Stat206: Random Graphs and Complex Networks

Lecture 18: Emergence of Giant Component

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18.1 Theorem 5.4 of [1]

Theorem 1 *Consider an Erdös-Renyi random graph $\mathcal{G}(n,p)$. Set $p = c/n$, fixed $c > 0$.*

Study $n \to \infty$ limits.

Assertions hold with prob$\to 1$ as $n \to \infty$.

(a) If $c < 1$ the largest component size is at most $\frac{3}{(1-c)^2}\log n$.

(b) If $c > 1$ there is a unique giant component with $(1 + o(1))\beta n$ vertices, where $\beta = \beta(c) > 0$ solves $\beta + e^{-\beta c} = 1$. The second largest component size is at most $\frac{16c}{(c-1)^2}\log n$.

We will use 3 ideas:

- Comparison with GWBP, Poisson(c) offspring.
- Exploring a component by "expanding" vertices.
- Chernoff (tail) bounds: quote
  If Y has Binomial(m, q) distribution:

$$\begin{aligned} P(Y \geq mq + t) &\leq \exp\left(-\frac{t^2}{2(mq + t/3)}\right), \ t \geq 0 \\ P(Y \leq mq - t) &\leq \exp\left(-\frac{t^2}{2mq}\right), \ t \geq 0 \end{aligned}$$

In a GWBP, offspring ξ with probability generating function $\Phi(\cdot)$, the extinction probability $\rho = \rho(\xi)$ solves $\rho = \Phi(\rho)$. For $\xi =_d$ Poissson(c), $c > 1$, we get $\rho = 1 - \beta$ for β in (b).

One can check that if $n' \sim n$ then

$$\rho[\text{Binomial}(n', c/n)] \to \rho[\text{Poisson}(c)] = 1 - \beta$$

"Expand" a vertex by writing its new neighbors (see Figure 1)

$$X_i = \#\text{ of new vertices which are new neighbors of } v_i =_d \text{Bin}(n - \#\text{ of vertices written already}, c/n)$$

Proof: **Case (a)**: Fix vertex v.

$$X_i \leq_{\text{Stoch}} \text{Bin}\left(n, \frac{c}{n}\right)$$

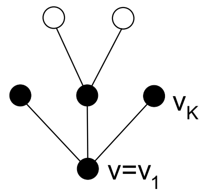

Figure 1: Exploring a component. Unshaded vertices haven't been explored yet.

$$\begin{aligned} P(\text{component } \ni v \text{ has size} \geq K) \quad &\leq P\left(1+\sum_{i=1}^{K} X_i \geq K\right) \leq P\left(\sum_{i=1}^{K} \mathrm{Bin}_i(n,c/n) \geq K-1\right) \\ &= P\left(\mathrm{Bin}(Kn,c/n) \geq K-1\right) \leq_{\text{Chernoff}} \exp\left(-\frac{[(1-c)K-1]^2}{2[cK+((1-c)K-1)/3]}\right) \\ &\leq_{algebra} \exp\left(-\frac{(1-c)^2K}{2}\right) \end{aligned}$$

where $mq = ck$ and $t = K-1-cK = (1-c)K-1$.

Then we can write:

$$P\left(\exists \text{ component of size } \geq K\right) \leq nP(\text{component } \ni v \text{ has size} \geq K) \leq n\exp\left(-\frac{(1-c)^2K}{2}\right)$$

and we can choose $K = K_n$ such that this upper bound goes to zero as $n \to \infty$.

Case (b): Idea: show that each component size is either: $\leq K_- \stackrel{\text{def}}{=} \frac{16c}{(c-1)^2\log n}$ or $\geq K_+ \stackrel{\text{def}}{=} n^{2/3}$.

Fix vertex v, fix $K \in [K_-, K_+]$, seek upper bound on chance of event:

$$E_K = \{v \text{ in component of size } \geq K, \text{and } \# \text{ of unexplored vertices } v_K \text{ is less than } (c-1)K/2\}$$

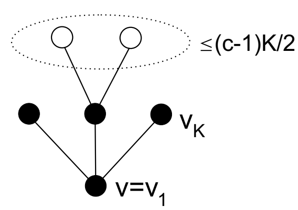

Figure 2: On the event E_K the number of unexplored vertices is $\leq (c-1)K/2$.

On this event the total # of vertices written $\leq K + (c-1)K/2 = (c+1)K/2$.

May suppose $X_i \geq_{Stoch} \mathrm{Bin}\,(n-(c+1)K_+/2, p)$. (If we see this many $(n-(c+1)K_+/2)$ extra children the event in question cannot occur). Then,

$$\begin{aligned} P(E_K) \quad &\leq \quad P\left(\sum_{i=1}^{K} X_i \leq K + \frac{(c-1)K}{2}\right) \leq P\left(\sum_{i=1}^{K} \mathrm{Bin}(n-(c+1)K_+/2,p) \leq K + \frac{(c-1)K}{2}\right) \\ &= \quad P\left(\mathrm{Bin}(K(n-(c+1)K_+/2),p) \leq K + \frac{(c-1)K}{2}\right) \end{aligned}$$

We can define: "Bad" event $\stackrel{\text{def}}{=}$ event E_K occurs for some v, some $K \in [K_-, K_+]$. Then, using Chernoff bound and algebra we have,

$$P(\text{Bad}) \le n \sum_{K=K_-}^{K_+} P(E_K) \le n \sum_{K=K_-}^{K_+} \exp\left[-\frac{(c-1)^2 K}{9c}\right] \le nK_+ \exp\left[-\frac{(c-1)^2 K_-}{9c}\right]$$

and we can choose $K_- = \frac{16c}{(c-1)^2} \log n$ to make this bound $\to 0$.

We now work on "Good" event, that is the complement of "Bad", in which case all components have size $\le K_-$ or $\ge K_+$. When exploring a component of size $\ge K_+$, at step K_+ there are at least $(c-1)K_+/2$ unexplored vertices.

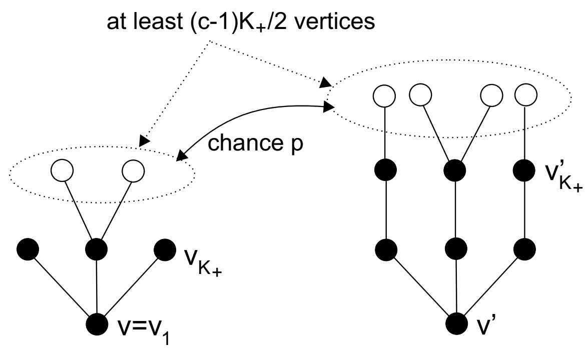

Figure 3: Exploring 2 disjoint components.

Fix v and v'. Explore components for K_+ steps. Suppose all vertices seen so far in the 2 components are disjoint. Then,

$$\begin{aligned}\text{Chance } v, v' \text{ in diff. components of } \mathcal{G}(n,p) &\le \text{Chance no edges between unexplored vertices of the 2 components} \\ &\le (1-p)^{[(c-1)K_+/2]^2} \le \exp\left(-\frac{p(c-1)^2 K_+^2}{4}\right) = \exp\left[\frac{(c-1)^2}{4} cn^{1/3}\right]\end{aligned}$$

This is a bound on $P(v, v'$ in different components of size $\ge K_+)$. So,

$$P(\exists \text{ different components of sizes } \ge K_+) \le n^2 \exp\left[\frac{(c-1)^2}{4} cn^{1/3}\right] \to 0$$

Therefore there exists at most 1 component of size $\ge K_+$.

We will now focus on determining the size of the giant component. Define

$$Y \stackrel{\text{def}}{=} \# \text{ of vertices in (small) components of size } \le K_-$$

Then the size of the giant component is $n - Y$. Write $\rho(n,p) =$ chance v in small component. Then,

$$\rho \le P(\text{GWBP, Bin}(n - K_-, p), \text{ goes extinct})$$

Note that $X_i \ge_{\text{Stoch}} \text{Bin}(n - K_-, p)$. Also,

$$\rho \ge P(\text{GWBP}, \text{Bin}(n,p), \text{ goes extinct with size } \le K_-) = P(\text{GWBP}, \text{Bin}(n,p), \text{ goes extinct}) - o(1)$$

because as $n \to \infty$, $\text{Bin}(n,p) \to \text{Poisson}(c)$ and $K_- \to \infty$. This implies $\rho(n,p) \to 1 - \beta$. Then

$$E(Y) = n\rho(n,p) \sim n(1-\beta)$$

We want to prove $\frac{Y}{E(Y)} \to 1$ in probability. This would imply part (b).

By Chebyshev's inequality it is enough to show that

$$E(Y^2) \leq (1+o(1))[E(Y)]^2 = (1+o(1))n^2(1-\beta)^2$$

For this we write:

$$\begin{aligned} E(Y^2) &= \sum_v \sum_{v'} P(v, v' \text{ in small components}) \\ &\leq (\text{same component}) n\rho K_- + (\text{diff. component}) n\rho n P(v' \text{ in diff. small component } | v \text{ in small component}) \end{aligned}$$

This last probability equals $\rho(n - O(K_-), p) \to (1-\beta)$ by continuity argument. Also $\rho \to 1-\beta$. So,

$$E(Y^2) \leq (1+o(1))n^2(1-\beta)^2$$

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References

[1] Svante Janson, Tomasz Luczak, Andrzej Rucinski, *Random Graphs*, Wiley, 2000.