5: Some analogs of epidemics, and some research suggestions

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There is a large literature modeling spread of opinions, adoption of innovations etc via social networks, and some of the models fit our FMIE setting. Instead of describing some standard model let me describe two new FMIE models suggested by the “social networks” context. Mathematically they are variants of Pandemic, and we will see in this lecture that on the complete graph their behavior can be derived from the precise understanding of Pandemic given by the “randomly shifted logistic” result.

Studying them in other geometries is an **Open Topic**, which opens up other questions to study about the short-term behavior of Pandemic.

Model: Deference

(i) The agents are labelled 1 through n. Agent i initially has opinion i.
(ii) When two agents meet, they adopt the same opinion, the smaller of the two opinion-labels.

Clearly opinion 1 spreads as Pandemic, so the "ultimate" behavior of Deference is not a new question.

As we shall see, easy to give analysis in complete graph model, as a consequence of the "randomly-shifted logistic" result for Pandemic. Study $(X_1^n(t), \ldots, X_k^n(t))$, where $X_k^n(t)$ is the proportion of the population with opinion k at time t.

Key insight: opinions 1 and 2 and ... and k combined behave as one infection in Pandemic, hence as a random time-shift of the logistic curve F.

We can rewrite "randomly-shifted logistic" result for Pandemic as

$$(X_1^n(\log n + s),\ -\infty < s < \infty) \xrightarrow{d} (F(C_1 + s),\ -\infty < s < \infty) \quad (1)$$

where F is the logistic function and $C_1 = \log(\xi_1)$, where $\xi_1 \stackrel{d}{=}$ Exponential(1) arises as the limit r.v. $e^{-t}N(t) \to \xi_1$ associated with the Yule process.

When we combine opinions 1 and 2 and ... and k and consider the sum $\sum_{i=1}^k X_i^{(n)}(\cdot)$, it has a representation of the same format (1) but with a different random shift C_k. Because the initial phase is just a collection of k independent Yule processes, we see that

$$C_k = \log(\xi_1 + \ldots + \xi_k),\ k \geq 1. \quad (2)$$

Here the (ξ_i) are i.i.d. Exponential(1).

So we get $n \to \infty$ limit behavior

$$((X_1^n(\log n + s), X_2^n(\log n + s), \ldots, X_k^n(\log n + s)),\ -\infty < s < \infty) \to \quad (3)$$

$$((F(C_1 + s), F(C_2 + s) - F(C_1 + s), , \ldots, F(C_k + s) - F(C_{k-1} + s)),\ -\infty < s < \infty)$$

where

$$C_j = \log(\xi_1 + \ldots + \xi_j),\ j \geq 1. \quad (4)$$

[see my amateur picture]

The Deference model envisages agents as "slaves to authority". Here is a conceptually opposite "slaves to fashion" model, whose analysis is mathematically surprisingly similar.

Model: Fashionista.

Take a general meeting model. At the times of a rate-λ Poisson process, a new fashion originates with a uniform random agent, and is time-stamped. When two agents meet, they each adopt the latest (most recent time-stamp) fashion.

There is an equilibrium distribution, for the random partition of agents into "same fashion".

For the complete graph geometry, we can copy the analysis of Deference. Combining all the fashions appearing after a given time, these behave (essentially) as one infection in Pandemic (over the pandemic window), hence as a random time-shift of the logistic curve F. So when we study the vector $(X_k^n(t), -\infty < k < \infty)$ of proportions of agents adopting different fashions k, we expect $n \to \infty$ limit behavior of the form

$$(X_k^n(\log n + s),\ -\infty < k < \infty) \to \tag{5}$$
$$(F(C_k + s) - F(C_{k-1} + s),\ -\infty < k < \infty)$$

where $(C_k,\ -\infty < k < \infty)$ are the points of some stationary process on $(-\infty, \infty)$.

Knowing this form for the $n \to \infty$ asymptotics, we can again determine the distribution of (C_i) by considering the initial stage of spread of a new fashion. It turns out that

$$C_i = \log\left(\sum_{j \le i} \exp(\gamma_j)\right) = \gamma_i + \log\left(\sum_{k \ge 1} \exp(\gamma_{i-k} - \gamma_i)\right) \tag{6}$$

where γ_j are the times of a rate-λ Poisson process on $(-\infty, \infty)$. The second expression makes it clear that (C_i) is a stationary process.

[see another amateur picture]

Here is the outline argument for (6). Consider the recent fashions at time $t = 0$ adopted by small but non-negligible proportions of the population. More precisely, consider fashions originating during the time interval $[-\log n + t_n, -\log n + 2t_n]$, where $t_n \to \infty$ slowly. For a fashion originating at time $-\log n + \eta$, the time-0 set of adopting agents will be a subset of the corresponding epidemic process, which we know has proportional size $\xi \exp(-\eta) = \exp(-\eta + \log \xi)$ where ξ has Exponential(1) distribution.

The times $-\log n + \eta_j$ of origination of different fashions form by assumption a rate-λ Poisson process, and after we impose IID shifts $\log \xi_j$ we note (as an elementary property of Poisson processes) that the shifted points $-\log n + \eta_j + \log \xi_j$ still form a rate-λ Poisson process, say γ_j, on $(-\infty, \infty)$. So the sizes of small recent fashion groups (that is letting $j \to -\infty$), for which overlap between fashions becomes negligible, are approximately $\exp(\gamma_j)$. Summing over $j \le i$ gives

$$\sum_{j \le i} \exp(\gamma_j) \approx F(C_i) \approx \exp(C_i)$$

and we end up with the representation (6).

Diversity statistic

Consider our "sum of squares of proportions" diversity statistic for the stationary distribution

$$s = s(N, \lambda) = \mathbb{P}(\text{two random agents have same fashion}).$$

The analysis above gives

$$s(N, \lambda) \to s(\infty, \lambda) = \mathbb{E} \sum_k \left(F(\lambda^{-1} C_k) - F(\lambda^{-1} C_{k-1}) \right)^2 .$$

We have (complicated explicit) expression for the right side. In particular we can see the $\lambda \to \infty$ behavior is

$$s(\infty, \lambda) \sim c\lambda^{-1}$$

for some (less complicated explicit) constant c.

Fashionista on the torus $\mathbb{Z}_m^2$

The "interesting" ($=$ analyzable) case of fashion-origination rate λ is

$$m^{-1} \ll \lambda \ll m^2$$

in which we can approximate by deterministic growth in $\mathbb{R}^2$ of the "ball" from the shape theorem. This is a variation on a common "stochastic geometry" style of model – the dead leaves model or [picture].

Again we study the "sum of squares of proportions" diversity statistic for the stationary distribution

$$s = s(m, \lambda) = \mathbb{P}(\text{two random agents have same fashion}).$$

If you are a physicist you can see without further calculation that

$$s(m, \lambda) \sim c\lambda^{-2/3} m^{-2/3}.$$

For mathematicians I will outline the analysis, ignoring constants. So assume ball is disc. Here is the relevant continuum model.

Space-time Poisson point process (rate μ) of "centers" of discs whose radius then grows deterministically at rate 1. Partition $\mathbb{R}^2$ into regions defined by
"$z \in \mathbb{R}^2$" belongs to the smallest disc containing z.

Let $s^*(\mu)$ be expected area of region containing origin then

$$s(m, \lambda) \sim m^{-2} s^*(\lambda / m^2).$$

But we can determine $s^*(\cdot)$ by scaling. Scaling space $(x, y) \to (ax, ay)$ transforms

$$s^*(\mu) \to a^2 s^*(\mu); \quad \mu \to a^{-3} \mu$$

which gives $s^*(\mu) = c\mu^{-2/3}$ and then

$$s(m, \lambda) \sim c\lambda^{-2/3} m^{-2/3}.$$

A few variants of the Voter model

Many such variants have been studied (and I'm not familiar with literature) but typically not with our "general geometry" focus.

A bathtub question – a half-specified model.

Each agent has one of two opinions (R or D) and an amount x of money.

When two D''s meet they follow Averaging process.
When two R's meet they follow Compulsive Gambler.
When R meets D then ?????????? (you specify: total money conserved)

If R/D follows Voter model then we can immediately see what eventually happens; so we want to include some bias in the interaction.

Sticky Voter model.

Modify the 2-opinion Voter model by:

you only change opinion when you meet two people in succession with the opposite opinion.

This is a 4-state FMIE process. Almost same as the (non-FMIE) process defined by

Change opinion at rate q^2 where $q :=$ proportion of neighbors with different opinion from yours.

Let's study over r-regular graphs; Start with IID Ber(1/2). The "mean-field" approximation for $p(t) :=$ proportion of opinion 1 is

$$p'(t) \approx 2p - 1$$

so that $p = 1/2$ is unstable.

Conjecture. ["Diffusive clustering"] There is a function $\rho_r^*(t) \to 0$ as $t \to \infty$, not depending on n or the detailed geometry, such that

$$\mathbb{P}(\text{ random n'bors have different opinions at } t) \leq \rho_r(t).$$

The iPod model. [embargoed]

S set of all songs. Each agent at each time has a probability distribution $(q(s), s \in \mathbf{S})$ indicating preferences (constantly listen in random order). When two agents meet they each choose one s and play it for other agent; we suppose this increases the other's liking for s. So let's invent the rule: Agent hearing song s' updates

$$q(s) \to (1-\eta)q(s) + \eta 1_{(s=s')}.$$

Mathematically simple because we can fix s and study the FMIE process on states $[0,1]$,

$$X_i(t) = \text{ agent } i\text{'s liking for } s \text{ at time } t$$

and the update rule is

$$(x_1, x_2) \to (x_1(1-\eta) + \eta \,\mathrm{Ber}(x_2), x_2(1-\eta) + \eta \,\mathrm{Ber}(x_1)).$$

This resembles the Voter model in that the sum $\sum_i X_i(t)$ is a martingale, and so converges to 0 or n as $t \to \infty$.

- How long?
- Study variant with new items and stationary distribution (cf. infinite alleles).

Starting point for analysis. Study

$$S(t) = \left(\sum_i X_i(t) \right)^2 .$$

Assume standardized rates $\sum_j \nu_{ij} = 1 \ \forall i$.
When i and j meet,

$$\mathbb{E}[\Delta S] = \eta^2 (x_1(1 - x_1) + x_2(1 - x_2)).$$

This has rate ν_{ij}, so summing over $\{i, j\}$

$$E(dS(t) | \mathbf{X}(t) = \mathbf{x}) = \eta^2 \ \sum_i x_i(1 - x_i) \ \ dt.$$

The geometry seems to have magically disappeared!

(but is hiding under invisibility cloak).

People sometime ask me where ideas for new math questions come from. Answer: everywhere. For instance

I can remember Bertrand Russell telling me of a horrible dream. He was in the top floor of the University Library, about A.D. 2100. A library assistant was going round the shelves carrying an enormous bucket, taking down books, glancing at them, restoring them to the shelves or dumping them into the bucket. At last he came to three large volumes which Russell could recognize as the last surviving copy of *Principia Mathematica*. He took down one of the volumes, turned over a few pages, seemed puzzled for a moment by the curious symbolism, closed the volume, balanced it in his hand and hesitated . . . .
(G. H. Hardy, *A Mathematician's Apology*)

Goal: A distributed algorithm which maintains a small number of copies of "information" (a book) in an unreliable network over times much longer than lifetimes of individual vertices. The algorithm doesn't know the current number of copies.

In our FMIE setting, with (large) n vertices, set μ ($= 10$, say) for desired average number of copies, then set $p = \mu/n$. We will define a process of copies such that, in the "reliable network" setting,

$$\text{the equilibrium distribution is independent Bernoulli}(p) \\ \text{conditioned on non-empty.} \tag{7}$$

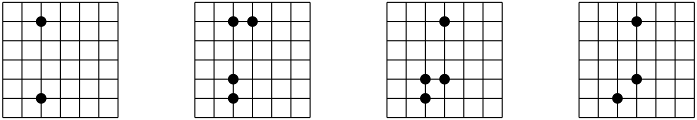
In fact such a process is known in statistical physics:

Kinetically constrained Ising model.
Use the directed meeting model. At a directed meeting $(i \to j)$,

- if i has a copy then j "resets" to have a copy with chance p and no copy with chance $1 - p$;
- if i has no copy then j does not change state.

Note (7) holds by checking the general criterion for a reversible equilibrium (and in particular, does not depend on the geometry).

What happens on a low-degree graph? View a copy as a "particle".



First-order effect: Isolated particles do RW at rate $p/2$.
Second-order effect: A particle splits into two non-adjacent particles at rate $O(p^2)$. Two particles becoming adjacent have chance $O(1)$ to merge.

Math Insight: Could directly define a process of particles doing RW, splitting, coalescing – but wouldn't know its equilibrium distribution.

This model (studied from different viewpoints on infinite lattice $\mathbb{Z}^d$ in statistical physics) has these qualitative properties and a simple equilibrium distribution. But no rigorous work on finite graphs (mixing times, etc).

Heuristically, copies should survive in unreliable network provided $p^2 \gg$ failure rate of node.

Many of the FMIE models we have seen could be interpretated as "ways to attain consensus". Here's another.

Naming game model.
Many papers by A. Baronchelli et al on arXiv; here is quote from one abstract.

...the system builds up non-trivial scale-invariant correlations, for instance in the distribution of competing synonyms, which display a Zipf-like law. These correlations make the system ready for the transition towards shared conventions, which, observed on the time-scale of collective behaviors, becomes sharper and sharper with system size. This surprising result not only explains why human language can scale up to very large populations but also ......

Naming game model.

Each agent starts with a different word. (Later they will have a collection of words).

When i meets j as $i \to j$, i speaks a random word from his collection.

If j doesn't have that word, she adds it to her collection. If she does, then both i and j decide (temporarily) to adopt this work, and so delete all other words.

I just mention this an illustration of one style of models in literature. Note that (like birthday problem) on complete graph, agents will initially build a collection of order $n^{1/2}$ words.

Show FRoA page ????????

The Contact process (SIS epidemic) on a graph, in our set-up, is Pandemic with rates $\nu_{ij} = \lambda$ for each edge (i,j), where now infected agents become cured at rate 1, but are then available to be re-infected *ad infinitum*. There is a short overview paper Durrett (2010) *Some features of the spread of epidemics and opinions on a random graph*. Here is one highlight. This result is especially interesting because the "mean-field" analysis by statistical physicists gave the wrong answer.

Theorem (Chatterjee - Durrett (2009))

Consider a Newman, Strogatz and Watts random graph G_n on the vertex set $\{1,2,...,n\}$, where the degrees d_i satisfy $\mathbb{P}(d_i = k) \sim Ck^{-\alpha}$ as $k \to \infty$ for some constant C and some $\alpha > 3$, and $\mathbb{P}(d_i \le 2) = 0$. Let $(\xi_t^1; t \ge 0)$ denote the contact process on the random graph G_n starting from all sites occupied. Then for any value of the infection rate $\lambda > 0$, there is a positive constant $p(\lambda)$ so that for any $\delta > 0$

$$\inf_{t \le \exp(n^{1-\delta})} \mathbb{P}\left(n^{-1}|\xi_t^1| \ge p(\lambda)\right) \to 1 \text{ as } n \to \infty.$$