1.9. Observations on subadditive subsequences.

Subadditive sequences are such handy tools that one is almost forced to explore the extent to which the subadditivity hypotheses can be modified without harm. Two particularly natural modifications are (1) to relax the requirement that subadditivity holds for all m and n and (2) to relax the subadditivity condition itself to a weaker inequality. Marvelously, these two possibilities are closely related, and in both cases, one can provide results that are the best possible.

THEOREM 1.9.1 (DeBruijn–Erdös (1952a)). *If the sequence* $\{a_n\}$ *of real numbers satisfies the subadditivity condition*

$$a_{m+n} \leq a_m + a_n \quad \text{over the restricted range} \quad \frac{1}{2}n \leq m \leq 2n,$$

then $\lim_{n\to\infty} a_n/n = \gamma$, *where* $-\infty \leq \gamma < \infty$ *and* $\gamma = \inf a_n/n$.

Proof. If we set $g(n) = a_n/n$, then subadditivity expresses itself as a convexity relation:

$$(1.36) \quad g(n) \leq kn^{-1}g(k) + (n-k)n^{-1}g(n-k) \quad \text{for all } \frac{1}{2} \leq k^{-1}(n-k) \leq 2.$$

The idea that drives the proof is that when n is chosen so that $g(n)$ is large, then (1.36) can be used to show $g(k)$ is large for many values of k.

To exploit this idea, we choose a subsequence n_τ so that as $\tau \to \infty$, we have $g(n_\tau) \to g^* = \limsup g(n)$. We then have for any $\epsilon > 0$ that there is an $N(\epsilon)$ such that $j \geq N(\epsilon)$ implies $g(j) \leq g^* + \epsilon$, and we can even require of $N(\epsilon)$, that for $n_\tau > N(\epsilon)$, we also have $g(n_\tau) \geq g^* - \epsilon$.

For k such that $n_\tau \leq 3k \leq 2n_\tau$, where $n_\tau \geq 3N(\epsilon)$, we have $n_\tau - k \geq N(\epsilon)$, so inequality (1.36) implies

$$g^* - \epsilon \leq g(n_\tau) \leq kg(k)/n_\tau + (n_\tau - k)(g^* + \epsilon)/n_\tau.$$

This bound simplifies to $g^* - 2n_\tau\epsilon/k \leq g(k)$, and hence for $n_\tau \geq 3N(\epsilon)$, we have

$$g^* - 6\epsilon \leq \min\{g(k) : \ n_\tau \leq 3k \leq 2n_\tau\}.$$

Next, we let $g_* = \liminf_{n\to\infty} g(n)$ and choose m so that $g(m) \leq g_* + \epsilon$. Taking $a_1 = m$, $a_2 = 2m$, and $a_{k+1} = a_k + a_{k-1}$ for $k \geq 3$, we have by (1.36) and induction that $g(a_k) \leq g_* + \epsilon$. We also have $a_{k+1}/a_k \leq \frac{3}{2}$ for $k \geq 3$ since $a_{k+1}/a_k \leq 1 + a_{k-1}/a_{k+1}$ and $a_{k+1} \geq 2a_{k-1}$.

If there is no element of $A = \{a_1, a_2, \ldots\}$ that is also in $\{j : \ n_\tau \leq 3j \leq 2n_\tau\}$, then there would be a k such that

$$a_{k+1}/a_k > \frac{(2n_\tau)/3}{(3n_\tau)/3} = \frac{2}{3};$$

hence we therefore find $\{j : n_\tau/3 \leq j \leq 2n_\tau/3\} \cap A \neq \phi$. By (1.36) and the fact that for all k we have $g(a_k) \leq g_* + \epsilon$, we find $g^* - 6\epsilon \leq g_* + \epsilon$. Since this inequality holds for all $\epsilon > 0$, we conclude that $g^* = g_*$.

A couple of easy points serve to illuminate this theorem. First, it is not hard to check that the hypothesis cannot be relaxed to $n/\alpha \le m \le \alpha n$ for any $\alpha < \frac{1}{2}$. Further, one should note that there is a three-term version of Theorem 1.9.1 where we require $f(n_1 + n_2 + n_3) \le f(n_1) + f(n_2) + f(n_3)$ for all n_i satisfying $\frac{1}{3} \le n_i/n_j \le 3$. Here one should note that for the restricted subadditivity theorem, the three-term result is not contained in the two-term result, but in the basic unrestricted case, the three-term version is just a special case of the two-term version.

The next theorem shows that the basic subadditivity condition can be relaxed in terms of quality as well as extent without doing damage to the conclusion. The theorem also illustrates the gains that are made by relaxing the range over which one needs the subadditive inequality. In many circumstances, it can be difficult (or impossible) to show subadditivity of a sequence over its whole range, and the proof of the next result illustrates that at the very least it can be much more convenient to build sequences that have subadditivity over a restricted set.

THEOREM 1.9.2 (DeBruijn–Erdös (1952a)). *Suppose ϕ is a positive and nondecreasing function that satisfies*

$$\int_1^\infty \frac{\phi(t)}{t^2}\,dt < \infty.$$

If $\{a_n\}$ satisfies the relaxed subadditivity relation

$$a_{n+m} \le a_n + a_m + \phi(n+m) \quad \textit{for } \frac{1}{2}n \le m \le 2n,$$

then as $n \to \infty$, a_n/n converges to $\gamma = \inf a_n/n$.

Proof. The natural idea is to add a term b_n to a_n so that the sum $a_n + b_n$ is subadditive. For this idea to lead to the convergence of a_n/n, one also needs for b_n to be small in the sense that $b_n = o(n)$. We will verify that a suitable choice is

$$b_n = 3n \int_n^\infty \phi(3t)t^{-2}\,dt.$$

By the convergence of the integral of $\phi(x)/t^2$, the added term is $o(n)$, so we just need to check the subadditivity of $c_n = a_n + b_n$. From the monotonicity of ϕ, we have the estimation

$$\int_a^b \phi(3t)t^{-2}\,dt \ge \phi(3a)\int_a^b t^{-2}\,dt = \phi(3a)\{1/a - 1/b\},$$

so substituting into the definition of $\{c_n\}$, we see that

$$\begin{aligned}
&c_{m+n} - c_m - c_n \\
&\quad = a_{n+m} - a_n - a_m - 3m\int_m^{n+m}\phi(3t)t^{-2}\,dt - 3n\int_n^{n+m}\phi(3t)t^{-2}\,dt \\
&\quad \le \phi(n+m) - 3m\phi(3m)\{1/m - 1/(n+m)\} - 3n\phi(3n)\{1/n - 1/(n+m)\}.
\end{aligned}$$

By the range restriction $\frac{1}{2}n \le m \le 2n$ and the monotonicity of ϕ, we have

$$\phi(n+m) \le \phi(3m) \quad \text{and} \quad \phi(n+m) \le \phi(3n),$$

so we have that $c_{m+n} - c_m - c_n$ is bounded above by

$$\phi(3m)\{1 - 3m(1/m - 1/(n+m))\} + \phi(3n)\{1 - 3n(1/n - 1/(n+m))\} \le 0.$$

There are many embellishments of the subadditive limit relation that might deserve to be developed at this point, but one stands out above the rest for the guidance that it offers in problems where one needs information on the rate of convergence of a_n/n to its limit. The central idea is that if one has *both* a subadditive relation and a superadditive relation, then a rate result of some type is guaranteed. One of the oldest and nicest of such results is due to Pólya and Szegö (1924).

LEMMA 1.9.1 (Subadditive Rate Result). *If a real sequence* $a_1, a_2, a_3, \ldots$ *satisfies*

$$(1.37) \qquad a_m + a_n - 1 < a_{m+n} < a_m + a_n + 1 \quad \textit{for all } m, n = 1, 2, 3, \ldots,$$

then there is a finite constant ω *such that*

$$|a_n/n - \omega| < 1 \quad \textit{for all } n.$$

Proof. The proof uses a different principle than the convexity and limit-supremum ideas of the previous results, and this difference is suggestive of how one might proceed in other problems where a two-sided condition is available.

We first note that from $2a_m - 1 < a_{2m} < 2a_m + 1$, we get a bound on the change from doubling the index:

$$(1.38) \qquad \left| \frac{a_{2m}}{2m} - \frac{a_m}{m} \right| < \frac{1}{2m}.$$

We then note that we have convergence of the series

$$\frac{a_1}{1} + \left(\frac{a_2}{2} - \frac{a_1}{a_1} \right) + \left(\frac{a_4}{4} - \frac{a_2}{a_2} \right) + \left(\frac{a_8}{8} - \frac{a_4}{a_4} \right) + \cdots = \lim_{n \to \infty} \frac{a_{2^n}}{2^n} = \omega$$

since in view of (1.38), it is majorized by

$$|a_1| + 2^{-1} + 2^{-2} + 2^{-3} + \cdots.$$

By the immediately preceding theorem, we already know that $a_n/n \to \inf a_n/n$, so we can identify the subsequence limit ω as $\inf a_n/n$. Finally, we note that by telescoping and (1.38), we have

$$\left| \omega - \frac{a_m}{m} \right| < \left| \frac{a_{2m}}{2m} - \frac{a_m}{m} \right| + \left| \frac{a_{4m}}{4m} - \frac{a_{2m}}{2m} \right| + \cdots < \frac{1}{2m} + \frac{1}{4m} + \frac{1}{8m} \cdots = \frac{1}{m}.$$

Naturally, the original proof of this lemma did not call on the theorem of DeBruijn and Erdös since that result came almost thirty years later. Pólya and Szegö (1924) instead used a very clever interpolation argument to extract convergence of a_n/n from the fact that one has convergence along the subsequence $n = 1, 2, 4, 8, 16, \ldots$.