

$$\Psi^{worst}(2) \leq 2\sqrt{2+\sqrt{2}} \quad (1)$$

$$\Psi^{worst}(\frac{3}{2}) \leq 2\sqrt{2+2\sqrt{2}} \quad (2)$$

$$\Psi^{worst}(\sqrt{2}) \leq 2\sqrt{2+3\sqrt{2}} \quad (3)$$

Proof. The proof is similar with the method in the paper. For fixed $0 < t_\infty < \infty$, choose $t = t(n) \to t_\infty$ such that $n^{1/2}/t(n)$ is an integer $m = m(n)$. The first step for the construction of the network is to divide the square into m^2 subsquares with side-length t. The total length of these grid roads (including the boundary of $[0, \sqrt{n}]^2$) is

$$\sqrt{n} \times 2(m+1) \sim 2n/t_\infty$$

Next, for each subsquare, construct the two diagonal roads. Consequently, now the network partitions the region $[0, \sqrt{n}]^2$ into $4m^2$ isosceles right triangle. Then these diagonal roads have total length

$$2\sqrt{2}t_\infty \times m^2 = 2\sqrt{2}n/t_\infty$$

Note that each city lies in at most one of the triangles. Finally, to form a connect graph that contains all n cities, construct three roads that are perpendicular to the three edges from the city across the edges. To estimate the length of such roads, first consider the following basic geometry problem:

For a isosceles right triangle with base length 1, for any fixed point in the triangle, construct three segment which are perpendicular to the three edges starting from the point. Use $h_i,\ i = 1, 2, 3$ to represent the length of these three segments. Then we have that

$$\frac{1}{2}(1 \cdot h_1 + \frac{\sqrt{2}}{2} \cdot h_2 + \frac{\sqrt{2}}{2} \cdot h_3) = \frac{1}{4} \Rightarrow h_1 + h_2 + h_3 \leq \frac{\sqrt{2}}{2}$$

Thus, for the added roads which make the network contain the cities, the total length is less than or equal to $\frac{\sqrt{2}}{2}nt_\infty$.

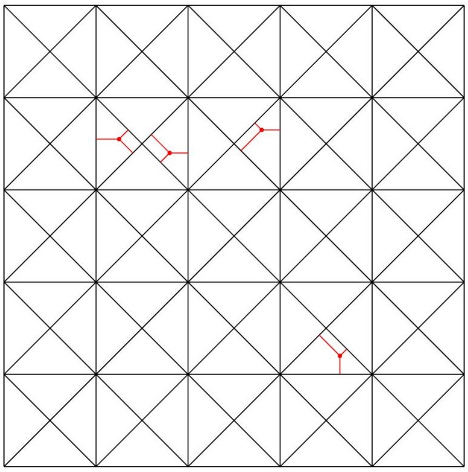

Figure 1: Part of the construction of network $\mathscr{N}_n^1$

Now we study the network $\mathscr{N}_n^1$ thus constructed. For the corresponding normalised total length, we have

$$n^{-1}\text{len}(\mathscr{N}_n^1) \to \frac{\sqrt{2}}{2}\left(t_\infty + \frac{2\sqrt{2}+4}{t_\infty}\right)$$

The key part is to consider the restriction of the stretch. As mentioned in the paper, i a right triangle with side-lengths a, b and $c = \sqrt{a^2 + b^2}$ we have

$$\frac{a+b}{c} \leq \sqrt{2}$$

To show that a city pair (i, j) has $r(i, j)/d(i, j) \leq \sqrt{2}$, we must first consider the shortest path connecting these two cities. In this construction of the network, the shortest path between two cities can be easily found in a direct way. The bound is clearly true in the following cases:

(i) the two cities are in the same square;

(ii) the two cities are in different rows and different columns;

(iii) the two cities are in adjacent squares;

Therefore, it remains to consider the final case:

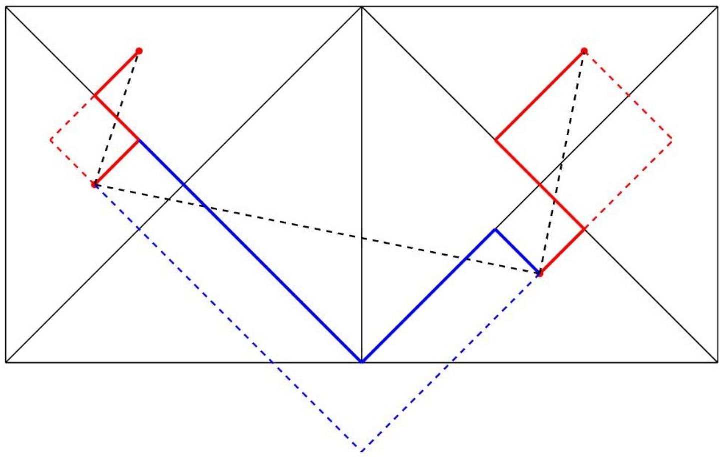

Figure 2: Illustration for these cases

(iv) the two cities are in squares in the same column (without loss of generality) separated by some number $k \geq 1$ of squares.

So for this network $\mathscr{N}_n^1$, it suffices to discuss the worst situation in case (iv). In the construction of network $\mathscr{N}_n^1$, the worst situation for a city pair (v, w) that maximise the ratio $r(v, w)/d(v, w)$ is quite clear. Specifically, this happens when $k = 1$. The intervening square contains no cities, and the two cities are arbitrarily close the the centers of the north edge and the south edge of the intervening square. For such two cities, we have $r(v, w)/d(v, w) = 2$ and this is the upper bound for case (iv). This implies that $\text{stretch}(\mathscr{N}_n^1) \leq 2$. Therefore, we have

$$n^{-1}\text{len}(\mathscr{N}_n^1) \to \frac{\sqrt{2}}{2}\left(t_\infty + \frac{2\sqrt{2}+4}{t_\infty}\right) \geq 2\sqrt{2+\sqrt{2}}$$

where $t_\infty = \sqrt{2\sqrt{2}+4}$ can reach the minimum. This just shows the upper bound (1).

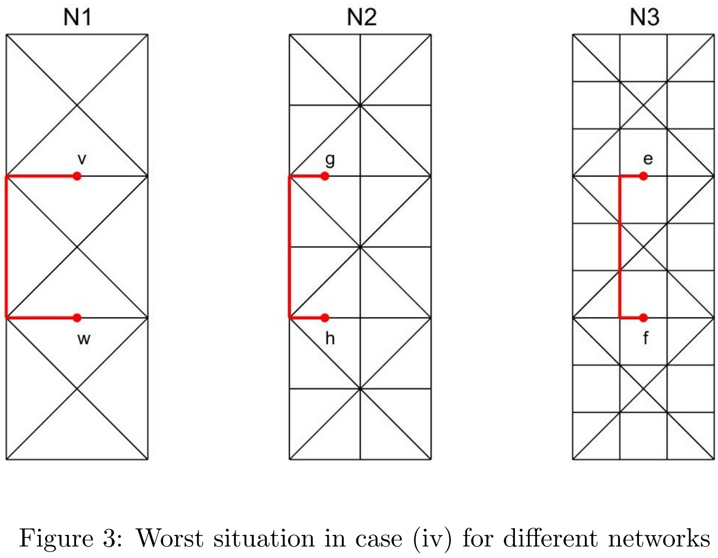

Figure 3: Worst situation in case (iv) for different networks

Now consider $\mathscr{N}_n^2$ obtained from $\mathscr{N}_n^1$ by adding, for each square, the N-S and the E-W interior roads across the square through the center of the square. Note that it's not necessary to add new diagonals for the smaller squares (otherwise we will just get the exactly same network, only with small side-length for squares). On this occasion, the case (iv) worst situation is where the two cities are arbitrarily close to a quarter of the way along the north and south edges of the intervening square. in this situation, it's easy to see that

$$r(g,h)/d(g,h) = 3/2 \Rightarrow \mathrm{stretch}(\mathscr{N}_n^2) \leq 3/2$$

In this setting, the total extra network length is

$$2t_\infty \times m^2 = 2n/t_\infty$$

This implies that

$$n^{-1}\mathrm{len}(\mathscr{N}_n^2) \to \frac{\sqrt{2}}{2}\left(t_\infty + \frac{4\sqrt{2}+4}{t_\infty}\right) \geq 2\sqrt{2+2\sqrt{2}}$$

where $t_\infty = \sqrt{4+4\sqrt{2}}$ can reach the minimum. This just shows the upper bound (2).

Finally consider the networks $\mathscr{N}_n^3$ obtained by adding, for each square, two N-S and two E-W interior roads partitioning the square into nine equal subsquares. Under this circumstance, the case (iv) worst situation is where the two cities are arbitrarily close to the half of the way along the north and the south edges of the intervening square. In this situation, it is clear that $r(e,f)/d(e,f) = 4/3$. However, this quantity is smaller than the upper bound of stretch in the case (i), (ii) and (iii), which is $\sqrt{2}$. So, in this setting, we have that $\mathrm{stretch}(\mathscr{N}_n^3) \leq \sqrt{2}$. The total extra network length with respect to $\mathscr{N}_n^1$ is

$$4t_\infty \times m^2 = 4n/t_\infty$$

This implies that

$$n^{-1}\mathrm{len}(\mathscr{N}_n^3) \to \frac{\sqrt{2}}{2}\left(t_\infty + \frac{6\sqrt{2}+4}{t_\infty}\right) \geq 2\sqrt{2+3\sqrt{2}}$$

where $t_\infty = \sqrt{6\sqrt{2}+4}$ can reach the minimum. This shows the upper bound (3).

As seen in the argument about the network $\mathscr{N}_n^3$, we don't need to divide the square into even smaller subsquares. This is because that although such procedure can make the upper bound of stretch for case (iv) become smaller, this bound will be less than $\sqrt{2}$. So in this kind of setting, the upper bound of the stretch corresponds to the other cases. This cannot give us a better bound for $\Psi^{worst}(\sqrt{2})$ than (3). □

Although these constructions help us get better upper bounds for the estimation of the worst case study, it is still unknown whether these results are optimal. Further works are still waiting to be conducted in order to find the optimal bounds.

Reference

1 Aldous, David; Lando, Tamar The stretch-length tradeoff in geometric networks: average case and worst case study. *Math. Proc. Cambridge Philos. Soc.* 159 (2015), no. 1, 125-151.