A Real-World Markov Chain arising in Recreational Volleyball

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January 21, 2021

Abstract

Card shuffling models have provided motivating examples for the mathematical theory of mixing times for Markov chains. As a complement we introduce a more intricate realistic model of a certain observable real-world scheme for mixing human players onto teams. We quantify numerically the effectiveness of this mixing scheme over the 7 or 8 steps performed in practice. We give a combinatorial proof of the non-trivial fact that the chain is indeed irreducible.

Key words: Markov chain, mixing time.
MSC subject classification: 60J10

1 Introduction

In introducing Markov chains at some elementary level, the first author always found it difficult to give a motivating example with a real-world story, a plausible probability model, and a fairly rich mathematical structure. Then he realized that he was a regular participant in one such story. A first thought was to write out the model for possible use as an instructional example in an introductory lecture. As often happens, things turned out to be more complicated than first imagined, so it was re-purposed as a basis for a challenging undergraduate project to study further aspects of the model. The second author took up the challenge.

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2 The model

The story concerns recreational volleyball, in a "drop-in" setting without fixed teams, and where one wants the team compositions to change from game to game, both as socialization and to avoid persistent large differences in team skill levels. Specifically there are 24 people, and at each stage there are two ongoing games on two courts, each game between two teams, each team with 6 players on a half-court. Over the 2 hour period there will be 7 or 8 successive such stages, everyone always playing. The rule[1] for changing team composition is very simple, exploiting a particular incidental feature of volleyball:

At the end of one stage, the players in the back row of each team stay in these positions for the start of the next game, while the front row players move (clockwise in the gym) to the same positions in the next half-court.

See Figure 1. The key point is that in volleyball, there are 6 "positions"[2], and players rotate one position each time their team regains the serve, and this happens a *random* number of times during a game. So, relative to initial positions, the 3 players who finish in the front row will be (to a good approximation) a *uniform* random choice over the 6 possibilities of 3 adjacent players.

To complete a mathematical model, note that the number of one-position rotations of two opposing teams can differ (because they alternate rotations) by at most one. So we model the final positions of players in opposing teams in a game as rotations by (C_1, C_2) where C_1 is uniform on $\{0, 1, \ldots, 5\}$ and $C_2 = C_1 - 1 + \text{Binomial}(2, 1/2)$ modulo 6, the Binomial term reflecting randomness of initial serving team and of final serving team, independently for the two courts.

This specifies a "big" Markov chain on the 24! states (assignment of players to positions). One step of the chain is from the starting positions in one game to the starting positions in the next game, as illustrated in Figure 1. This model is conceptually loosely related to some card shuffling models[3] such as [3, 7] in that a "rotation" of team players corresponds to a cut-shuffle of a 6-card deck. But unlike playing cards, the volleyball players care about their positions relative to other players for various reasons[4] and

[1]Actually used in the gym where the first author plays; I don't know how common it is.

[2]By convention numbered 1 to 6, starting in serving position (back right, as facing the net) and ordered counter-clockwise. Because players rotate clockwise, this indicates serving order.

[3]See further discussion in section 7.

[4]Friendly rivalry between spikers/blockers; more talented setters enable sophisticated

A	F	E	i	vi	v	
B	C	D	ii	iii	iv	start game 0
d	c	b	δ	γ	β	
e	f	a	ε	ϕ	α	

F	E	D	iv	iii	ii	
A	B	C	v	vi	i	end game 0
b	a	f	β	α	ϕ	
c	d	e	γ	δ	ε	

F	E	D	iv	iii	ii	
f	a	b	A	B	C	start game 1
β	α	ϕ	i	vi	v	
c	d	e	γ	δ	ε	

Figure 1: One step of the big chain

this suggests actual observables for study in the model.

3 Results

The central, albeit vague, question is

> how effective is this scheme at mixing up the teams?

In a lecture course, this provides a real-world example for later discussion of the *mixing times* topic. It seems intuitively obvious that this scheme would mix perfectly in the long run. Basic finite Markov chain theory[5] identifies "mix perfectly in the long run" with *irreducible and aperiodic*, which implies convergence of time-t distributions to a unique stationary distribution, which

fast plays; attractive members of opposite sex; ...

[5] In many textbooks such as [5, 8].

in our model is clearly the uniform distribution on all 24! states. Theory also tells us that *irreducible* is equivalent to the property

> the directed graph of all possible transitions on the 24! states is strongly connected.

This property is purely combinatorial – the numerical values of the non-zero transition probabilities do not matter – and we give a constructive proof in section 4.

Our second set of results concern numerical calculation or simulation of statistics relating to the realistic short term in this story – 7 or 8 steps. Some basic observables involve the *friend chain* indicating the relative positions of two players. A variety of numerical results are shown in section 5. For instance, if your friend does not start on your team, then the probability that you are never on the same team over 8 games varies between 0.251 and 0.403 depending on initial relative positions (Table 5). A pedagogic point is that, for numerical calculations, we don't want to work with $24! \times 24!$ transition matrices, but instead exploit symmetry to reduce to question-specific small-state chains. For instance the way a given player moves between games is simple: with chance 1/2 they stay, with chance 1/2 they move to the next half-court. In jargon, the lazy cyclic walk [7].

The bottom line is that, as regards simple observables, this scheme does a reasonable job of mixing up the teams over 8 games. However the central point of sophisticated *mixing time* theory [7] is to go beyond the unquantified "eventually" implied by irreducibility, and instead to quantify when the step-t distribution is close to (in our case) the uniform distribution. The usual quantification involves *variation distance* between distributions on the 24! states, and this is the context of the famous Bayer-Diaconis "7 shuffles suffice" result [2] for riffle shuffles. Studying variation distance for our big chain, either numerically of via analytic bounds, remains a challenging open problem. We give some preliminary observations in section 6.

`xxx We need to say something about` *`aperiodic`*`.`

4 The big chain is irreducible

4.1 Notation

To prove that the "big" chain is irreducible, we will show that it is possible to move from any one given state to any other given state via some sequence of allowable transitions of the chain.

Label the four half-courts A, B, C, D as shown in Figure 2. The change in configuration, from the start of a game to the end of that game, can be represented symbolically in the form

$$A^{x_1} C^{x_2} B^{x_3} D^{x_4},$$

where $0 \leq x_i \leq 5$ indicates the number of positions (modulo 6) rotated by the team in the relevant half-court. Figure 3 gives an illustration.

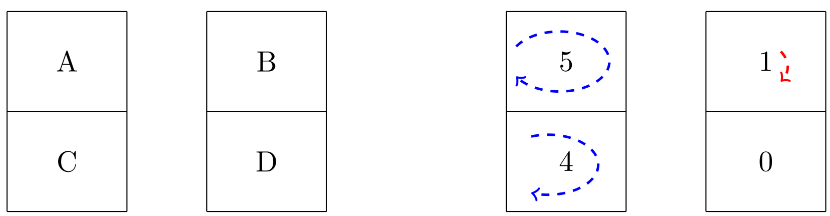

Figure 2: (Left) Labelling of the 4 half-courts. (Right) Rotations involved in step A^5C^4BE.

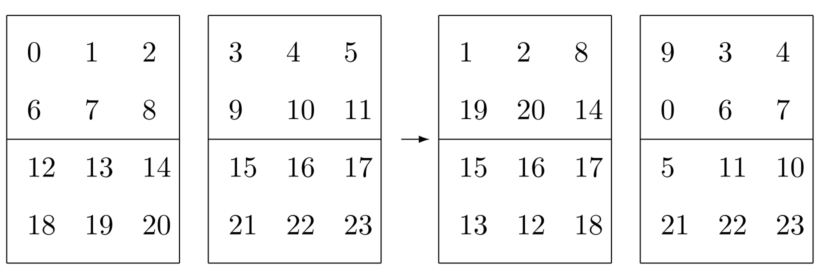

Figure 3: The effect of step A^5C^4BE.

So the allowable values are $x_1 - x_2 \in \{-1, 0, 1\}$ and $x_3 - x_4 \in \{-1, 0, 1\}$ modulo 6. We then append a symbol E to indicate the final movement (the front row players in each half-court move to the same positions in the next half-court). This provides a coding of a step of the chain. For brevity we omit any $x_i = 0$ term and write B instead of B^1. So a typical step is coded in a format like A^5C^4BE. The reader may check that the Figure 1 example is $A^5C^4B^3D^2E$.

Then a sequence of steps can be specified by concatenation: so A^5C^4BEEDE represents 3 steps of the chain, the second (EE) step indicating a game with zero (modulo 6) rotations of each team before the front row switch. We will name certain sequences later as X, F, G, H in describing the construction. The number of steps in a sequence is just the number of E's, when expanded fully. In writing the sequences (such as the definition of X below) we often include spaces for visual clarity but the spaces have no mathematical significance.

One aspect of this notation may be confusing. The sequence $EEEE$ would code the identity move. That means that $BE\ EEEE$ has the same effect as BE. But note that $BEEEE$ is different, and in fact will be a useful device because it has the effect of rotating the players in half-court B while fixing all other players – see Figure 4. Note also that B^5EEEE is the analogous back-rotation. This syntax issue explains why we sometimes (e.g. in the definition of F below) need to include an initial $EEEE$ in the definition.

0	1	2	3	4	5		0	1	2	9	3	4
6	7	8	9	10	11	→	6	7	8	10	11	5
12	13	14	15	16	17		12	13	14	15	16	17
18	19	20	21	22	23		18	19	20	21	22	23

Figure 4: The effect of sequance $BEEEE$.

4.2 High level description 1

It is an elementary fact that any permutation of a card deck can be obtained by a sequence of transpositions of two adjacent cards. Indeed the "random adjacent transposition" shuffling scheme is one of the original and most deeply studied examples in the modern theory of mixing times [1, 6, 9]. By analogy, we start by showing that any transposition of two players on the same half-court can be obtained by some sequence of steps. There are 3 cases, depending on the initial distance between the two players, illustrated in Figure 5, and we will exhibit sequences F, G, H for each case. We will show the first case (adjacent players) in detail.

0	1	2	3	4	5		1	0	2	3	4	5
6	7	8	9	10	11	→	6	7	8	9	10	11
12	13	14	15	16	17		12	13	14	15	16	17
18	19	20	21	22	23		18	19	20	21	22	23

(a) Transpose two adjacent players in the same quadrant (F)

0	1	2	3	4	5		2	1	0	3	4	5
6	7	8	9	10	11	→	6	7	8	9	10	11
12	13	14	15	16	17		12	13	14	15	16	17
18	19	20	21	22	23		18	19	20	21	22	23

(b) Transpose two players in the same quadrant with one space in between (G)

0	1	2	3	4	5		8	1	2	3	4	5
6	7	8	9	10	11	→	6	7	0	9	10	11
12	13	14	15	16	17		12	13	14	15	16	17
18	19	20	21	22	23		18	19	20	21	22	23

(c) Transpose two players in the same quadrant with two spaces in between (H)

Figure 5: Transpositions achieved by specific sequences F, G, H.

4.3 Sequences that transpose two players

We start by introducing a 16 step sequence X defined as

$$(X :=) \quad AE \;\; B^2D^3EEE \;\; A^2C^3E \;\; B^3D^3EEE \;\; A^5E \;\; B^5EEE \;\; AE \;\; B^5EEE \;\; A.$$

The step-by-step trajectory of sequence X is shown in Figure YYY below, which demonstrates that the effect of X is as shown in Figure 6. The introduction of this X is somewhat magical and hard to explain, but note that for some players it is like a reverse step of the chain.

0	1	2	3	4	5		1	0	2	4	7	8
6	7	8	9	10	11	→	5	3	6	9	10	11
12	13	14	15	16	17		20	19	18	15	16	17
18	19	20	21	22	23		14	13	12	21	22	23

Figure 6: The effect of sequence X.

(xxx somehow I cannot put Figures YYY and ZZZ into the `figure` environment).

1. AE (separate 2)

6	0	1	3	4	5
14	13	12	7	8	2
15	16	17	11	10	9
18	19	20	21	22	23

2-4. B^2D^3EEE (move 2 to the upper left/bottom left corner of I)

6	0	1	8	7	3
2	5	4	21	22	23
12	13	14	15	16	17
18	19	20	9	10	11

5. A^2C^3E (separate 0 and 1 from 2

5	2	6	8	7	3
18	19	20	4	1	0
15	16	17	23	22	21
14	13	12	9	10	11

6-8. B^3D^3EEE (move 0 to the left of 2)

5	2	6	0	1	4
3	7	8	9	10	11
20	19	18	15	16	17
14	13	12	21	22	23

9. A^5E

2	6	8	0	1	4
18	19	20	5	3	7
15	16	17	11	10	9
14	13	12	21	22	23

10-12. B^5EEE

2	6	8	1	4	7
0	5	3	9	10	11
20	19	18	15	16	17
14	13	12	21	22	23

13. AE

0	2	6	1	4	7
18	19	20	5	3	8
15	16	17	11	10	9
14	13	12	21	22	23

14-16. B^5EEE (move 1 to the left of 02)

0	2	6	4	7	8
1	5	3	9	10	11
20	19	18	15	16	17
14	13	12	21	22	23

A

1	0	2	4	7	8
5	3	6	9	10	11
20	19	18	15	16	17
14	13	12	21	22	23

Figure YYY: Step-by-step trajectory of sequence X.

We can now define the sequence F that transposes the two adjacent players at the left corner of the back row of half-court A, as shown in Figure 5a. Essentially it is just 3 applications of X. Precisely

$$(F :=) \quad EEEEX \;\; X \;\; XEEEE.$$

So F involves 56 steps of the chain. Figure ZZZ shows the step-by-step trajectory of sequence F.

1. $EEEEX$

1	0	2	4	7	8
5	3	6	9	10	11
20	19	18	15	16	17
14	13	12	21	22	23

2. X

0	1	2	7	3	6
8	4	5	9	10	11
12	13	14	15	16	17
18	19	20	21	22	23

3. $XEEEE$

1	0	2	3	4	5
6	7	8	9	10	11
20	19	18	15	16	17
14	13	12	21	22	23

Figure ZZZ: Step-by-step trajectory of sequence F.

The other transposition sequences. To transpose the two players at the back row of half-court A with one space in between, as shown in Figure 5b, we use the sequence G defined as

$$(G :=) \quad EEEE \;\; A^5FA \;\; F \;\; A^5FA \;\; EEEE.$$

This works because the effect of these sequences is to alter the back row as

$$012 \to 021 \to 201 \to 210.$$

Finally, to transpose the players at the upper left corner and at the lower right corner in half-court A, as shown in Figure 5c, we use the sequence H defined as

$$(H :=) \quad FA^5 \;\; FA^5 \;\; FAFAF.$$

The reader may check that this works, and is one place where the initial $EEEE$ in the definition of F is needed.

4.4 High level description 2

By symmetry, to prove irreducibility it is enough to prove that, from any initial state, one can reach (by some sequence of allowable steps) the reference state shown in Figure 7, where each player i is in position i. Because any permutation of the 6 players in half-court A can be derived from a sequence of transpositions, from the existence of transposition sequences (section 4.3) it suffices to show that we can move, by some sequence of steps, player i to position i, for every player in all the other half-courts B, C, D. We will move players to positions row by row, in the following order

- back row of D: (21, 22, 23)
- back row of C: (18, 19, 20)
- front row of D: (15, 16, 17)
- front row of C: (12, 13, 14)
- back row of B: (3, 4, 5)
- front row of B: (9, 10, 11)

Each row in turn is *fixed*, in that it remains in place after each subsequent row has been moved to its position.

0	1	2		3	4	5
	A				B	
6	7	8		9	10	11
12	13	14		15	16	17
	C				D	
18	19	20		21	22	23

Figure 7: Reference State

In the next section we will describe the algorithm in words. A key point is that we will use half-court A as a kind of temporary stopover for players in transit.

4.5 The algorithm

As noted in Figure 4 a sequence like $BEEE$ has the effect of rotating a given half-court by one position, and one can repeat such a sequence. So we can use phrases such as "*rotate* player i to position j" (in the same half-court). We will call E the *migration* step, so by repeating step E we can use "migrate player i to position j" (where player i has the same relative position (e.g. front, right) as j, in the front row. (But remember that all 12 front-row players migrate.) In both cases we of course only do the move if necessary, that is if the player is not already in the desired position.

The algorithm is based on variations of the following "Procedure P", where P is one of the half-courts B, C, D, and where the procedure acts to move the required 3 players to the back row of P.

Procedure P.

(**1**) Label the back row positions of P as a, b, c, left-to-right.
(So for $P = D$ we have $(a, b, c) = (21, 22, 23)$).

(**2**) Rotate (if in back row) player a to front row. Migrate player a to half-court A and rotate to front right position. Migrate player a to the front right position in P.

(**3**) Rotate player a to back right position in P.

(**4a**) If player b is not in P, repeat actions (**2**) for player b. This moves player b to the front right position in P.

(**4b**) Else: rotate P so that player b is in front row while player a is in back row. Migrate player b to half-court A. Rotate A to move player b to front right. Rotate P so that player a is returned to back right position in P. Migrate player b to the front right position in P.

(**5**) Now players (a, b) are in (back right, front right) positions in P: rotate one space to (back center, back right) positions.

(Next we move player c in essentially the same way as player b, Here are the details.)

(**6a**) If player c is not in P, repeat actions (**2**) for player c. This moves player c to the front right position in P.

(**6b**) Else: rotate P so that player c is in front row while players a, b are in back row. Migrate player c to half-court A. Rotate A to move player c to front right. Rotate P so that players (a, b) are returned to (back center, back right) positions in P. Migrate player c to the front right position in P.

(**7**) Rotate players (a, b, c) to (back left, back center, back right) positions in P.

end Procedure P.

Procedure P moves the required 3 players to the back row of P. In the algorithm below, once a back row is placed, it is "fixed" and those players are never moved subsequently. This is because that would require a rotation of P at some subsequent use of (**2**), which cannot happen because of the *if in back row* condition in (**2**): these players are already in place.

The algorithm.

(**8**) Apply Procedure D, then Procedure C.

(This fixes the back rows of C and D. For the third row (the front row in D) we will use a little trick, to first arrange them on a back row.)

(**9**) Here we consider players (15,16,17) who need to be moved to the front row of D. But we label them as $(a, b, c) = (3, 4, 5)$ and use Procedure B to move them to the back row of B. Then revert to labels $(15, 16, 17)$.

(**10**) Rotate players (15,16,17) three positions so they become the front row of B.

(**11**) Migrate one step, so players (15,16,17) become the front row in D.

(The previous back rows remain fixed. Next we consider the fourth row, the front row of C. Here another small complication arises; any migration will move the front row of D (fixed above), so we need to ensure it is migrated back into place before finishing.)

(**12**) Here we are considering players (12,13,14). As in (**9**), label them as $(a, b, c) = (3, 4, 5)$ and use Procedure B to move them to the back row of B. Then revert to labels (12,13,14). This involves a certain number of migrates, but players (15, 16, 17) remain the front row in some half-court and the fixed back court players (18-23) are not moved.

(**13**) Migrate players (15, 16, 17) to make them the front row in half-court A. Rotate players (12,13,14) in B three positions so they become the front row in B. Migrate two turns.

(Now all players 12-23 in half-courts C and D have been moved to their reference positions.)

(**14**) Apply Procedure B. This moves players (3,4,5) to the back row of B, as required. It involves some number of migrates, but the players (12-17) assigned to front rows of C and D will remain as front rows of adjacent half-courts, so can be migrated back to their required positions.

(Now we have fixed all rows of B, C, D except the front row of B. For the front row of B we do a quite different scheme, using the transposition sequences from section 4.3.)

(**15**) Here we are considering players (9,10,11). Our first goal is to move them to target positions (2,1,0) in the back row of A. Any of those players (9,10,11) in A can be moved to target position via transpositions. Remaining players amongst (9,10,11) must be in the front row of B. Migrate the front row of B to the front row of A, transpose relevant players to target positions, and migrate back.

(**16**) Players (9,10,11) are at positions (2,1,0) in the back row of A. Rotate to front row of A, and migrate to front row of B.

(Now all rows of B, C, D are fixed, as required.)

End Algorithm

This completes the proof of irreducibility.

5 The friend chain

Perhaps the most natural observable to study concerns the positions of two players, say *ego* and *friend*. Naively this would require a 24×23 state chain, but we can exploit some symmetries to make a 26 state chain indicating *relative* positions of the two players. Doing so requires some care; the states are indicated in Figure 8, as explained next.

First note that our process is not invariant under a quarter-turn of the 4 half-courts[6], but is invariant under a half-turn, so as in Figure 8, we can assume *ego* is in the left court.

[6] A friend on the same team in the first half court might be an opponent in the next game; this cannot happen on the second half-court.

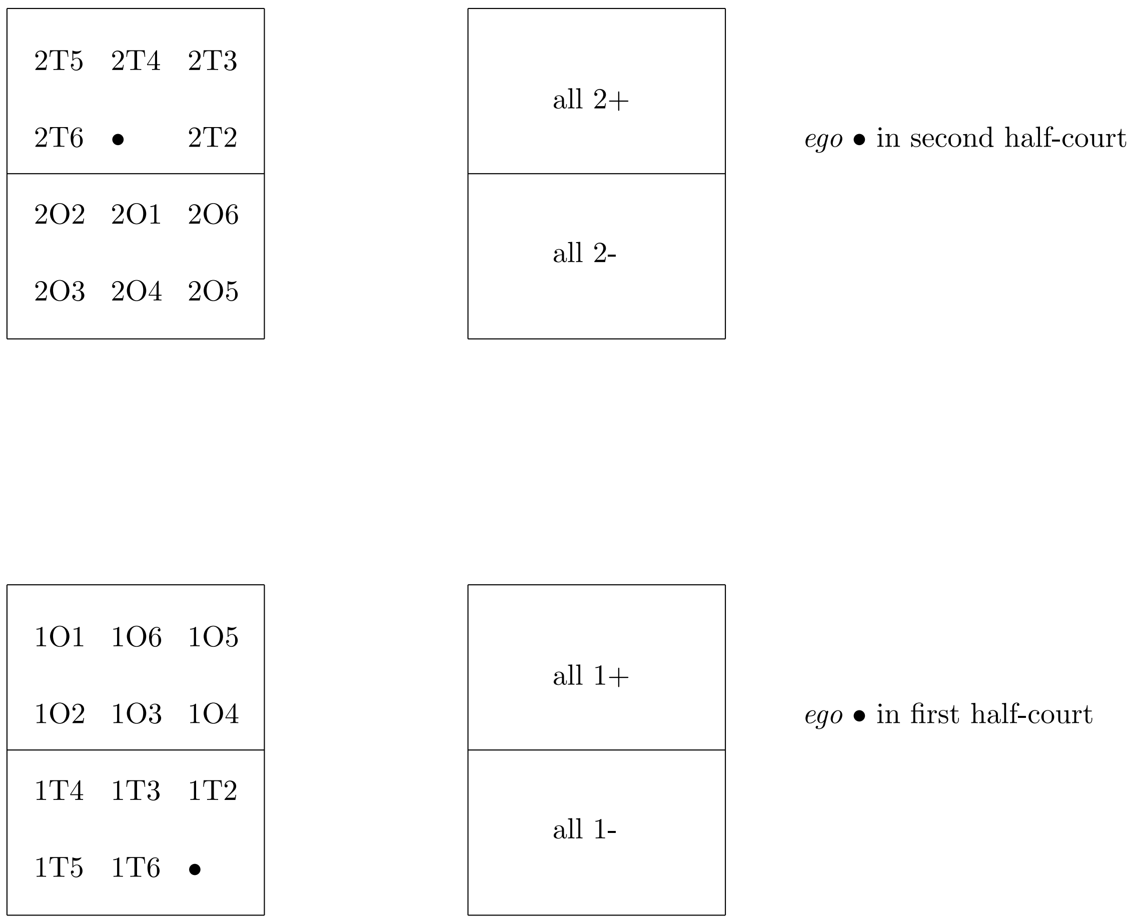

Figure 8: Positions of *friend* relative to *ego* •.

The states of what we will call the *friend chain* indicate relative positions at the start of a game. To describe the state, first record which half-court *ego* is in (denoted as initial 1 or 2) and then whether *friend* is in the same team (denoted T) or the current opposing team (denoted O), or on the other court. If on the other court, we need only note which half-court (because the position will be randomized during the game), so denote by one of (1+, 1-, 2+, 2-) as illustrated. Finally, writing temporarily ego* for the opponent in the same position as ego, we indicate a friend's position as 1, 2, 3, 4, 5, or 6, counter-clockwise from ego or ego*.

This adds up to 26 states, and one can check that the "big" chain rules define a Markov chain on these states, with transition matrix $\mathbf{P}$ as shown in Figures 9 and 10. Its stationary distribution π is induced from the uniform stationary distribution of the "big" chain: probability $\frac{6}{46}$ for each of the states $1+, 1-, 2+, 2-$ and probability $\frac{1}{46}$ for each of the 22 remaining states.

5.1 Numerics for the friend chain

Let us investigate the mixing properties of the friend chain. Standard theory quantifies "closeness to stationarity after n steps" via *variation distance* $d^*(n)$ or *separation distance* $s^*(n)$ from worst-case start, that is via

$$\begin{aligned} d^*(n) &:= \max_i \tfrac{1}{2} \sum_j |p_{ij}^n - \pi_j| \\ s^*(n) &:= \max_{i,j} (1 - p_{ij}^n / \pi_j) \end{aligned}$$

for the n-step transition matrix $\mathbf{P}^n$. Another measure of distance to stationarity is the L^2 or the χ^2 distance. The L^2 distance between P_i and the stationary distribution π after n steps is

$$\|P^n(i, \cdot) - \pi\|_2 = \sqrt{\sum_j \frac{(p_{ij}^n - \pi_j)^2}{\pi_j}}$$

and

$$\|P^n - \pi\|_2 = \max_i \|P^n(i, \cdot) - \pi\|_2.$$

Shown in Table 1 are the numerical values for these distances.

But these are not "observable" quantities. More relevant to players is the mean number of games is which *friend* is on the same team, or on the opposing team, as *ego*. This is a simple calculation involving only matrix powers, and shown in Tables 2 and 3 are the mean number of games in which *friend* is on the opponent team and when *friend* is on the same team.

	$1+$	$1-$	$1T2$	$1T3$	$1T4$	$1T5$	$1T6$	$1O1$	$1O2$	$1O3$	$1O4$	$1O5$	$1O6$
$1+$	1/4	1/4											
$1-$		1/4	1/36	2/36	3/36	2/36	1/36						
$1T2$			1/3						1/6				
$1T3$				1/6						1/3			
$1T4$											1/2		
$1T5$						1/6						1/3	
$1T6$							1/3						1/6
$1O1$	1/12							1/4	1/12				1/12
$1O2$	1/6							3/24	1/6	1/24			
$1O3$	1/3								1/12	1/12			
$1O4$	5/12									1/24		1/24	
$1O5$	1/3											1/12	1/12
$1O6$	1/6							3/24				1/24	1/6
$2T2$		1/6	1/3										
$2T3$		1/3		1/6									
$2T4$		1/2											
$2T5$		1/3				1/6							
$2T6$		1/6					1/3						
$2O1$	1/12	5/12											
$2O2$	1/6	1/3											
$2O3$	1/3	1/6											
$2O4$	5/12	1/12											
$2O5$	1/3	1/6											
$2O6$	1/6	1/3											
$2+$			1/36	2/36	3/36	2/36	1/36	3/36	2/36	1/36		1/36	2/36
$2-$	1/4								1/36	2/36	3/36	2/36	1/36

Figure 9: Transition matrix of the friend chain (first part).

	$2T2$	$2T3$	$2T4$	$2T5$	$2T6$	$2O1$	$2O2$	$2O3$	$2O4$	$2O5$	$2O6$	$2+$	$2-$
$1+$												$1/4$	$1/4$
$1-$						$3/36$	$2/36$	$1/36$		$1/36$	$2/36$		$1/4$
$1T2$	$1/3$						$1/6$						
$1T3$		$1/6$						$1/3$					
$1T4$									$1/2$				
$1T5$				$1/6$						$1/3$			
$1T6$					$1/3$						$1/6$		
$1O1$	$1/24$				$1/24$							$10/24$	
$1O2$	$1/12$	$1/12$										$1/3$	
$1O3$	$1/24$	$1/6$	$1/8$									$1/6$	
$1O4$		$1/12$	$3/12$	$1/12$								$1/12$	
$1O5$			$3/24$	$1/6$	$1/24$							$1/6$	
$1O6$				$1/12$	$1/12$							$1/3$	
$2T2$	$1/3$											$1/6$	
$2T3$		$1/6$										$1/3$	
$2T4$												$1/2$	
$2T5$				$1/6$								$1/3$	
$2T6$					$1/3$							$1/6$	
$2O1$	$1/24$				$1/24$	$1/4$	$1/12$				$1/12$		
$2O2$	$1/12$	$1/12$				$3/24$	$1/6$	$1/24$					
$2O3$	$1/24$	$1/6$	$3/24$				$1/12$	$1/12$					
$2O4$		$1/12$	$3/12$	$1/12$				$1/24$		$1/24$			
$2O5$			$3/24$	$1/6$	$1/24$					$1/12$	$1/12$		
$2O6$				$1/12$	$1/12$	$3/24$				$1/24$	$1/6$		
$2+$												$1/4$	$1/4$
$2-$							$1/36$	$2/36$	$3/36$	$2/36$	$1/36$		$1/4$

Figure 10: Transition matrix of the friend chain (second part).

n	1	2	3	4	5	6	7	8	9
$d^*(n)$	0.957	0.638	0.375	0.263	0.180	0.122	0.083	0.058	0.040
$s^*(n)$	1	1	1	0.933	0.508	0.374	0.297	0.223	0.160
$L^2(n)$	4.690	2.254	1.544	1.05	0.71	0.492	0.339	0.233	0.159

Table 1: Measures of distance to stationarity for the friend chain, after n games.

Start	1+	1-	1T2	1T3	1T4	1T5	1T6	1O1	1O2	1O3	1O4	1O5	1O6
OT	1.607	1.962	1.803	2.107	2.222	2.107	1.803	3.059	2.894	2.606	2.482	2.606	2.894
ST	1.093	1.515	3.773	2.725	2.314	2.725	3.773	1.421	1.499	1.550	1.523	1.550	1.499

Table 2: Mean number of games (out of 8) in which *friend* is on the opposite team OT (and the same team ST) as *ego*, who starts in the first half-court.

Start	2T2	2T3	2T4	2T5	2T6	2O1	2O2	2O3	2O4	2O5	2O6	2+	2-
OT	1.493	1.678	1.700	1.678	1.493	3.059	2.894	2.606	2.482	2.606	2.894	1.962	1.940
ST	3.778	2.698	2.297	2.698	3.778	1.421	1.499	1.550	1.523	1.550	1.499	1.515	1.103

Table 3: Mean number of games (out of 8) in which *friend* is on the opposite team OT (and the same team ST) as *ego*, who starts in the second half-court.

Notice that there is a symmetry property visible in Tables 2 and 3, i.e. the values under $1T2$ to $1T6$, $1O2$ to $1O6$, $2T2$ to $2T6$, and $2O2$ to $2O6$ are exactly invariant under reversal. Moreover, the values under $1-$ and $2+$ are the same. This shows that if *friend* is playing with or against *ego*, the mean number of games in which *friend* is on the opposite team (or the same team) can be computed depending on the distance from *friend's* initial position to *ego* or *temporary ego's* initial position. In other words, we can reduce the number of states, and the states will depend on *friend's* shortest distance (counterclockwise or clockwise) from *ego*.

A related question is the chance that you *never* play as an opponent (or as teammate) to your friend. Shown in Tables 4 and 5 are the numerical values for "opponent" and "teammate", respectively, omitting the cases where this is zero (initial opponent, or $1T4$).

Comparison with random teams. There are several ways one could compare the effect of the "mixing" scheme we study with the alternate scheme of randomly assigning players to team for every game. For instance, under random mixing, if *friend* starts somewhere which is not the opposite team as *ego*, the probability that *friend* will never be on the opposite team

Start	1+	1-	1T2	1T3	1T5	1T6
Probability	0.098	0.057	0.141	0.026	0.026	0.141

Start	2T2	2T3	2T4	2T5	2T6	2+	2-
Probability	0.168	0.081	0.082	0.081	0.168	0.057	0.057

Table 4: Probability that *friend* is never an opponent of *ego* over 8 games.

Start	1+	1-	1O1	1O2	1O3	1O4	1O5	1O6
Probability	0.403	0.292	0.344	0.317	0.271	0.251	0.271	0.317
Start	2O1	2O2	2O3	2O4	2O5	2O6	2+	2-
Probability	0.344	0.317	0.271	0.251	0.271	0.317	0.292	0.393

Table 5: Probability that *friend* is never a teammate of *ego* over 8 games.

as *ego* over eight games is (by calculation) 0.121. We see from Table 4 that this is less than the values under our scheme, if the starting position is only one position away from *ego* or the corresponding position of *ego* on the other half court. In the remaining cases, it is much greater. Similarly, if *friend* starts somewhere which is not the same team as *ego*, then (under random mixing) the probability that *friend* will never be on the same team as *ego* for eight games is 0.180. From Table 5 this is always less than the values under our scheme.

5.2 Monte Carlo Simulations

More complicated "observables" can most easily be addressed via Monte Carlo simulation of the process. For instance

> What is the probability that *ego* will encounter (as either teammate or opponent *all* of the other 23 players during an 8-game sequence?

By Monte Carlo, the probability ≈ 0.595 if *ego* starts at the 1^{st} half court, or ≈ 0.675 if *ego* starts in the 2^{nd} half court. (Intuitively, these figures differ because in the former case one has more overlap between opponents in the first and second games.) Over 10 games, these probabilities increase to 0.814 and 0.857.

6 Mixing time for the big chain

At a research level there has been extensive study of mixing times for many different card-shuffling models, usually in the asymptotic setting (as the size n of card deck $\to \infty$). Our model is rather specific to the $n = 24$ case, so we have not sought to embed it into some family allowing large n.

One can get lower bounds on mixing time by considering specific functions of the chain, and the variation distances in Table 1 for the friend chain are a lower bound for the distances in the big chain. A first step [1] in studying some "local moves" shuffles such as random adjacent transpositions was to obtain lower bounds by studying motion of initially adjacent cards. In our model the "friends" maximal-start variation distance in Table 1 is indeed[7] from the case of initial adjacent players.

Recall that the progress of ego around the 4 half-courts is just the lazy cyclic walk, for which variation distance to stationarity in recorded in Table 6.

n	1	2	3	4	5	6	7	8	9
$d(n)$	0.5	0.25	0.125	0.125	0.0625	0.0625	0.083	0.0313	0.0313

Table 6: Variation distance for the lazy cyclic walk.

These values are slightly less than the values in Table 1 for the friend chain – both are *a priori* lower bounds for variation distance for the big chain.

To find a better lower bound for the mixing time for the "big" chain, we can combine the two aforementioned ideas and form a 52- state ("big friend") chain, where *ego* can be in any of the 4 half-courts. In this new chain, we label the positions with respect to *ego* similar to the friend chain (see Figures 8 and 11). We first record which half-court *ego* is in (denoted as 1, 2, 3 or 4) and then whether friend is in the same team (denoted T) or the current opposing team (denoted O), or on the other court. If on the other court, we only note which half-court, so denote by one of + or -. Finally, writing temporarily ego* for the opponent in the same position as ego, we indicate a friend's position as 1, 2, 3, 4, 5, or 6, counter-clockwise from ego or ego*. Its stationary distribution π is induced from the uniform stationary distribution of the "big friend" chain: probability $\frac{6}{92}$ for each of the states $1+, 1-, 2+$, $2-$, $3+$, $3-$, $4+$, and $4-$, and probability $\frac{1}{92}$ for each of the 44 remaining states.

[7]except for opening games

The transition matrix for this “big friend” chain is easy to obtain from the transition matrix of the “friend” chain. Consider the transition matrix of the “friend” chain in Figures 9 and 10. We can view it as a matrix composed of smaller matrices as in Table 7, where T_{11}, T_{12}, T_{21}, and T_{22} are 13×13 matrices. The states used in the transition matrix for the “big friend” chain are arranged as follows: 1+, 1−, $1T2$ to $1T6$, $1O1$ to $1O6$, $2T2$ to $2T6$, $2O1$ to $2O6$, 2+, 2−, 3+, 3−, $3T2$ to $3T6$, $3O1$ to $3O6$, $4T2$ to $4T6$, $4O1$ to $4O6$, 4+, and 4−, and we can view the transition matrix for the “big friend” chain as Table 8.

	1	2
1	T_{11}	T_{12}
2	T_{21}	T_{22}

Table 7: Transition matrix for the friend chain

	1	2	3	4
1	T_{11}	T_{12}	0	0
2	0	T_{22}	T_{21}	0
3	0	0	T_{11}	T_{12}
4	T_{21}	0	0	T_{22}

Table 8: Transition matrix for the big friend chain

We can now investigate the mixing properties of the big friend chain. Shown in Table 9 are the distances from stationarity for the big chain.

n	1	2	3	4	5	6	7	8	9
$d^*(n)$	0.978	0.713	0.520	0.340	0.242	0.168	0.125	0.085	0.058
$s^*(n)$	1	1	1	1	0.827	0.681	0.461	0.391	0.272
$L^2(n)$	6.708	2.977	1.868	1.228	0.827	0.563	0.387	0.266	0.183

Table 9: Measures of distance to stationarity for the “big friend” chain, after n games

Notice that the values we got for $d^*(n)$ in the “big friend” chain are larger than that of the “friend” chain, improving the lower bound of the “big friend” chain.

7 Final remarks

We have mentioned analogies with card shuffling several times, because "rotations" of team players correspond to a cut-shuffle of a 6-card deck. Our model is equivalent to a certain (not very easily implemented physically) random shuffle of a 24-card deck via first breaking into 4 sub-decks. Persi Diaconis (personal communication) remarks that casinos and some fantasy games involve shuffling decks much larger than the usual 52-card deck, and this is often done via some scheme involving breaking into sub-decks, shuffling each in some way, and recombining in some way. Such schemes (thereby loosely analogous to our model) have generally not been studied in mathematical probability, an exception being the "casino shelf shuffling machines" studied by Diaconis-Fulman-Holmes [4].

We have interpreted the underlying question

> how effective is this scheme at mixing up the teams?

in terms of mixing times, that is implicitly by comparison with the alternative of randomly assigning players to teams for each game. An opposite alternative would be some analog of "design of statistical experiments" schemes, deterministically assigning players to teams for all games in such a way that relative positions of two players were as uniformly spread as possible. At a practical level, our scheme is much easier and faster to implement than either alternative. Moreover its implementation is *robust* to small variation in number of players, which is common in informal settings: an extra player will rotate off court, or a 5-player team always has 3 players deemed front row.

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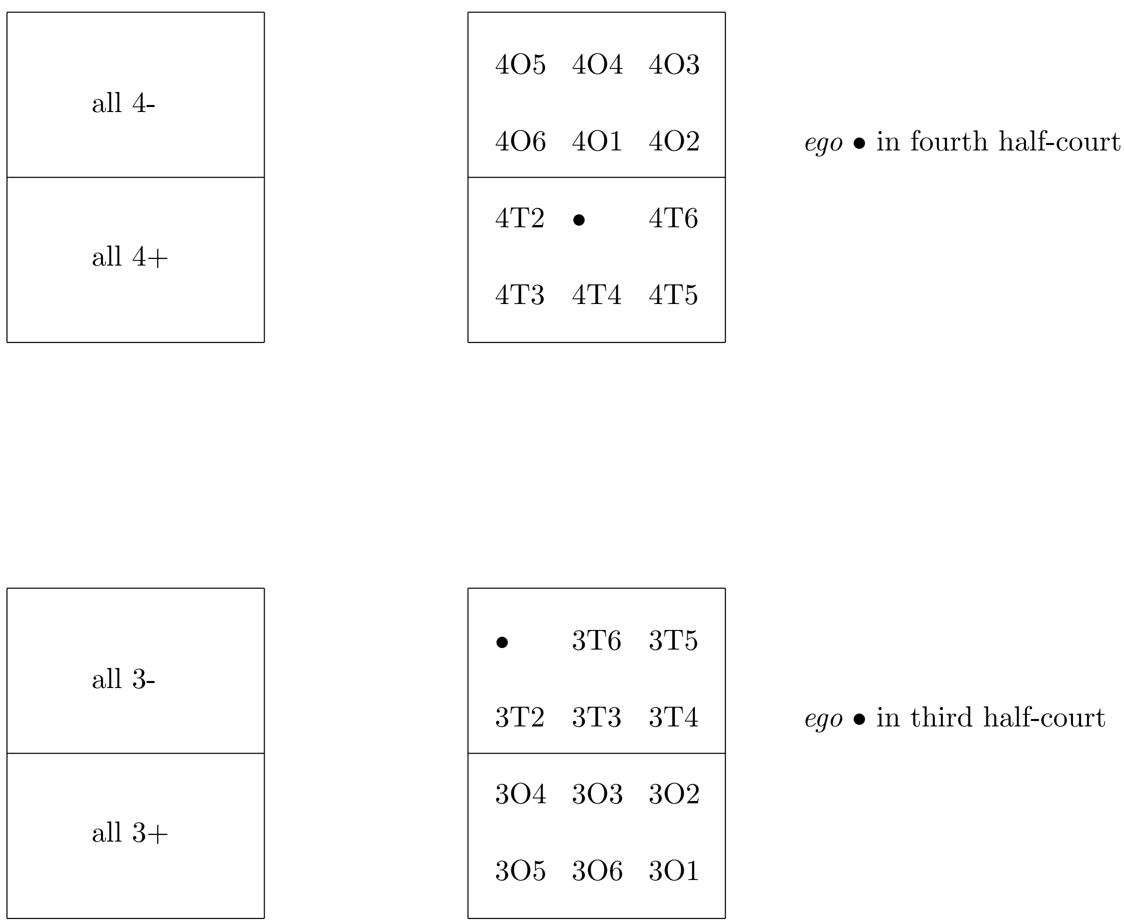

Figure 11: Positions of *friend* relative to *ego* •.