When Knowing Early Matters: Gossip, Percolation and Nash Equilibria

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May 27, 2010

Abstract

Continually arriving information is communicated through a network of n agents, with the value of information to the j'th recipient being a decreasing function of j/n, and communication costs paid by recipient. Regardless of details of network and communication costs, the social optimum policy is to communicate arbitrarily slowly. But selfish agent behavior leads to Nash equilibria which (in the $n \to \infty$ limit) may be efficient (Nash payoff = social optimum payoff) or wasteful ($0 <$ Nash payoff $<$ social optimum payoff) or totally wasteful (Nash payoff = 0). We study the cases of the complete network (constant communication costs between all agents), the grid with only nearest-neighbor communication, and the grid with communication cost a function of distance. The main technical tool is analysis of the associated first passage percolation process or SI epidemic (representing spread of one item of information) and in particular its "window width", the time interval during which most agents learn the item.

Many arguments are just outlined, not intended as complete rigorous proofs. This version was written in July 2007 to accompany a talk at the ICTP workshop "Common Concepts in Statistical Physics and Computer Science", and intended as a starting point for future thesis projects which could explore these and many variant problems in detail. One of the topics herein (first passage percolation on the $N \times N$ torus with short and long range interactions; section 6.2) has now been studied rigorously by Chatterjee and Durrett [4], and so it seems appropriate to make this version publicly accessible.

*Research supported by N.S.F. Grant DMS0704159

1 Introduction

A topic which one might loosely call "random percolation of information through networks" arises in many different contexts, from epidemic models [2] and computer virus models [10] to *gossip algorithms* [8] designed to keep nodes of a decentralized network updated about information needed to maintain the network. This topic differs from *communication networks* in that we envisage information as having a definite source but no definite destination.

In this paper we study an aspect where the vertices of the network are agents, and where there are costs and benefits associated with the different choices that agents may make in communicating information. In such "economic game theory" settings one anticipates a *social optimum* strategy that maximizes the total net payoff to all agents combined, and an (often different) *Nash equilibrium* characterized by the property that no one agent can benefit from deviating from the Nash equilibrium strategy followed by all other agents (so one anticipates that any reasonable process of agents adjusting strategies in a selfish way will lead to some Nash equilibrium). Of course a huge number of different models of costs, benefits and choices could fit the description above, but we focus on the specific setting where the value to you of receiving information depends on how few people know the information before you do. Two familiar real world examples are *gossip* in social networks and *insider trading* in financial markets. In the first, the gossiper gains perceived social status from transmitting information, and so is implicitly willing to pay for communicate to others; in the second the owner of knowledge recognizes its value and implictly expects to be paid for communication onwards. Our basic model makes the simpler assumption that the value to an agent attaches at the time information is received, and subsequently the agent takes no initiative to communicate it to others, but does so freely when requested, with the requester paying the cost of communication. In our model the benefits come from, and communication costs are paid to, the outside world: there are no payments between agents.

Remark. Many arguments are just outlined, not intended as complete rigorous proofs. This version was written in July 2007, and intended as a starting point for future thesis projects which could explore these and many variant problems in detail. One of the topics herein (first passage percolation on the $N \times N$ torus with short and long range interactions; section 6.2) has now been studied rigorously by Chatterjee and Durrett [4], and so it seems appropriate to make this version publicly accessible.

1.1 The general framework: a rank-based reward game

There are n agents (our results are in the $n \to \infty$ limit). The basic two rules are:

Rule 1. New items of information arrive at times of a rate-1 Poisson process; each item comes to one random agent.

Information spreads between agents by virtue of one agent calling another and learning all items that the other knows (details are case-specific, described later), with a (case-specific) communication cost paid by the *receiver* of information.

Rule 2. The j'th person to learn an item of information gets reward $R(\frac{j}{n})$.

Here $R(u),\ 0 < u \leq 1$ is a function such that

$$R(u) \text{ is decreasing; } R(1) = 0; \quad 0 < \bar{R} := \int_0^1 R(u)du < \infty. \tag{1}$$

Assuming information eventually reaches each agent, the total reward from each item will be $\sum_{j=1}^{n} R(\frac{j}{n}) \sim n\bar{R}$. If agents behave in some "exchangeable" way then the average net payoff (per agent per unit time) is

$$\text{payoff} = \bar{R} - (\text{average communication cost per agent per unit time}). \tag{2}$$

Now the average communication cost per unit time can be made arbitrarily small by simply communicating less often (because an agent learns *all* items that another agent knows, for the cost of one call. Note the calling agent does not know in advance whether the other agent has any new items of information). Thus the "social optimum" protocol is to communicate arbitrarily slowly, giving payoff arbitrarily close to $\bar{R}$. But if agents behave selfishly then one agent may gain an advantage by paying to obtain information more quickly, and so we seek to study Nash equilibria for selfish agents. In particular there are three qualitative different possibilities. In the $n \to \infty$ limit, the Nash equilibrium may be

- efficient (Nash payoff = social optimum payoff)
- or wasteful (0 < Nash payoff < social optimum payoff)
- or totally wasteful (Nash payoff = 0).

1.2 Methodology

Allowing agents' behaviors to be completely general makes the problems rather complicated (e.g. a subset of agents could seek to coordinate their actions) so in each specific model we restrict agent behavior to be of a specified form, making calls at random times with a rate parameter θ; the agent's "strategy" is just a choice of θ, and for this discussion we assume θ is a single real number. If all agents use the same parameter value θ then the spread of one item of information through the network is as some model-dependent first passage percolation process (see section 2.2). So there is some function $F_{\theta,n}(t)$ giving the proportion of agents who learn the item within time t after the arrival of the information into the network. Now suppose one agent **ego** uses a different parameter value ϕ and gets some payoff-per-unit-time, denoted by payoff(ϕ, θ). The Nash equilibrium value θ^{Nash} is the value of θ for which **ego** cannot do better by choosing a different value of ϕ, and hence is the solution of

$$\frac{d}{d\phi}\text{payoff}(\phi,\theta)\Big|_{\phi=\theta} = 0. \tag{3}$$

Obtaining a formula for payoff(ϕ, θ) requires knowing $F_{\theta,n}(t)$ and knowing something about the geometry of the sets of informed agents at time t – see (19,26) for the two basic examples. The important point is that where we know the exact $n \to \infty$ limit behavior of $F_{\theta,n}(t)$ we get a formula for the exact limit θ^{Nash}, and where we know order of magnitude behavior of $F_{\theta,n}(t)$ we get order of magnitude behavior of θ^{Nash}.

Note that we have assumed that in a Nash equilibrium each agent uses the same strategy. This is only a sensible assumption when the network cost structure has enough symmetry (is *transitive* – see section 7.1) and the non-transitive case is an interesting topic for future study.

It turns out (section 4) that for determining the qualitative behavior of the Nash equilibria, the important aspect is the size of the *window width* $w_{\theta,n}$ of the associated first passage percolation process, that is the time interval over which the proportion of agents knowing the item of information increases from (say) 10% to 90%. While this is well understood in the simplest examples of first passage percolation on finite sets, it has not been studied for very general models and our game-theoretic questions provide motivation for future such study.

To interpret later formulas it turns out to be convenient to work with the derivative of R. Write $R'(u) = -r(u)$, so that $R(u) = \int_u^1 r(s)ds$ and (1) becomes

$$r(u) \geq 0; \quad 0 < \bar{R} := \int_0^1 ur(u)du < \infty. \tag{4}$$

1.3 Summary of results

1.3.1 The complete graph case

Network communication model: Each agent i may, at any time, call any other agent j (at cost 1), and learn all items that j knows.

Poisson strategy. The allowed strategy for an agent i is to place calls, at the times of a Poisson (rate θ) process, to a random agent.

Result (section 2). In the $n \to \infty$ limit the Nash equilibrium value of θ is

$$\theta^{\text{Nash}} = \int_0^1 (1 + \log(1-u))R(u)du = \int_0^1 r(u)g(u)du \tag{5}$$

where $g(u) = -(1-u)\log(1-u) > 0$.

Our assumptions (1) on $R(u)$ imply $0 < \theta^{\text{Nash}} < \bar{R}$. Because an agent's average cost per unit time equals his value of θ, from (2) the Nash equilibrium payoff $\bar{R} - \theta^{\text{Nash}}$ is strictly less than the social optimum payoff $\bar{R}$ but strictly greater than 0. So this is a "wasteful" case.

1.3.2 The nearest neighbor grid

Network communication model: Agents are at the vertices of the $N \times N$ torus (i.e. the grid with periodic boundary conditions). Each agent i may, at any time, call any of the 4 neighboring agents j (at cost 1), and learn all items that j knows.

Poisson strategy. The allowed strategy for an agent i is to place calls, at the times of a Poisson (rate θ) process, to a random neighboring agent.

Result (section 3). The Nash equilibrium value of θ is such that as $N \to \infty$

$$\theta_N^{\text{Nash}} \sim N^{-1} \int_0^1 g(u)r(u)du \tag{6}$$

where $g(u) > 0$ is a certain complicated function – see (28).

So here the Nash equilibrium payoff $\bar{R} - \theta_N^{\text{Nash}}$ tends to $\bar{R}$; this is an "efficient" case.

1.3.3 Grid with communication costs increasing with distance

Network communication model. The agents are at the vertices of the $N \times N$ torus. Each agent i may, at any time, call any other agent j, at cost $c(N, d(i,j))$, and learn all items that j knows.

Here $d(i,j)$ is the distance between i and j. We treat two cases, with different choices of $c(N,d)$. In section 5 we take cost function $c(N,d) = c(d)$ satisfying

$$c(1) = 1; \quad c(d) \uparrow \infty \text{ as } d \to \infty \tag{7}$$

and

Poisson strategy. An agent's strategy is described by a sequence $(\theta(d);\ d = 1, 2, 3, \ldots)$; where for each d:

at rate $\theta(d)$ the agent calls a random agent at distance d.

In this case a simple abstract argument (section 5) shows that the Nash equilibrium is efficient (without calculating what the equilibrium strategy or payoff actually is) for any $c(d)$ satisfying (7).

In section 6 we take

$$\begin{aligned} c(N,d) &= 1; \quad d = 1 \\ &= c_N; \quad d > 1 \end{aligned}$$

where $1 \ll c_N \ll N^3$, and

Poisson strategy. An agent's strategy is described by a pair of numbers $(\theta_{\text{near}}, \theta_{\text{far}}) = \theta$:

at rate θ_{near} the agent calls a random neighbor
at rate θ_{far} the agent calls a random non-neighbor.

In this case we show (42) that the Nash equilibrium strategy satisfies

$$\theta^{\text{Nash}}_{\text{near}} \sim \zeta_1 c_N^{-1/2}; \quad \theta^{\text{Nash}}_{\text{far}} \sim \zeta_2 c_N^{-2}$$

for certain constants ζ_1, ζ_2 depending on the reward function. So the Nash equilibrium cost $\sim \zeta_1 c_N^{-1/2}$, implying that the equilibrium is efficient.

1.3.4 Plan of paper

The two basic cases (complete graph, nearest-neighbor grid) can be analyzed directly using known results for first passage percolation on these structures; we do this analysis in sections 2 and 3. There are of course simple arguments for order-of-magnitude behavior in those cases, which we recall in section 4 (but which the reader may prefer to consult first) as a preliminary to the more complicated model "grid with communication costs increasing with distance", for which one needs to understand orders of magnitude before embarking on calculations.

1.4 Variant models and questions

These results suggest many alternate questions and models, a few of which are addressed briefly in the sections indicated, the others providing suggestions for future research.

- Are there cases where the Nash equilibrium is totally wasteful? (section 2.1)
- Wouldn't it be better to place calls at regular time intervals? (section 7.2)
- Can one analyze more general strategies?

- In the grid context of section 1.3.3, what is the equilibrium strategy and cost for more general costs $c(N, d)$?
- What about the symmetric model where, when i calls j, they exchange information? (section 7.1)
- In formulas (5,6) we see decoupling between the reward function $r(u)$ and the function $g(u)$ involving the rest of the model – is this a general phenomenon?
- In the nearest-neighbor grid case, wouldn't it be better to cycle calls through the 4 neighbors?
- What about non-transitive models, e.g. social networks where different agents have different numbers of friends, so that different agents have different strategies in the Nash equilibrium?
- To model gossip, wouldn't it be better to make the reward to agent i depend on the number of other agents who learn the item from agent i? (section 7.3)
- To model insider trading, wouldn't it be better to say that agent j is willing to pay some amount $s(t)$ to agent i for information that i has possessed for time t, the function $s(\cdot)$ not specified in advance but a component of strategy and hence with a Nash equilibrium value?

1.5 Conclusions

As the list above suggests, we are only scratching the surface of a potentially large topic. In the usual setting of information communication networks, the goal is to communicate quickly, and our two basic examples (complete graph; nearest-neighbor grid) are the extremes of rapid and slow communication. It is therefore paradoxical that, in our rank-based reward game, the latter is efficient while the former is inefficient. One might jump to the conclusion that in general efficiency in the rank-based reward game was inversely related to network connectivity. But the examples of the grid with long-range interaction show the situation is not so simple, in that agents *could* choose to make long range calls and emulate a highly-connected network, but in equilibrium they do not do so very often.

2 The complete graph

The default assumptions in this section are

Network communication model: Each agent i may, at any time, call any other agent j (at cost 1), and learn all items that j knows.

Poisson strategy. The allowed strategy for an agent i is to place calls, at the times of a Poisson (rate θ) process, to a random agent.

2.1 Finite number of rewards

Before deriving the result (5) in our general framework, let us step outside that framework to derive a very easy variant result. Suppose that only the first two recipients of an item of information receive a reward, of amount w_n say. Agent strategy cannot affect the first recipient, only the second. Suppose **ego** uses rate ϕ and other agents use rate θ. Then (by elementary properties of Exponential distributions)

$$P(\textbf{ego} \text{ is second to receive item}) = \frac{\phi}{\phi + (n-2)\theta} \tag{8}$$

and so

$$\text{payoff}(\phi, \theta) = \frac{w_n}{n} + \frac{\phi w_n}{\phi + (n-2)\theta} - \phi.$$

We calculate

$$\frac{d}{d\phi}\text{payoff}(\phi, \theta) = \frac{(n-2)\theta w_n}{(\phi + (n-2)\theta)^2} - 1$$

and then the criterion (3) gives

$$\theta_n^{\text{Nash}} = \frac{(n-2)w_n}{(n-1)^2} \sim \frac{w_n}{n}.$$

To compare this variant with the general framework, we want the total reward available from an item to equal n, to make the social optimum payoff $\to 1$, so we choose $w_n = n/2$. So we have shown that the Nash equilibrium payoff is

$$\text{payoff} = 1 - \theta_n^{\text{Nash}} \to \tfrac{1}{2}. \tag{9}$$

So this is a "wasteful" case.

By the same argument we can study the case where (for fixed $k \geq 2$) the first k recipients get reward n/k. In this case we find

$$\theta_n^{\text{Nash}} \sim \frac{k-1}{k}$$

and the Nash equilibrium payoff is

$$\text{payoff} \to \tfrac{1}{k} \tag{10}$$

while the social optimum payoff $= 1$. Thus by taking $k_n \to \infty$ slowly we have a model in which the Nash equilibrium is "totally wasteful".

2.2 First passage percolation : general setup

The classical setting for first passage percolation, surveyed in [11], concerns nearest neighbor percolation on the d-dimensional lattice. Let us briefly state our general setup for first passage percolation (of "information") on a finite graph. There are "rate" parameters $\nu_{ij} \geq 0$ for undirected edges (i, j). There is an initial vertex v_0, which receives the information at time 0. At time t, for each vertex i which has already received the information, and each neighbor j, there is chance $\nu_{ij} dt$ that j learns the information from i before time $t + dt$. Equivalently, create independent Exponential(ν_{ij}) random variables V_{ij} on edges (i, j). Then each vertex v receives the information at time

$$T_v = \min\{V_{i_0 i_1} + V_{i_1 i_2} + \ldots + V_{i_{k-1} i_k}\}$$

minimized over paths $v_0 = i_0, i_1, i_2, \ldots, i_k = v$.

2.3 First passage percolation on the complete graph

Consider first passage percolation on the complete n-vertex graph with rates $\nu_{ij} = 1/(n-1)$. Pick k random agents and write $\bar{S}^n_{(1)}, \ldots, \bar{S}^n_{(k)}$ for the times at which these k agents receive the information. The key fact for our purposes is that as $n \to \infty$

$$(\bar{S}^n_{(1)} - \log n, \ldots, \bar{S}^n_{(k)} - \log n) \ \stackrel{d}{\to}\ (\xi + S_{(1)}, \ldots, \xi + S_{(k)}) \tag{11}$$

where the limit variables are independent, ξ has double exponential distribution $P(\xi \le x) = \exp(-e^{-x})$ and each $S_{(i)}$ has the *logistic* distribution with distribution function

$$F_1(x) = \frac{e^x}{1+e^x}, \quad -\infty < x < \infty. \tag{12}$$

Here $\stackrel{d}{\to}$ denotes convergence in distribution. To outline a derivation of (11), fix a large integer L and decompose the percolation times as

$$\bar{S}^n_{(i)} - \log n = (\tau_L - \log L) + (\bar{S}^n_{(i)} - \tau_L + \log(L/n)) \tag{13}$$

where τ_L is the time at which some L agents have received the information. By the Yule process approximation (see e.g. [1]) to the fixed-time behavior of the first passage percolation, the number $N(t)$ of agents possessing the information at fixed large time t is approximately distributed as We^t, where W has Exponential(1) distribution, and so

$$P(\tau_L \le t) = P(N(t) \ge L) \approx P(We^t \ge L) = \exp(-Le^{-t})$$

implying $\tau_L - \log L \approx \xi$ in distribution, explaining the first summand on the right side of (11). Now consider the proportion $H(t)$ of agents possessing the information at time $\tau_L + t$. This proportion follows closely the deterministic logistic equation $H' = H(1-H)$ whose solution is (12) shifted to satisfy the initial condition $H(0) = L/n$, so this solution approximates the distribution function of $S_{(i)} - \log(L/n)$. Thus the time $\bar{S}^n_{(i)}$ at which a random agent receives the information satisfies

$$(\bar{S}^n_{(i)} - \tau_L + \log(L/n)) \approx S_{(i)} \text{ in distribution}$$

independently as i varies. Now the limit decomposition (11) follow from the finite-n decomposition (13)..

We emphasize (11) instead of more elementary derivations (using methods of [9, 13]) of the limit distribution for $\bar{S}^n_{(1)} - \log n$ because (11) gives the correct dependence structure for different agents. Because only *relative* order of gaining information is relevant to us, we may recenter by subtracting ξ and suppose that the times at which different random agents gain information are independent with logistic distribution (12).

2.4 Analysis of the rank-based reward game

We now return to our general reward framework

> The j'th person to learn an item of information gets reward $R(\frac{j}{n})$

and give the argument for (5).

Suppose all agents use the Poisson(θ) strategy. In the case $\theta = 1$, the way that a single item of information spreads is exactly as the first passage percolation process above; and the general-θ

case is just a time-scaling by θ. So as above, we may suppose that (all calculations in the $n \to \infty$ limit) the recentered time S_θ to reach a random agent has distribution function

$$F_\theta(x) = F_1(\theta x) \tag{14}$$

which is the solution of the time-scaled logistic equation

$$\frac{F'_\theta}{1 - F_\theta} = \theta F_\theta \tag{15}$$

(Recall F_1 is the logistic distribution (12)). Now consider the case where all other agents use a value θ but **ego** uses a different value ϕ. The (limit, recentered) time $T_{\phi,\theta}$ at which **ego** learns the information now has distribution function $G_{\phi,\theta}$ satisfying an analog of (15):

$$\frac{G'_{\phi,\theta}}{1 - G_{\phi,\theta}} = \phi F_\theta. \tag{16}$$

To explain this equation, the left side is the rate at time t at which **ego** learns the information; this equals the rate ϕ of calls by **ego**, times the probability $F_\theta(t)$ that the called agent has received the information. To solve the equation, first we get

$$1 - G_{\phi,\theta} = \exp\left(-\phi \int F_\theta\right).$$

But we know that in the case $\phi = \theta$ the solution is F_θ, that is we know

$$1 - F_\theta = \exp\left(-\theta \int F_\theta\right),$$

and so we have the solution of (16) in the form

$$1 - G_{\phi,\theta} = (1 - F_\theta)^{\phi/\theta}. \tag{17}$$

If **ego** gets the information at time t then his percentile rank is $F_\theta(t)$ and his reward is $R(F_\theta(t))$. So the expected reward to **ego** is

$$ER(F_\theta(T_{\phi,\theta})); \quad \text{where } \mathrm{dist}(T_{\phi,\theta}) = G_{\phi,\theta}.$$

We calculate

$$\begin{aligned} P(F_\theta(T_{\phi,\theta}) \le u) &= G_{\phi,\theta}(F_\theta^{-1}(u)) \\ &= 1 - (1 - F_\theta(F_\theta^{-1}(u)))^{\phi/\theta} \text{ by (17)} \\ &= 1 - (1 - u)^{\phi/\theta} \end{aligned} \tag{18}$$

and so

$$ER(F_\theta(T_{\phi,\theta})) = \int_0^1 r(u)\ (1 - (1 - u)^{\phi/\theta}) du.$$

This is the mean reward to **ego** from one item, and hence also the mean reward per unit time in the ongoing process. So, including the "communication cost" of ϕ per unit time, the net payoff (per unit time) to **ego** is

$$\text{payoff}(\phi, \theta) = -\phi + \int_0^1 r(u)\ (1 - (1 - u)^{\phi/\theta}) du. \tag{19}$$

The criterion (3) for θ to be a Nash equilibrium is, using the fact $\frac{d}{d\phi} x^{\phi/\theta} = \frac{\log x}{\theta} x^{\phi/\theta}$,

$$1 = \tfrac{1}{\theta} \int_0^1 r(u)\ (-\log(1 - u))\ (1 - u) du. \tag{20}$$

This is the second equality in (5), and integrating by parts gives the first equality.

Remark. For the linear reward function

$$R(u) = 2(1-u); \quad \bar{R} = 1$$

result (5) gives Nash payoff $= 1/2$. Consider alternatively

$$R(u) = \tfrac{1}{u_0} 1_{(u \le u_0)}; \quad \bar{R} = 1.$$

Then the $n \to \infty$ Nash equilibrium cost is

$$\theta^{\text{Nash}}(u_0) = \frac{1}{u_0} \int_0^{u_0} (1 + \log(1-u))\, du.$$

In particular, the Nash payoff $1 - \theta^{\text{Nash}}(u_0)$ satisfies

$$1 - \theta^{\text{Nash}}(u_0) \to 0 \text{ as } u_0 \to 0.$$

In words, as the reward becomes concentrated on a smaller and smaller proportion of the population then the Nash equilibrium becomes more and more wasteful. In this sense result (5) in the general framework is consistent with the "finite number of rewards" result (10).

3 The $N \times N$ torus, nearest neighbor case

Network communication model. There are N^2 agents at the vertices of the $N \times N$ torus. Each agent i may, at any time, call any of the 4 neighboring agents j (at cost 1), and learn all items that j knows.

Poisson strategy. The allowed strategy for an agent i is to place calls, at the times of a Poisson (rate θ) process, to a random neighboring agent.

We will derive formula (6). As remarked later, the function $g(u)$ is ultimately derived from fine structure of first passage percolation in the plane, and seems impossible to determine as an explicit formula. But of course the main point is that (in contrast to the complete graph case) the Nash equilibrium payoff $\bar{R} - \theta_N^{\text{Nash}} = \bar{R} - O(N^{-1})$ tends to the social optimum $\bar{R}$.

3.1 Nearest-neighbor first passage percolation on the torus

Consider (nearest-neighbor) first passage percolation on the $N \times N$ torus, started at a uniform random vertex, with rates $\nu_{ij} = 1$ for edges (i,j). Write $(T_i^N,\ 1 \le i \le 4)$ for the information receipt times of the 4 neighbors of the origin (using paths not through the origin), and write $Q^N(t)$ for the number of vertices informed by time t. Write $T_*^N = \min(T_i^N,\ 1 \le i \le 4)$.

The key point is that we expect a $N \to \infty$ limit of the following form

$$(T_i^N - T_*^N, 1 \le i \le 4;\ N^{-2} Q^N(T_*^N);\ (N^{-1}(Q^N(T_*^N + t) - Q^N(T_*^N)), 0 \le t < \infty))$$

$$\xrightarrow{d} (\tau_i, 1 \le i \le 4;\ U;\ (Vt, 0 \le t < \infty)) \tag{21}$$

where $\tau_i, 1 \le i \le 4$ are nonnegative with $\min_i \tau_i = 0$; U has uniform$(0,1)$ distribution; $0 < V < \infty$; with a certain complicated joint distribution for these limit quantities.

To explain (21), first note that as $N \to \infty$ the differences $T_i^N - T_*^N$ are stochastically bounded (by the time to percolate through a finite set of edges) but cannot converge to 0 (by linearity of growth rate in the shape theorem below), so we expect some non-degenerate limit distribution $(\tau_i, 1 \leq i \leq 4)$. Next consider the time T_0^N at which the origin is wetted. By uniformity of starting position, $Q^N(T_0^N)$ must have uniform distribution on $\{1, 2, \ldots, N^2\}$, and it follows that $N^{-2}Q^N(T_*^N) \stackrel{d}{\to} U$. The final assertion

$$(N^{-1}(Q^N(T_*^N + t) - Q^N(T_*^N)), 0 \leq t < \infty) \stackrel{d}{\to} (Vt, 0 \leq t < \infty) \tag{22}$$

is related to the *shape theorem* [11] for first-pasage percolation on the infinite lattice started at the origin. This says that the random set $\mathcal{B}_s$ of vertices wetted before time s grows linearly with s, and the spatially rescaled set $s^{-1}\mathcal{B}_s$ converges to a limit deterministic convex set $\mathcal{B}$:

$$s^{-1}\mathcal{B}_s \to \mathcal{B}. \tag{23}$$

It follows that

$$N^{-2}Q^N(sN) \to q(s) \text{ as } N \to \infty$$

where $q(s)$ is the area of $s\mathcal{B}$ regarded as a subset of the continuous torus $[0, 1]^2$. Because $N^{-2}Q^N(T_0^N) \stackrel{d}{\to} U$ we have

$$T_*^N \approx T_0^N \approx N^2 q^{-1}(U)$$

where $q^{-1}(\cdot)$ is the inverse function of $q(\cdot)$. Writing $Q'^N(\cdot)$ for a suitably-interpreted local growth rate of $Q^N(\cdot)$ we deduce

$$(N^{-2}Q^N(T_*^N), N^{-1}Q'^N(T_*^N)) \stackrel{d}{\to} (U, q'(q^{-1}(U)))$$

and so (22) holds for $V = q'(q^{-1}(U))$.

3.2 Analysis of the rank-based reward game

We want to study the case where other agents call some neighbor at rate θ but **ego** (at the origin) calls some neighbor at rate ϕ. To analyze rewards, by scaling time we can reduce to the case where other agents call *each* neighbor at rate 1 and **ego** calls each neighbor at rate $\lambda = \phi/\theta$. We want to compare the rank M_λ^N of **ego** (rank $= j$ if **ego** is the j'th person to receive the information) with the rank M_1^N of **ego** in the $\lambda = 1$ case. As noted above, M_1^N is uniform on $\{1, 2, \ldots, N^2\}$. Writing $(\xi_i^\lambda, \ 1 \leq i \leq 4)$ for independent Exponential(λ) r.v.'s, the time at which the origin receives the information is

$$T_*^N + \min_i (T_i^N - T_*^N + \xi_i^\lambda)$$

and the rank of the origin is

$$M_\lambda^N = Q^N(T_*^N) + N\widetilde{Q}^N(\min_i (T_i^N - T_*^N + \xi_i^\lambda))$$

where

$$\widetilde{Q}^N(t) = N^{-1}(Q^N(T_*^N + t) - Q^N(T_*^N)).$$

Note we can construct $(\xi_i^\lambda, \ 1 \leq i \leq 4)$ as $(\lambda^{-1}\xi_i^1, \ 1 \leq i \leq 4)$. Now use (22) to see that as $N \to \infty$

$$(N^{-2}M_1^N, N^{-1}(M_\lambda^N - M_1^N)) \stackrel{d}{\to} (U, VZ(\lambda)) \tag{24}$$

where

$$Z(\lambda) := \min_i (\tau_i + \xi_i^\lambda) - \min_i (\tau_i + \xi_i^1). \tag{25}$$

Now in the setting where **ego** calls at rate ϕ and others at rate θ we have

$$\text{payoff}(\phi,\theta) - \text{payoff}(\theta,\theta) + (\phi - \theta) = E\left[R\left(\frac{M^N_{\phi/\theta}}{N^2}\right) - R\left(\frac{M^N_1}{N^2}\right)\right]$$

and it is straightforward to use (24) to show this

$$\sim N^{-1}\int_0^1 (-r(u))\ z_u(\phi/\theta)du, \ \text{ for } z_u(\lambda) := E(VZ(\lambda)|U=u). \tag{26}$$

The Nash equilibrium condition

$$\left.\frac{d}{d\phi}\text{payoff}(\phi,\theta)\right|_{\phi=\theta} = 0$$

now implies

$$\theta_N^{\text{Nash}} \sim N^{-1}\int_0^1 (-r(u))\ z'_u(1)du. \tag{27}$$

Because $Z(\lambda)$ is decreasing in λ we have $z'_u(1) < 0$ and this expression is of the form (6) with

$$g(u) = -z'_u(1) = -\tfrac{d}{d\lambda}\ E(VZ(\lambda)|U=u)|_{\lambda=1} \tag{28}$$

Remark. The distribution of V depends on the function $q(\cdot)$ which depends on the limit shape in nearest neighbor first passage percolation, which is not explicitly known. Also $Z(\lambda)$ involves the joint distribution of (τ_i), which is not explicitly known, and also is (presumably) correlated with the direction from the percolation source which is in turn not independent of V. This suggests it would be difficult to find an explicit formula for $g(u)$.

4 Order of magnitude arguments

Here we mention simple order of magnitude arguments for the two basic cases we have already analyzed. As mentioned in the introduction, what matters is the size of the *window width* $w_{\theta,n}$ of the associated first passage percolation process We will re-use such arguments in sections 5 and 6.1, in more complicated settings.

Complete graph. If agents call at rate $\theta = 1$ then by (11) the window width is order 1; so if θ_n is the Nash equilibrium rate then the window width w_n is order $1/\theta_n$. Suppose $w_n \to \infty$. Then **ego** could call at some fixed slow rate ϕ and (because this implies many calls are made near the start of the window) the reward to **ego** will tend to $R(0)$, and **ego**'s payoff $R(0) - \phi$ will be larger than the typical payoff $\bar{R} - \theta_n$. This contradicts the definition of Nash equilibrium. So in fact we must have w_n bounded above, implying θ_n bounded below, implying the Nash equilbrium in wasteful.

Nearest neighbor torus. If agents call at rate $\theta = 1$ then by the shape theorem (23) the window width is order N. The time difference between receipt time for different neighbors of **ego** is order 1, so if **ego** calls at rate 2 instead of rate 1 his rank (and hence his reward) increases by order $1/N$. By scaling, if the Nash equilibrium rate is θ_N and **ego** calls at rate $2\theta_N$ then his increased reward is again of order $1/N$. His increased cost is θ_N. At the Nash equilibrium the increased reward and cost must balance, so θ_N is order $1/N$, so the Nash equilibrium is efficient.

5 The $N \times N$ torus with general interactions: a simple criterion for efficiency

Network communication model. The agents are at the vertices of the $N \times N$ torus. Each agent i may, at any time, call any other agent j, at cost $c(d(i,j))$, and learn all items that j knows.

Here $d(i,j)$ is the distance between i and j, and we assume the cost function $c(d)$ satisfies

$$c(1) = 1; \quad c(d) \uparrow \infty \text{ as } d \to \infty. \tag{29}$$

Poisson strategy. An agent's strategy is described by a sequence $(\theta(d);\ d = 1, 2, 3, \ldots)$; and for each d:

at rate $\theta(d)$ the agent calls a random agent at distance d.

A simple argument below shows

$$\text{under condition (29) the Nash equilibrium is efficient.} \tag{30}$$

Consider the Nash strategy, and suppose first that the window width w_N converges to a limit $w_\infty < \infty$. Consider a distance d such that the Nash strategy has $\theta^{\text{Nash}}(d) > 0$. Suppose **ego** uses $\theta(d) = \theta^{\text{Nash}}(d) + \phi$. The increased cost is $\phi c(d)$ while the increased benefit is at most $O(w_\infty \phi)$, because this is the increased chance of getting information earlier. So the Nash strategy must have $\theta^{\text{Nash}}(d) = 0$ for sufficiently large d, not depending on N. But for first passage percolation with bounded range transitions, the shape theorem (23) remains true and implies that w_N scales as N.

This contradiction implies that the window width $w_N \to \infty$. Now suppose the Nash equilibrium were inefficient, with some Nash cost $\bar{\theta} > 0$. Suppose **ego** adopts the strategy of just calling a random neighbor at rate ϕ_N, where $\phi_N \to 0$, $\phi_N w_N \to \infty$. Then **ego** obtains asymptotically the same reward $\bar{R}$ as his neighbor, a typical agent. But **ego**'s cost is $\phi_N \to 0$. This is a contradiction with the assumption of inefficiency. So the conclusion is that the Nash equilibrium is efficient and $w_N \to \infty$.

Remarks. Result (30) is striking. but does not tell us what the Nash equilibrium strategy and cost actually are. It is a natural open problem to study the case of (29) with $c(d) = d^\alpha$. Instead we study a simpler model in the next section.

6 The $N \times N$ torus with short and long range interactions

Network communication model. The agents are at the vertices of the $N \times N$ torus. Each agent i may, at any time, call any of the 4 neighboring agents j (at cost 1), or call any other agent j at cost $c_N \geq 1$, and learn all items that j knows.

Poisson strategy. An agent's strategy is described by a pair of numbers $(\theta_{\text{near}}, \theta_{\text{far}}) = \theta$:
at rate θ_{near} the agent calls a random neighbor
at rate θ_{far} the agent calls a random non-neighbor.

This model obviously interpolates between the complete graph model ($c_N = 1$) and the nearest-neighbor model ($c_N = \infty$).

First let us consider for which values of c_N the nearest-neighbor Nash equilibrium (θ_{near} is order N^{-1}, $\theta_{\mathrm{far}} = 0$) persists in the current setting. When **ego** considers using a non-zero value of θ_{far}, the cost is order $c_N \theta_{\mathrm{far}}$. The time for information to reach a typical vertex is order $N/\theta_{\mathrm{near}} = N^2$, and so the benefit of using a non-zero value of θ_{far} is order $\theta_{\mathrm{far}} N^2$. We deduce that

if $c_N \gg N^2$ then the Nash equilibrium is asymptotically the same as in the nearest-neighbor case; in particular, the Nash equilibrum is efficient.

Let us study the more interesting case

$$1 \ll c_N \ll N^2.$$

The result in this case turns out to be, qualitatively

$$\theta^{\mathrm{Nash}}_{\mathrm{near}} \text{ is order } c_N^{-1/2} \text{ and } \theta^{\mathrm{Nash}}_{\mathrm{far}} \text{ is order } c_N^{-2}. \text{ In particular, the Nash equilibrum is efficient.} \quad (31)$$

"Efficient" because the cost $c_N \theta_{\mathrm{far}} + \theta_{\mathrm{near}}$ is order $c_N^{-1/2}$. See (42) for the exact result.

We first do the order-of-magnitude calculation (section 6.1), then analyze the relevant first passage percolation process (section 6.2), and finally do the exact analysis in section 6.3.

6.1 Order of magnitude calculation

Our order of magnitude argument for (31) uses three ingredients (32,33,34). As in section 4 we consider the *window width* w_N of the associated percolation process. Suppose **ego** deviates from the Nash equilibrium $(\theta^{\mathrm{Nash}}_{\mathrm{near}}, \theta^{\mathrm{Nash}}_{\mathrm{far}})$ by setting his $\theta_{\mathrm{far}} = \theta^{\mathrm{Nash}}_{\mathrm{far}} + \delta$. The chance of thereby learning the information earlier, and hence the increased reward to **ego**, is order δw_N and the increased cost is δc_N. At the Nash equilibrium these must balance, so

$$w_N \asymp c_N \quad (32)$$

where $\asymp$ denotes "same order of magnitude". Now consider the difference ℓ_N between the times that different neighbors of **ego** are wetted. Then ℓ_N is order $1/\theta^{\mathrm{Nash}}_{\mathrm{near}}$. Write $\delta = \theta^{\mathrm{Nash}}_{\mathrm{near}}$ and suppose **ego** deviates from the Nash equilibrium by setting his $\theta_{\mathrm{near}} = 2\delta$. The increased benefit to **ego** is order ℓ_N / w_N and the increased cost is δ. At the Nash equilibrium these must balance, so $\delta \asymp \ell_N / w_N$ which becomes

$$\theta^{\mathrm{Nash}}_{\mathrm{near}} \asymp w_N^{-1/2} \asymp c_N^{-1/2}. \quad (33)$$

Finally we need to calculate how the window width w_N for FPP depends on $(\theta_{\mathrm{near}}, \theta_{\mathrm{far}})$, and we show in the next section that

$$w_N \asymp \theta_{\mathrm{near}}^{-2/3} \theta_{\mathrm{far}}^{-1/3}. \quad (34)$$

Granted this, we substitute (32,33) to get

$$c_N \asymp c_N^{1/3} \theta_{\mathrm{far}}^{-1/3}$$

which identifies $\theta_{\mathrm{far}} \asymp c_N^{-2}$ as stated at (31).

6.2 First passage percolation on the $N \times N$ torus with short and long range interactions

We study the model (call it *short-long FPP*, to distinguish it from nearest-neighbor FPP) defined by rates

$$\begin{aligned} \nu_{ij} &= \tfrac{1}{4}, && j \text{ a neighbor of } i \\ &= \lambda_N / N^2, && j \text{ not a neighbor of } i \end{aligned}$$

where $1 \gg \lambda_N \gg N^{-3}$.

Recall the shape theorem (23) for nearest neighbor first passage percolation; let A be the area of the limit shape $\mathcal{B}$. Define an artificial distance ρ such that $\mathcal{B}$ is the unit ball in ρ-distance; so nearest neighbor first passage percolation moves at asymptotic speed 1 with respect to ρ-distance. Consider short-long FPP started at a random vertex of the $N \times N$ torus. Write F_{N,λ_N} for the proportion of vertices reached by time t and let $T_{(0,0)}$ be the time at which the origin is reached. The event $\{T_{(0,0)} \leq t\}$ corresponds asymptotically to the event that at some time $t-u$ there is percolation across some long edge (i,j) into some vertex j at ρ-distance $\leq u$ from $(0,0)$ (here we use the fact that nearest neighbor first passage percolation moves at asymptotic speed 1 with respect to ρ-distance). The rate of such events at time $t-u$ is approximately

$$N^2 F_{N,\lambda_N}(t-u) \times Au^2 \times \lambda_N/N^2$$

where the three terms represent the number of possible vertices i, the number of possible vertices j, and the percolation rate ν_{ij}. Since these events occur asymptotically as a Poisson process in time, we get

$$1 - F_{N,\lambda_N}(t) \approx P(T_{(0,0)} \leq t) \approx \exp\left(-A\lambda_N \int_0^\infty u^2 F_{N,\lambda_N}(t-u)\ du\right). \tag{35}$$

This motivates study of the equation (for an unknown distribution function F_λ)

$$1 - F_\lambda(t) = \exp\left(-\lambda \int_{-\infty}^t (t-s)^2 F_\lambda(s)\ ds\right), \quad -\infty < t < \infty \tag{36}$$

whose solution should be unique up to centering. Writing F_1 for the $\lambda = 1$ solution, the general solution scales as

$$F_\lambda(t) := F_1(\lambda^{1/3} t).$$

So by (35), up to centering

$$F_{N,\lambda_N}(t) \approx F_1((A\lambda_N)^{1/3} t). \tag{37}$$

To translate this result into the context of the rank-based rewards game, suppose each agent uses strategy $\theta_N = (\theta_{N,\text{near}}, \theta_{N,\text{far}})$. Then the spread of one item of information is as first passage percolation with rates

$$\begin{aligned} \nu_{ij} &= \theta_{N,\text{near}}/4,\ j \text{ a neighbor of } i \\ &= \theta_{N,\text{far}}/(N^2-5),\ j \text{ not a neighbor of } i. \end{aligned}$$

This is essentially the case above with $\lambda_N = \theta_{N,\text{far}}/\theta_{N,\text{near}}$, time-scaled by $\theta_{N,\text{near}}$, and so by (37) the distribution function F_{N,θ_N} for the time at which a typical agent receives the information is

$$F_{N,\theta_N}(t) \approx F_1\left(A^{1/3} \theta_{N,\text{far}}^{1/3} \theta_{N,\text{near}}^{2/3} t\right). \tag{38}$$

In particular the window width is as stated at (34).

6.3 Exact equations for the Nash equilibrium

The equations will involve three quantities:
(i) The solution F_1 of (36).
(ii) The area A of the limit set $\mathcal{B}$ in the shape theorem (23) for nearest-neighbor first passage

pecolation.
(iii) The limit distribution (cf. (21))

$$(T_i^r - T_*^r,\ 1 \le i \le 4) \ \xrightarrow{d}\ (\tau_i,\ 1 \le i \le 4) \text{ as } r \to \infty \tag{39}$$

for relative receipt times of neighbors of the origin in nearest-neighbor first passage pecolation, where now we start the percolation at a random vertex of ρ-distance $\approx r$ from the origin.

To start the analysis, suppose all agents use rates $\theta = (\theta_{N,\mathrm{near}}, \theta_{N,\mathrm{far}})$. Consider the quantities

S is the first time that **ego** receives the information from a non-neighbor

T is the first time that **ego** receives the information from a neighbor

$F = F_{N,\theta_N}$ is the distribution function of T.

With probability $\to 1$ as $N \to \infty$ **ego** will actually receive the information first from a neighbor, and so F is asymptotically the distribution function of the time at which **ego** receives the information.

Now suppose **ego** uses a different rate $\phi_{N,\mathrm{far}} \neq \theta_{N,\mathrm{far}}$ for calling a non-neighbor. This does not affect T but changes the distribution of S to

$$P(S > t) \approx \exp\left(-\phi_{N,\mathrm{far}} \int_{-\infty}^{t} F(s)\ ds\right)$$

by the natural Poisson process approximation. Because $\theta_{N,\mathrm{far}}$ is small we can approximate

$$P(S \le t) \approx \phi_{N,\mathrm{far}} \int_{-\infty}^{t} F(s)\ ds.$$

The mean reward to **ego** for one item, as a function of $\phi_{N,\mathrm{far}}$, varies as

$$E(R(F(S)) - R(F(T))1_{(S<T)}\ +\ \text{constant}.$$

Because $U = F(T)$ is uniform on $(0,1)$, in the $N \to \infty$ limit

$$\begin{aligned}
E(R(F(S)) - R(F(T))1_{(S<T)} &= E(R(F(S)) - R(U))1_{(F(S)<U)} \\
&= \int_0^1 du\ E(R(F(S)) - R(u))1_{(F(S)<u)} \\
&= \int_0^1 du\ E \int_{\min(F(S),u}^{u} r(y)dy \\
&= \int_0^1 dy\ (1-y)r(y)P(F(S) \le y) \\
&= \int_0^1 dy\ (1-y)r(y)P(S \le F^{-1}(y)) \\
&= \phi_{N,\mathrm{far}} \int_0^1 dy\ (1-y)r(y) \int_{-\infty}^{F^{-1}(y)} F(s)ds.
\end{aligned}$$

The cost associated with using $\phi_{N,\mathrm{far}}$ is $c_N \phi_{N,\mathrm{far}}$, and at the Nash equilibrium the cost and reward must balance, so at the Nash equilibrium $F = F_{N,\theta_N}$ must satisfy

$$c_N \sim \int_0^1 dy\ (1-y)r(y) \int_{-\infty}^{F^{-1}(y)} F(s)ds. \tag{40}$$

Now suppose instead that **ego** uses a different rate $\phi_{N,\text{near}} \neq \theta_{N,\text{near}}$ for calling a neighbor. As in section 3.2, we set $\lambda = \phi_{N,\text{near}}/\theta_{N,\text{near}}$ so that we can use rate-1 nearest-neighbor first passage pecolation as comparsion. For (τ_i) at (39) and independent Exponential(λ) random variables (ξ_i^λ) write (as at (25))

$$Z(\lambda) := \min_i(\tau_i + \xi_i^\lambda) - \min_i(\tau_i + \xi_i^1).$$

So $Z(\lambda)$ is the time difference for **ego** receiving the information, caused by **ego** using $\phi_{N,\text{near}}$ instead of $\theta_{N,\text{near}}$. This time difference is measured after time-rescaling; in real time units the time difference is $Z(\lambda)/\theta_{N,\text{near}}$.

As above, write T for receipt time for **ego** using $\theta_{N,\text{near}}$, and $F = F_{N,\theta_N}$ for its distribution function. Then receipt time for **ego** using $\phi_{N,\text{near}}$ is $T + Z(\lambda)/\theta_{N,\text{near}}$, so **ego**'s rank becomes $\approx F(T)+F'(T)Z(\lambda)/\theta_{N,\text{near}}$, and setting $U = F(T)$ the rank of **ego** is $\approx U+F'(F^{-1}(U))Z(\lambda)/\theta_{N,\text{near}}$. The associated mean reward change for **ego** is asymptotically

$$\tfrac{z(\lambda)}{\theta_{N,\text{near}}} \times \int_0^1 r(u)F'(F^{-1}(u))\ du; \quad \lambda = \phi_{N,\text{near}}/\theta_{N,\text{near}}$$

where $z(\lambda) = EZ(\lambda)$. Because the cost of using rate $\phi_{N,\text{near}}$ equals $\phi_{N,\text{near}}$, the Nash equilibrium condition (3) implies

$$\theta^2_{N,\text{near}} \sim z'(1) \int_0^1 r(u)F'(F^{-1}(u))\ du. \tag{41}$$

We have now obtained the desired two equations for F_{N,θ_N} at the Nash equilibrium θ_N. Use (38) to rewrite these equations (40,41) in terms of F_1 as

$$\begin{aligned} c_N &\sim A^{-1/3}\theta_{N,\text{far}}^{-1/3}\theta_{N,\text{near}}^{-2/3} \int_0^1 dy\ (1-y)r(y) \int_{-\infty}^{F_1^{-1}(y)} F_1(s)ds \\ \theta^2_{N,\text{near}} &\sim A^{1/3}\theta_{N,\text{far}}^{1/3}\theta_{N,\text{near}}^{2/3} z'(1) \int_0^1 r(u)F_1'(F_1^{-1}(u))\ du. \end{aligned}$$

Solving for $\theta_{N,\text{near}}, \theta_{N,\text{far}}$ we find

$$\theta_{N,\text{near}} \sim Q^{1/2}c_N^{1/2}, \quad \theta_{N,\text{far}} \sim A^{-1}Q^{-1}c_N^{-2} \tag{42}$$

for

$$Q = z'(1)\left(\int_0^1 dy\ (1-y)r(y)\int_{-\infty}^{F_1^{-1}(y)} F_1(s)ds\right)\left(\int_0^1 r(u)F_1'(F_1^{-1}(u))\ du\right).$$

7 Variants

7.1 Transitivity and the symmetric variant

The examples we have studied so far have a certain property called *transitivity* in graph theory [3]. Informally, transitivity means "the network looks the same to each agent"; formally, it means that for any two agents i, j there is an automorphism of the network that preserves the network cost structure and maps i to j. This is what allows us to assume that in a Nash equilibrium each agent uses same strategy.

The general framework of section 1.1 uses the *asymmetric* model in which agent i calls agent j (at a certain cost to i) and learns all items that j knows. In the *symmetric* variant, agent i calls agent j (at a certain cost to i), and each tells the other all items they know.

For the transitive networks we have studied there is a simple relationship between the Nash equilibrium values of the asymmetric and symmetric variants of the Poisson strategies:

$$\theta^{\mathrm{Nash}}_{\mathrm{sym}} = \tfrac{1}{2}\theta^{\mathrm{Nash}}_{\mathrm{asy}}. \tag{43}$$

The point is that the percolation process in the symmetric variant is just the percolation process in the asymmetric variant, run at twice the speed, and this leads to the following relationship between the reward when **ego** uses rate ϕ and other agents use rate θ:

$$\mathrm{reward}_{\mathrm{sym}}(\phi, \theta) = \mathrm{reward}_{\mathrm{asy}}(\phi + \theta, 2\theta).$$

Because $\mathrm{payoff}(\phi, \theta) = \mathrm{reward}(\phi, \theta) - \phi$ in each case, we get

$$\mathrm{payoff}_{\mathrm{sym}}(\phi, \theta) = \mathrm{payoff}_{\mathrm{asy}}(\phi + \theta, 2\theta) + \theta$$

and therefore

$$\frac{d}{d\phi}\mathrm{payoff}_{\mathrm{sym}}(\phi, \theta) = \frac{d}{d\phi}\mathrm{payoff}_{\mathrm{asy}}(\phi + \theta, 2\theta).$$

The criterion (3) leads to (43).

7.2 Communication at regular intervals

We have studied "Poisson rate θ" calling strategies because these are simplest to analyze explicitly. A natural alternative is the "regular, rate θ" strategy in which agent i calls a random other agent at times

$$U_i, U_i + \tfrac{1}{\theta}, U_i + \tfrac{2}{\theta}, \ldots \tag{44}$$

where U_i is uniform on $(0, \frac{1}{\theta})$.

Consider first the complete graph case, and the setting (section 2.1) where (for fixed $k \geq 2$) the first k recipients get reward n/k. In this case, for $k = 2$ formula (8) is replaced by

$$P(\textbf{ego} \text{ is second to receive item}) = \int_0^{\min(\frac{1}{\phi}, \frac{1}{\theta})} (1 - \theta u)^{n-2}\ \phi\ du$$

and repeating the analysis in section 2.1 gives exactly the same asymptotics (9,10) as in the Poisson case. Consider instead the general reward framework

The j'th person to learn an item of information gets reward $R(\frac{j}{n})$.

If all agents use rate θ then the distribution function F_θ for receipt time for a typical agent satisfies (as an analog of the logistic equation (15))

$$1 - F_\theta(t) = \int_{t - \frac{1}{\theta}}^{t} \prod_{i \geq 0} \left(1 - F_\theta(s - \tfrac{i}{\theta})\right)\ \theta\ ds. \tag{45}$$

If **ego** switches to rate ϕ then the distribution function $G_{\phi,\theta}$ for **ego**'s receipt time satisfies (as an analog of (16))

$$1 - G_{\phi,\theta}(t) = \int_{t - \frac{1}{\phi}}^{t} \prod_{i \geq 0} \left(1 - F_\theta(s - \tfrac{i}{\phi})\right)\ \phi\ ds. \tag{46}$$

One can now continue the section 2.4 analysis; we do not get useful explicit solutions but the qualitative behavior is similar to the "Poisson calls" case, and in particular the Nash equilibrium is wasteful.

Similarly, on the $N \times N$ grid with nearest neighbor interaction, switching from the "Poisson calls" case to the "regular calls" case preserves the order N^{-1} value of the Nash equilibrium rate θ^{Nash}_N and hence preserves its efficiency.

7.3 Gossip with reward based on audience size

Perhaps a more realistic model for gossip is to replace Rule 2 by

Rule 3. An agent i gets reward c whenever another agent learns an item from i.

For the complete graph and Poisson(θ) strategies we can re-use the section 2.4 analysis to calculate the Nash equilibrium. First suppose all agents use the same rate θ and consider an agent i who receives the information at percentile u. For $j > un$ the j'th agent to receive the information has chance $\frac{1}{j}$ to receive it from agent i, and so the mean reward to agent i is (calculations in the $n \to \infty$ limit) $c \int_u^1 \frac{1}{x}\, dx = -c \log(1-u)$. Suppose now **ego** switches to rate ϕ. Then (calls incur unit cost)

$$\text{payoff}(\phi, \theta) = -\phi + cE(-\log(1 - F_\theta(T_{\phi,\theta})))$$

where the time $T_{\phi,\theta}$ at which **ego** receives the information has distribution function $G_{\phi,\theta}$ at (17), and where F_θ at (14) is the distribution function of the time at which a typical agent receives the information. Now

$$\begin{aligned} E(-\log(1 - F_\theta(T_{\phi,\theta}))) &= \int_0^1 \tfrac{1}{1-u} P(F_\theta(T_{\phi,\theta}) \le u)\ du \\ &= \int_0^1 \tfrac{1}{1-u} (1 - (1-u))^{\phi/\theta}\ du \text{ by (18)} \\ &= \int_0^1 y^{-1} (1-y)^{\phi/\theta}\ dy \end{aligned}$$

and then we calculate

$$\frac{d}{d\phi}\text{payoff}(\phi, \theta) = -1 - c \int_0^1 \tfrac{\log(1-y)}{\theta}\ \tfrac{(1-y)^{\phi/\theta}}{y}\ dy.$$

Now the Nash equilibrium criterion (3) implies

$$\theta_n^{\text{Nash}} \to -c \int_0^1 \tfrac{1-y}{y}\ \log(1-y)\ dy. \tag{47}$$

So switching to this "Rule 3" model preserves the wastefulness of the Nash equilibrium on the compete graph.

However, for the $N \times N$ grid with nearest neighbor interaction, switching to the "Rule 3" models changes the efficient (θ_N^{Nash} is order N^{-1}) Nash equilibrium to a wasteful equilibrium with θ_N^{Nash} becoming order 1.

7.4 Related literature

We do not know any literature closely related to our model. As well as the epidemic and the gossip algorithm topics mentioned in the introduction, and classic applied probability work on *stochastic rumors* [5], other loosely related work includes

- models where agents form networks under conditions where there are costs for maintaining network edges and benefits from being part of a large network [7].
- Prisoners' Dilemma games between neighboring agents on a graph [6].

One can add many other topics which are harder to model mathematically, e.g. diffusion of technological innovations [12] or of ideologies.

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