

The Shape Theorem for Route-lengths in Connected Spatial Networks on Random Points

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Abstract

For a connected network on Poisson points in the plane, consider the route-length $D(r, \theta)$ between a point near the origin and a point near polar coordinates (r, θ) , and suppose $ED(r, \theta) = O(r)$ as $r \rightarrow \infty$. By analogy with the shape theorem for first-passage percolation, for a translation-invariant and ergodic network one expects $r^{-1}D(r, \theta)$ to converge as $r \rightarrow \infty$ to a constant $\rho(\theta)$. It turns out there are some subtleties in making a precise formulation and a proof. We give one formulation and proof via a variant of the subadditive ergodic theorem wherein random variables are sometimes infinite.

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Short title: Shape Theorem for Spatial Networks.

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1 Introduction

This paper is a technical part of a broader project investigating connected random spatial networks, in particular networks built over a Poisson process of points ξ in the plane. See [3] for the least technical overview. In any such network there is a (shortest-) route-length $d(\xi, \xi')$ between each pair of points of the Poisson process, which by connectivity is finite. Under weak assumptions (see [1] for a sufficient condition) one expects the mean route-length to grow only linearly with Euclidean distance. Consider a (deliberately vague, for now) notion

$D(r, \theta)$ is the route-length between a point ξ near the origin and a point ξ' near polar coordinates (r, θ) ,

and suppose we know $ED(r, \theta) = O(r)$ – that is, suppose we have a linear upper bound on mean route-length. If the random network has translation-invariant and ergodic distribution, then we intuitively expect that there should be a limit constant $\rho(\theta) = \lim_{r \rightarrow \infty} r^{-1}ED(r, \theta)$ and that in some sense renormalized random route-lengths should converge to the limit constant: $r^{-1}D(r, \theta) \rightarrow \rho(\theta)$. This intuition arises in part from an analogy with the shape theorem for first-passage percolation [6, 7] on the edges of the grid $\mathbb{Z}^2$. In the usual such model the times $\tau(e)$ attached to edges e are assumed i.i.d., but the proof (based on the subadditive ergodic theorem) extends to the setting where the $\tau(e)$ are assumed only to be ergodic translation-invariant. Studying route-lengths in random networks built over Poisson point processes is perhaps the most natural continuum analog of studying first-passage times in such lattice models. Two previously studied special continuum models, superficially different, can be fitted into our general setup – see section 1.3.

1.1 Formulating a theorem

In the broader project we visualize a spatial network as having vertices and edges; in most contexts, summary statistics such as “mean edge-length per unit area” are natural and important. In the specific context of this paper, only the induced route-lengths $d(\xi, \xi')$ are of interest, so we will dispense with other structure and work within the following set-up throughout this paper.

- (A1) There is a Poisson process of points $\Xi = \{\xi\}$ of intensity one, on $\mathbb{R}^2$.
- (A2) On each realization of Ξ there are non-negative finite “route-lengths” $d(\xi, \xi') = d(\xi', \xi)$ which are assumed (only) to satisfy the triangle inequality.

(A3) The distribution of the whole structure $\{\xi; d(\xi, \xi')\}$ is translation-invariant. That is, invariant w.r.t. the action of the group $(T_{a,b}; (a,b) \in \mathbb{R}^2)$ where $T_{a,b}(x,y) = (x+a, y+b)$. Moreover for each $(a,b) \neq (0,0)$ the action of $T_{a,b}$ is ergodic.

Some specific examples are mentioned briefly in section 1.2, though our emphasis is on the generality of the assumptions. So it is worth mentioning what we are not assuming. We are **not** assuming

(B1) rotational invariance

(B2) that $d(\xi, \xi') \geq |\xi - \xi'|$ (implicit in the underlying “route-lengths” story)

(B3) any kind of “locality” for the route-lengths $d(\xi, \xi')$.

In particular, for a nearby pair ξ, ξ' the route-length $d(\xi, \xi')$ may depend on the entire configuration Ξ . Finally, we often describe points in $\mathbb{Z}^2$ by their radial coordinates. So (r, θ) denotes a point $z \in \mathbb{Z}^2$; conversely, given $z \in \mathbb{Z}^2$ we write (r_z, θ_z) for its radial coordinates.

At first sight it looks easy to state and prove a theorem under assumptions (A1-A3) – just find a suitable formalization of the vague notion $D(r, \theta)$ above, to which the subadditive ergodic theorem can be applied. But actually carrying this through seems surprisingly subtle. One attempt is to condition on points being planted at the origin and at (r, θ) : the remaining points being still distributed as the Poisson point process, one can define a conditioned network and then define $D(r, \theta)$ to be the route-lengths between the planted points in the conditioned network. However, for rather trivial reasons the desired result is simply not true in this interpretation (see section 2.2). A second attempt is to interpret $D(r, \theta)$ as originally stated for the point *nearest* the origin and the point *nearest* (r, θ) ; this makes a precise definition but it seems hard to work directly with this definition. A third attempt is to start by finding some feature to which one can apply the subadditive ergodic theorem. For instance, let ξ_n be the leftmost point in the semi-infinite strip $[n, \infty) \times [-1, 1]$. One can certainly apply the subadditive ergodic theorem to the array $(d(\xi_m, \xi_n))$ to conclude that (under an integrability assumption) $n^{-1}d(\xi_0, \xi_n)$ converges a.s. to a constant limit $\rho(0)$. At first sight this approach seems to resolve the whole issue. But the precise hypotheses and conclusions are tied to the particular feature initially chosen, and it seems technically hard to reconcile the results from different choices.

We adopt a fourth approach, aimed at a more natural type of conclusion. Write A, B for bounded subsets of $\mathbb{R}^2$ and for $z \in \mathbb{R}^2$ write $z + B = \{z + z' : z' \in B\}$. To motivate the precise definitions (1,2) below, consider $\xi \in A$ and $\xi' \in z + B$ with r_z large; the route-length $d(\xi, \xi')$ provides one interpretation of our initial vague notion $D(r_z, \theta_z)$, which we

want to prove is approximately the (deterministic) length $r_z \rho(\theta_z)$. To avoid conditioning on existence of points in sets, we sum: $\sum_{\xi \in A} \sum_{\xi' \in z+B} d(\xi, \xi')$ should be $N(A)N(z+B)r_z \rho(\theta_z) \pm o(r_z)$ where $N(\cdot)$ is the counting process of Ξ . But we can avoid writing $N(\cdot)$ by rewriting the approximation as $\sum_{\xi \in A} \sum_{\xi' \in z+B} |d(\xi, \xi') - r_z \rho(\theta_z)| = o(r_z)$. This prompts the following definitions.

For $c \geq 0$ define a random variable

$$S(A, B; c) := \sum_{\xi \in A} \sum_{\xi' \in B} |d(\xi, \xi') - c|. \quad (1)$$

Say the random network has the L^1 *shape property* if there exist constants $\rho(\theta)$ such that, for all bounded A, B ,

$$r_z^{-1} E S(A, z+B; r_z \rho(\theta_z)) \rightarrow 0 \text{ as } r_z \rightarrow \infty. \quad (2)$$

Because S is an additive set function, it is enough to prove (2) when A and B are sufficiently large (or sufficiently small) squares centered at the origin; and in the latter case we see how this notion provides a formalization of the idea behind $D(r, \theta)$.

Having decided on the conclusion we seek, what hypotheses do we need? Obviously it is necessary that the corresponding linear upper bound holds: for all bounded A, B ,

$$E \sum_{\xi \in A} \sum_{\xi' \in z+B} d(\xi, \xi') = O(r_z) \text{ as } r_z \rightarrow \infty. \quad (3)$$

We conjecture that (3) is sufficient (see section 2.1 for precise statement). However in this paper we work under the analogous, but stronger, L^2 assumption: for all bounded A, B ,

$$\sup_z \frac{E \sum_{\xi \in A} \sum_{\xi' \in z+B} d^2(\xi, \xi')}{\max(1, r_z^2)} < \infty. \quad (4)$$

Again, it is enough to verify this when A and B are sufficiently small squares centered at the origin.

Theorem 1 *Under the standing assumptions (A1 - A3), if hypothesis (4) holds then the L^1 shape property (2) holds. Moreover*

$$\sup_{\theta_2 \neq \theta_1} \frac{|\rho(\theta_2) - \rho(\theta_1)|}{|\theta_2 - \theta_1|} < \infty. \quad (5)$$

This is proved in section 2, though the main work is delegated to a new “sub-additive ergodic theorem with missing values”, Proposition 4, stated and proved in section 3. A conjectured stronger “a.s. shape theorem” conclusion is discussed briefly in section 2.1. Obviously, if we add the assumption of rotational invariance then $\rho(\theta)$ is constant.

1.2 Examples using route-lengths

In all these examples, the $d(\xi, \xi')$ are minimal route-lengths within given networks.

Proximity graphs [5, 3]. This family of graphs (our main example) is defined by:

(ξ, ξ') is an edge iff the set $A(\xi, \xi')$ contains no other point of the Poisson process

for different choices of $A(\xi, \xi')$, a fundamental choice (giving the *relative neighborhood graph*) being

$A(\xi, \xi')$ is the intersection of the disc with center ξ and radius $|\xi' - \xi|$ and the disc with center ξ' and radius $|\xi' - \xi|$.

Other graphs in the family use subsets of this A and hence are supergraphs of the relative neighborhood graph and hence can only have smaller route-lengths. The purpose of the companion paper [1] is to give a general property that implies our present condition (4), and to verify this property for the relative neighborhood graph. It follows that (4) holds for the relative neighborhood graph (and hence for every proximity graph) on a Poisson point process. Because edges are defined by a deterministic rule, proximity graphs inherit the stationary ergodicity property (A3) from the trivial tail σ -field property of the Poisson process. So our Theorem 1 applies, and by rotational invariance $\rho(\theta)$ is a constant ρ , depending on the model. Monte Carlo estimates of ρ (around 1.4 for the relative neighborhood graph) can be seen in [3] but we do not know any explicit rigorous upper bound.

The general condition in [1] might be applicable to other models, but the examples below can be handled more directly.

Lattice-based networks. One can start with (for instance) the square grid lattice as a network, and simply connect each Poisson point to the nearest grid point. One can see directly that this random network satisfies

the shape property with $\rho(\theta) = |\cos \theta| + |\sin \theta|$. This conclusion remains true if we make the network be translation-invariant and ergodic by replacing the deterministically-spaced grid lines by randomly-spaced ones.

Asymptotically efficient networks. It is not surprising that there are networks which are “optimal” in the sense $\rho(\cdot) \equiv 1$. It is at first sight surprising that one can find such networks whose length-per-unit-area is arbitrarily close to the minimum possible (over all connected networks – attained by the Steiner tree) length-per-unit-area. But this can be achieved by the simple device of superimposing, over the Steiner tree, a sparse Poisson line process. This construction is studied in detail in [2].

The Hammersley network. This network, introduced in [3], has the remarkable property that at each point ξ there are exactly 4 edges, one in each of the four quadrant directions (i.e. between East and North, etc). This network has not been studied carefully, but it is plausible one can use known properties of the underlying Hammersley process to prove directly that the shape property holds with $\rho(\theta) = \rho_0(|\cos(\theta - \pi/4)| + |\sin(\theta - \pi/4)|)$ for some constant ρ_0 .

1.3 Other examples

Suppose we remove the “satisfy the triangle inequality” requirement from (A2), to get instead

(A2*) On each realization of Ξ there are “costs” $0 < c(\xi, \xi^*) = c(\xi^*, \xi) \leq \infty$.

One can now define $d(\xi, \xi')$ as the cost of the minimum-cost path from ξ to ξ' , and this makes d satisfy the triangle inequality. So, provided d is always finite, (A2) holds. If the other hypotheses of Theorem 1 hold for d , then the conclusion of Theorem 1 gives the L^1 shape property for d . The following two particular cases have been studied previously by direct methods which establish the a.s. shape theorem; our Theorem 1 applies (assuming second moments in (a)) to give the L^1 shape theorem.

(a) Take the Delaunay triangulation on the Poisson points, and then take $c(\xi, \xi')$ to be i.i.d. with finite mean on the edges of the triangulation (and $= \infty$ elsewhere): [9].

(b) Fix $\alpha > 1$ and set $c(\xi, \xi') = |\xi - \xi'|^\alpha$: [4].

2 Reducing the proof of Theorem 1 to a subadditivity result

Write A, B for bounded subsets of $\mathbb{R}^2$ with non-zero area. Write $N(A)$ for the number of points of the Poisson point process Ξ in A and write $G(A)$ for the “good” event $\{N(A) \geq 1\}$. On $G(A)$ let ξ_A be a uniform random point of $\Xi \cap A$. Note that hypothesis (4) implies

$$E[d(\xi_{z_1+A}, \xi_{z_2+B}) 1_{G(z_1+A) \cap G(z_2+B)}] \leq \kappa(A, B) \max(1, |z_1 - z_2|) \quad (6)$$

where $\kappa(A, B) < \infty$ depends only on A, B .

Fix $\theta \in [0, 2\pi)$ and fix a bounded subset $A \subset \mathbb{R}^2$ of non-zero area. Write z_n for the point with radial coordinates (nr_0, θ) , where r_0 is sufficiently large that the sets $z_n + A$ are disjoint. So $G(z_n + A)$ is the event $\{N(z_n + A) \geq 1\}$, and on $G(z_n + A)$ let ξ_{z_n+A} be a uniform random point of $\Xi \cap (z_n + A)$.

Consider the array of random variables

$$\begin{aligned} X_{mn} &= d(\xi_{z_m+A}, \xi_{z_n+A}) \text{ on } G(z_m + A) \cap G(z_n + A) \\ &= \infty \text{ otherwise.} \end{aligned} \quad (7)$$

Note (A2) implies the triangle inequality

$$\text{for } \ell < m < n, X_{\ell n} \leq X_{\ell m} + X_{mn} \text{ on } 1_{G(z_\ell+A) \cap G(z_m+A) \cap G(z_n+A)}. \quad (8)$$

Proposition 4, stated and proved in section 3, is tailored to this setting. Specifically, hypothesis (i) is (7), (ii) is (8), (iii) follows from Poisson independence, (iv) from (A3) and (v) from (4). The conclusion of Proposition 4 is that there exists a constant $0 \leq \rho(\theta) < \infty$ such that

$$E \left[\left| \frac{d(\xi_A, \xi_{z_n+A})}{nr_0} - \rho(\theta) \right| 1_{G(A) \cap G(z_n+A)} \right] \rightarrow 0. \quad (9)$$

This is the main ingredient of the proof; the argument below continues with the details of converting (9) into the stated conclusion of Theorem 1. The typography in (9) is potentially confusing; note we are multiplying an absolute value by an indicator, not taking a conditional expectation.

A priori the limit constant $\rho(\theta)$ in (9) might depend on A and on r_0 . We first show it does not depend on r_0 ; more precisely, we will show

$$E \left[\left| \frac{d(\xi_A, \xi_{(r,\theta)+A})}{r} - \rho(\theta) \right| 1_{G(A) \cap G((r,\theta)+A)} \right] \rightarrow 0 \text{ as } r \rightarrow \infty. \quad (10)$$

Let us give the argument for (10) in some detail, intending to omit similar details later. Write $r = (n_1 + n_2)r_0 + \gamma$ for some $0 \leq \gamma < r_0$. Let $r \rightarrow \infty$ while choosing $n_1 = n_1(r) \rightarrow \infty$ and $n_2 = n_2(r) \rightarrow \infty$. By (9) and translation-invariance

$$E \left[\left| \frac{d(\xi_A, \xi_{(n_1 r_0, \theta) + A}) - n_1 r_0 \rho(\theta)}{r} \right| 1_{G(A) \cap G((n_1 r_0, \theta) + A)} \right] \rightarrow 0$$

$$E \left[\left| \frac{d(\xi_{(n_1 r_0 + \gamma, \theta) + A}, \xi_{(r, \theta) + A}) - n_2 r_0 \rho(\theta)}{r} \right| 1_{G((n_1 r_0 + \gamma, \theta) + A) \cap G((r, \theta) + A)} \right] \rightarrow 0.$$

Combining these with (6) applied to $\xi_{(n_1 r_0, \theta) + A}$ and $\xi_{(n_1 r_0 + \gamma, \theta) + A}$, and using the triangle inequality for $d(\cdot, \cdot)$,

$$E \left[\left| \frac{d(\xi_A, \xi_{(r, \theta) + A})}{r} - \rho(\theta) \right| 1_{G(A) \cap G((r, \theta) + A) \cap G((n_1 r_0, \theta) + A) \cap G((n_1 r_0 + \gamma, \theta) + A)} \right] \rightarrow 0 \text{ as } r \rightarrow \infty.$$

This expression differs from (10) only by the inclusion of the restriction to $G((n_1 r_0, \theta) + A) \cap G((n_1 r_0 + \gamma, \theta) + A)$, an event which has probability at least $1 - p$ for some $p = p(r_0, A) < 1$. Given k we can choose (for large r) k different values of n_1 such that the k corresponding events are independent because the underlying sets are disjoint; it follows that

$$E \left[\left| \frac{d(\xi_A, \xi_{(r, \theta) + A})}{r} - \rho(\theta) \right| 1_{G(A) \cap G((r, \theta) + A) \cap H(r, k)} \right] \rightarrow 0 \text{ as } r \rightarrow \infty$$

for certain events $H(r, k)$ such that $P(H(r, k)) \geq 1 - p^k$ for large r . Letting $k \rightarrow \infty$ and appealing to the L^2 bound (4) establishes (10).

Now consider two subsets $A \subset A'$. Could the two constants $\rho(\theta)$ and $\rho'(\theta)$ in (10) be different? When we make independent choices of random points ξ_{z+A} and $\xi_{z+A'}$ there is some fixed probability $p(A, A') > 0$ that the two random points are the same, and it easily follows that the limit constants must be equal. That is, $\rho(\theta)$ does not depend on choice of A .

Next we prove the Lipschitz property (5). Fix θ_1 and θ_2 . The triangle inequality and (6) give

$$E[|d(\xi_A, \xi_{(r, \theta_1) + A}) - d(\xi_A, \xi_{(r, \theta_2) + A})| 1_{G(A) \cap G((r, \theta_1) + A) \cap G((r, \theta_2) + A)}] \leq$$

$$E[d(\xi_{(r, \theta_1) + A}, \xi_{(r, \theta_2) + A})| 1_{G(A) \cap G((r, \theta_1) + A) \cap G((r, \theta_2) + A)}] \leq \kappa(A, A) \min(1, r|\theta_2 - \theta_1|).$$

Applying (10),

$$|\rho(\theta_2) - \rho(\theta_1)| P^3(G(A)) \leq \kappa(A, A) |\theta_2 - \theta_1|$$

and now any choice of A establishes (5).

Next we want to prove the analog of (10) where the angle is not fixed. That is, for $z = (r_z, \theta_z)$ with $r_z \rightarrow \infty$, we claim

$$E \left[\left| \frac{d(\xi_A, \xi_{z+A})}{r_z} - \rho(\theta_z) \right| 1_{G(A) \cap G(z+A)} \right] \rightarrow 0 \text{ as } r_z \rightarrow \infty. \quad (11)$$

By compactness and continuity of $\rho(\cdot)$ we can reduce to the case (r_n, θ_n) where $\theta_n \rightarrow \theta$, and it is enough to prove

$$E \left[\left| \frac{d(\xi_A, \xi_{(r_n, \theta_n)+A})}{r_n} - \rho(\theta) \right| 1_{G(A) \cap G((r_n, \theta_n)+A)} \right] \rightarrow 0 \text{ as } r_n \rightarrow \infty. \quad (12)$$

Here we repeat the format of the argument for (10). Take $r_n^* = (1 + o(1))r_n$. Apply the triangle inequality to $\xi_A, \xi_{(r_n^*, \theta)+A}, \xi_{(r_n, \theta_n)+A}$, apply the fixed- θ result (10) to the first distance and apply (6) to the second distance; we deduce the analog of (12) with the extra term $1_{G((r_n^*, \theta)+A)}$. But this is true for each of multiple possible choices for r_n^* , so we can deduce (12) and thence (11).

To complete the proof we need to convert (11) into an assertion involving the sums $S(A, z + B; r_z \rho(\theta_z))$ appearing in (2). Let us state the underlying logical structure carefully; note there is no assumption that the $(Y_{ij}^{(n)})$ are independent of $(N_1^{(n)}, N_2^{(n)})$.

Lemma 2 *Fix λ_1, λ_2 . For each n let $(Y_{ij}^{(n)}, 1 \leq i \leq N_1^{(n)}, 1 \leq j \leq N_2^{(n)})$ be an array of nonnegative random variables, and suppose that $N_1^{(n)}$ and $N_2^{(n)}$ are independent with $\text{Poisson}(\lambda_1)$ (resp. λ_2) distributions. On the event $\{N_1^{(n)} \geq 1, N_2^{(n)} \geq 1\}$, and conditional on the entire collection $(Y_{ij}^{(n)}, 1 \leq i \leq N_1^{(n)}, 1 \leq j \leq N_2^{(n)})$, take $(U_1^{(n)}, U_2^{(n)})$ to be independent with $\text{Uniform}[1, 2, \dots, N_1^{(n)}]$ and $\text{Uniform}[1, 2, \dots, N_2^{(n)}]$ distributions. Suppose*

$$Y_{U_1^{(n)}, U_2^{(n)}}^{(n)} 1_{(N_1^{(n)} \geq 1, N_2^{(n)} \geq 1)} \rightarrow 0 \text{ in probability as } n \rightarrow \infty.$$

Then

$$\sum_{i=1}^{N_1^{(n)}} \sum_{j=1}^{N_2^{(n)}} Y_{ij}^{(n)} \rightarrow 0 \text{ in probability as } n \rightarrow \infty.$$

Proof. It is enough to prove the conclusion restricted to $\{1 \leq N_1^{(n)} \leq L, 1 \leq N_2^{(n)} \leq L\}$ for fixed L . But with this restriction, the hypothesis

implies $\max_{ij} Y_{ij}^{(n)} \rightarrow 0$ in probability, which in turn implies the conclusion. $\blacksquare$

Now consider the setting of (11). As $r_z \rightarrow \infty$ the array

$$\left(\left| \frac{d(\xi, \xi')}{r_z} - \rho(\theta_z) \right|, \xi \in A, \xi' \in z + A \right)$$

satisfies the assumptions of Lemma 2, and the conclusion is

$$r_z^{-1} S(A, z + A; r_z \rho(\theta_z)) \rightarrow 0 \text{ in probability.}$$

The L^2 bound (4) extends this to

$$r_z^{-1} ES(A, z + A; r_z \rho(\theta_z)) \rightarrow 0$$

which is enough to establish the L^1 shape property.

2.1 The conjectured a.s. shape theorem

Implicit in the underlying picture of route-lengths in spatial networks is that route-lengths are at least as big as Euclidean distance:

$$d(\xi, \xi') \geq |\xi - \xi'|. \quad (13)$$

This was not assumed for Theorem 1; assuming it here, we see $\rho(\theta) \geq 1$. Using “triangle inequality” arguments as in the previous section, it is easy to check that

$$\mathbf{B} := \{z = (r, \theta) : r \leq 1/\rho(\theta)\}$$

defines a *convex* subset of the unit disc. A natural informal statement of a shape theorem is that, if we plant one Poisson point ξ_0 at the origin, then for large ℓ the set of points at route-length at most ℓ from ξ_0 is approximately the set of points within $\ell \mathbf{B}$. So one can formalize the *a.s. shape property* as follows, in the context of a planted point ξ_0 at the origin. For each $\varepsilon > 0$ there exists random $L(\varepsilon) < \infty$ such that for all $\ell > L(\varepsilon)$

$$\Xi \cap (1 - \varepsilon)\ell \mathbf{B} \subseteq \{\xi \in \Xi : d(\xi, \xi_0) \leq \ell\} \subseteq (1 + \varepsilon)\ell \mathbf{B}.$$

This has been proved by direct methods in the two special models of section 1.3. Because our Theorem 1 conclusion involves L^1 convergence instead of a.s. convergence, it implies only a somewhat weaker result; and also our “ L^2 bounded” assumption is stronger than seems necessary. In other words, the natural conjecture suggested by the analogy with the shape theorem for first-passage percolation is as follows.

Conjecture 3 *Under the standing assumptions (A1 - A3), and (3) and (13), there exists a convex set $\mathbf{B}$ such that the a.s. shape property holds.*

And though we work throughout with an underlying Poisson point process, such a result might be expected to hold for any ergodic translation-invariant point process.

2.2 A minor counter-example

Take a model to which Theorem 1 applies with $\rho(\theta) > 1$. Choose $r_k \uparrow \infty$ fast and $\delta_k \downarrow 0$. Modify the model by putting a straight line link between each pair of points whose distance apart is in $\cup_k [r_k, r_k + \delta_k]$. By making the $\delta_k \downarrow 0$ sufficiently fast and appealing to the “minimum cost path” device of section 1.3, the hypotheses and conclusion of Theorem 1 remain true with the same limit $\rho(\theta)$ – the extra links make no difference to route-length between typical pairs. But if we had attempted to formulate the theorem using “ $D(r, \theta)$ = distance between two points at distance r apart” then we would not get a $r \rightarrow \infty$ limit, because of the exceptional $r \in \cup_k [r_k, r_k + \delta_k]$.

3 A subadditive ergodic theorem with missing values

We develop a variation of Kingman’s subadditive ergodic theorem (see e.g. [8]) in which the random variables $(X_{ij}, 0 \leq i < j < \infty)$ are sometimes undefined (in which case we will set the value to ∞ , though that isn’t quite the natural interpretation in our application). Consider a sequence $(G_i, 0 \leq i < \infty)$ of “good” events, and write I_i for the indicator 1_{G_i} and write $I_{ij} = I_i I_j = 1_{G_i \cap G_j}$. Our assumptions are

- (i) $0 \leq X_{ij} \leq \infty$; $X_{ij} < \infty$ on $G_i \cap G_j$.
- (ii) For $i < j < k$, $X_{ik} \leq X_{ij} + X_{jk}$ on $G_i \cap G_j \cap G_k$.
- (iii) The process $(I_i, 0 \leq i < \infty)$ is independent Bernoulli(δ) for fixed $0 < \delta < 1$.
- (iv) Setting $\mathbf{X}_i = (I_i, X_{i,i+k}, 1 \leq k < \infty)$, the process $(\mathbf{X}_i, 0 \leq i < \infty)$ is stationary and ergodic.
- (v) $\sup_{n \geq 1} n^{-2} E[X_{0n}^2 I_{0n}] < \infty$.

Mostly these are the obvious analogs of the usual assumptions [8]. Note that in the usual setting we have a trivial implication

$$\text{if } EX_{01} < \infty \text{ then } \sup_n n^{-1} EX_{0n} \leq EX_{01} < \infty$$

whereas in our setting the implication:

$$\text{if } E[X_{01}I_{01}] < \infty \text{ then } \sup_n n^{-1}E[X_{0n}I_{0n}] < \infty$$

is not trivial (we don't know if it is true). The latter would be the natural hypothesis in our setting, but to make our straightforward proof technique work we make the stronger L^2 assumption (v). Also to keep matters simple, we assume ergodicity and seek only L^1 convergence.

Proposition 4 *Assume (i)-(v). Then there exists a constant $0 \leq c < \infty$ such that $E[|\frac{X_{0n}}{n} - c|I_{0n}] \rightarrow 0$.*

Proof. We compare the given process with another process in which one is allowed to use the “bad” indices, but with high penalty. Fix large K . Define

$$\begin{aligned} \tilde{X}_{ij} &= X_{ij} \text{ on } G_i \cap G_j \\ \tilde{X}_{i,i+1} &= K \text{ on the complement } (G_i \cap G_{i+1})^c \text{ of } G_i \cap G_{i+1} \\ \tilde{X}_{ij} &\text{ undefined, otherwise.} \end{aligned}$$

Now define a process

$$Y_{ij} = \min \left(\tilde{X}_{i_0, i_1} + \tilde{X}_{i_1, i_2} + \dots + \tilde{X}_{i_{m-1}, i_m} \right),$$

the minimum over $i = i_0 < i_1 < i_2 < \dots < i_m = j$ such that each $\tilde{X}_{i_{u-1}, i_u}$ is defined. Observe that Y_{ij} is always defined and finite, and is subadditive. Also, because $Y_{01} = X_{01}I_{01} + K(1 - I_{01})$ we have $EY_{01} < \infty$. So we can apply Kingman's subadditive ergodic theorem to deduce there exists a constant $0 \leq c^{(K)} < \infty$ such that

$$E \left| \frac{Y_{0n}^{(K)}}{n} - c^{(K)} \right| \rightarrow 0 \text{ as } n \rightarrow \infty \quad (14)$$

where we now write $Y^{(K)}$ to emphasize dependence on K .

Note that the L^1 convergence in (14) implies

$$\delta^2 c^{(K)} = \lim_{n \rightarrow \infty} E[c^{(K)}I_{0n}] = \lim_{n \rightarrow \infty} E\left[\frac{Y_{0n}^{(K)}}{n}I_{0n}\right].$$

Now by assumption (v) we have

$$B_1 := \sup_n n^{-1}E[X_{0n}I_{0n}] < \infty$$

and from the definition

$$Y_{0n}^{(K)} \leq X_{0n} \text{ on } G_0 \cap G_n. \quad (15)$$

So $\delta^2 c^{(K)} \leq B_1$. Also from the definition we see that $Y_{0n}^{(K)}$ is non-decreasing in K . Hence so is $c^{(K)}$, and so we can define the limit

$$c := \lim_{K \rightarrow \infty} c^{(K)}$$

for which $0 \leq c < \infty$.

The essential issue is to bound the difference in (15). Fix K and n . By definition there is some path $0 = i_0 < i_1 < i_2 < \dots < i_m = n$ for which

$$Y_{0n}^{(K)} = \tilde{X}_{i_0, i_1} + \tilde{X}_{i_1, i_2} + \dots + \tilde{X}_{i_{m-1}, i_m}. \quad (16)$$

Some steps on the path are of the form (for some i)

$$(i_u, i_{u+1}) = (i, i+1) \text{ and } G_i \cap G_{i+1} \text{ fails, so } \tilde{X}_{i_u, i_{u+1}} = K.$$

Write $\mathcal{I}$ for the random set of i for which this occurs (for some i_u), and note for later use

$$Y_{0n}^{(K)} \geq K|\mathcal{I}|. \quad (17)$$

Now consider a maximal run $[b, b']$ of bad events; that is, G_i occurs for $i = b-1$ and $i = b'+1$ but not for $b \leq i \leq b'$. Then the path in (16) must either use all the edges $(b-1, b)$, $(b, b+1)$, $\dots$, $(b'-1, b')$, $(b', b'+1)$ or none of them. If it uses all of them, replace the path segment $b-1 \rightarrow b \rightarrow \dots \rightarrow b' \rightarrow b'+1$ by the single edge $b-1 \rightarrow b'+1$, that is replace the part $\tilde{X}_{b-1, b} + \dots + \tilde{X}_{b', b'+1}$ of the sum (16) by $X_{b-1, b'+1}$. Make this replacement for each bad run touched by the path, and assume we are on $G_0 \cap G_n$ so there are no endpoint issues. This converts (16) into a new sum

$$Z_{0n}^{(K)} = X_{j_0, j_1} + X_{j_1, j_2} + \dots + X_{j_{q-1}, j_q} \quad (18)$$

where all the steps are between good indices, and so by the subadditivity assumption (ii) we have

$$X_{0n} \leq Z_{0n}^{(K)} \text{ on } G_0 \cap G_n.$$

The net effect of this conversion can be written precisely as

$$Z_{0n}^{(K)} - Y_{0n}^{(K)} = \sum_{(b, b') \in \mathcal{B}} X_{b-1, b'+1} - K|\mathcal{I}| \text{ on } G_0 \cap G_n$$

where $\mathcal{B}$ is the set of bad runs touched by the path. So we can bound the difference in (15) rather crudely as

$$E \left[(X_{0n} - Y_{0n}^{(K)}) I_{0n} \right] \leq E \left[I_{0n} \sum_{(b,b') \in \mathcal{B}} X_{b-1,b'+1} \right]. \quad (19)$$

Recall hypothesis (v):

$$B_2 := \sup_m m^{-2} E[X_{0m}^2 I_{0m}] < \infty.$$

For $0 \leq i, j \leq n$, $j - i \geq 2$ write Λ_{ij} for the event $\{(i+1, j-1) \in \mathcal{B}\}$ and note $\Lambda_{ij} \subset I_{ij}$. Using the Cauchy-Schwarz inequality.

$$\begin{aligned} E[1_{\Lambda_{ij}} X_{ij} I_{0n}] &\leq \sqrt{P(\Lambda_{ij} \cap I_{0n})} \sqrt{E[X_{ij}^2 I_{ij}]} \\ &\leq \sqrt{P(\Lambda_{ij} \cap I_{0n})} (j-i) B_2^{1/2} \end{aligned} \quad (20)$$

the second inequality by the stationarity assumption (iv).

Set $p_{ij} = P(\Lambda_{ij} \cap I_{0n})$ and note

$$p_{ij} \leq P(\Lambda_{ij}) \leq P(G_{i+1}^c \cap \dots \cap G_{j-1}^c) = (1 - \delta)^{j-i-1}$$

by assumption (iii). Because the sum in (19) can be written as $\sum_{ij} 1_{\Lambda_{ij}} X_{ij}$, we can combine (19) and (20) to get

$$E[(X_{0n} - Y_{0n}^{(K)}) I_{0n}] \leq B_2^{1/2} \sum_{i=0}^{n-2} \sum_{j=i+2}^n (j-i) p_{ij}^{1/2}. \quad (21)$$

Now consider the double sum above with $p_{ij}^{1/2}$ replaced by p_{ij} . That is, consider

$$\sum_{i=0}^{n-2} \sum_{j=i+2}^n (j-i) p_{ij} = E I_{0n} \sum_{i=0}^{n-2} \sum_{j=i+2}^n (j-i) 1_{\Lambda_{ij}} \quad (22)$$

$$\begin{aligned} &= E I_{0n} |\mathcal{I}| \text{ for the random set } \mathcal{I} \text{ in (17)} \\ &\leq K^{-1} E I_{0n} Y_{0n}^{(K)} \text{ by (17)} \\ &\leq K^{-1} E I_{0n} X_{0n} \text{ by (15)} \\ &\leq B_1 n / K. \end{aligned} \quad (23)$$

Now an elementary inequality (stated and proved as Lemma 5 below) bounds the right side of (21) in terms of the left side of (22). Combining this inequality with (21,23) gives: for any $J \geq 2$,

$$B_2^{1/2} n^{-1} E[(X_{0n} - Y_{0n}^{(K)}) I_{0n}] \leq \sum_{j=J+1}^{\infty} j(1-\delta)^{(j-1)/2} + J\sqrt{B_1/K}.$$

Taking $J = K^{1/2}$ we see

$$\lim_{K \rightarrow \infty} \sup_n n^{-1} E[(X_{0n} - Y_{0n}^{(K)}) I_{0n}] = 0$$

which by the L^1 convergence (14) and the inequality (15) implies

$$\lim_{K \rightarrow \infty} \limsup_n E[|n^{-1} X_{0n} - c^{(K)}| I_{0n}] = 0$$

establishing Proposition 4 for $c = \lim_{K \rightarrow \infty} c^{(K)}$ which was previously shown to be finite.

Lemma 5 *Let $0 < \eta < 1$ and let $(p_{ij}, 0 \leq i, j \leq n, j - i \geq 2)$ be constants such that $0 \leq p_{ij} \leq \eta^{j-i-1}$. Then for any $J \geq 2$*

$$n^{-1} \sum_{i=0}^{n-2} \sum_{j=i+2}^n (j-i) p_{ij}^{1/2} \leq \sum_{j=J+1}^{\infty} j \eta^{(j-1)/2} + J n^{-1/2} \sqrt{\sum_{i=0}^{n-2} \sum_{j=i+2}^n (j-i) p_{ij}}.$$

Proof. Fix i and set $q_j = p_{i,i+j}$ for $2 \leq j \leq n-i$. Then

$$\begin{aligned} \sum_{j=2}^{n-i} j q_j^{1/2} &\leq \sum_{j=J+1}^{\infty} j \eta^{(j-1)/2} + \sum_{j=2}^J j q_j^{1/2} \\ &\leq \sum_{j=J+1}^{\infty} j \eta^{(j-1)/2} + J \sqrt{\sum_{j=2}^J j q_j} \end{aligned}$$

by the Cauchy-Schwarz inequality. Setting $d_i = \sum_{j=2}^J j q_j$, another use of Cauchy-Schwarz gives

$$\sum_{i=0}^{n-2} \sqrt{d_i} \leq n^{1/2} \sqrt{\sum_{i=0}^{n-2} d_i}$$

and the result follows.

4 Shape Theorem

Assumptions:

for all bounded A, B ,

$$\sup_z \frac{E \sum_{\xi \in A} \sum_{\xi' \in z+B} d^p(\xi, \xi')}{\max(1, r_z^p)} < \infty. \quad (24)$$

For $x \in \mathbb{R}^2$, denote by x^* the nearest point from x in the Poisson point process, then define $D^*(x, y) = d(x^*, y^*)$. Thus, D^* is a random distance which is defined on the whole plane. Also, for each $z \in \mathbb{R}^2$, the law of the joint law $(D^*(x+, y+z))_{x, y \in \mathbb{R}^2}$ does not depend on z , which allows to use Kingman's subadditive theorem as soon as the integrability assumptions are satisfied.

We have the following results:

- if (24) holds for some $p > 1$, then there exists a semi-norm μ such that for $x \in \mathbb{R}^2$, $D^*(0, nx)/n$ almost surely converges to $\mu(x)$. $D^*(0, nx)/n$ also converges in $L^{p'}$ to $\mu(x)$ for $p' < p$.
- if (24) holds for some $p > 5$, then

$$\lim_{\|x\| \rightarrow +\infty} \frac{D^*(0, x) - \mu(x)}{\|x\|} = 0 \text{ a.s.}$$

(This is the Shape Theorem.)

The scheme of the proof is as follows:

- Step 1: prove that for $p' < p$, the family $(D^*(0, x)/(1 + \|x\|))_{x \neq 0}$ is bounded in $L^{p'}$.
- Step 2: Apply Kingman's theorem for the existence of a directional limit: $\mu(x) = \lim_{n \rightarrow +\infty} D^*(0, nx)/n$.
- Step 3: Prove the Shape Theorem.

Proof.

For each $R > 0$, we have

$$\begin{aligned} \mathbb{P}(D^*(0, x) \geq \theta(1 + \|x\|)) &\leq \mathbb{P}(\|0^*\| > R) + \mathbb{P}(\|x - x^*\| > R) \\ &\quad + \mathbb{P}\left(\sum_{\xi \in B_R^o} \sum_{\xi' \in B_R^x} d(\xi, \xi')^p \geq \theta^p(1 + \|x\|)^p\right) \\ &\leq 2 \exp(-R^2) + \theta^{-p}(1 + \|x\|)^{-p} \mathbb{E}\left[\sum_{\xi \in B_R^o} \sum_{\xi' \in B_R^x} d(\xi, \xi')^p\right] \end{aligned}$$

B_R^o and B_R^x can be covered by at most $(2R+1)^2$ copies of translations of the unit square $[0, 1] \times [0, 1]$. The distance between two copies is at most $\|x\| + 2R + 2$, so

$$\mathbb{P}(D^*(0, x) \geq \theta(1 + \|x\|)) \leq 2 \exp(-R^2) + C_p \theta^{-p} (2R+1)^4 \left(\frac{\|x\| + 2R + 2}{1 + \|x\|} \right)^p$$

Taking for instance $R = (p \log \theta)^{1/2}$, we get

$$\mathbb{P}(D^*(0, x) \geq \theta(1 + \|x\|)) \leq D \frac{(1 + \log \theta)^2}{\theta^p},$$

which gives

$$\mathbb{E} \left(\frac{D^*(0, x)}{1 + \|x\|} \right)^{p'} \leq M_p = 1 + \int_1^{+\infty} p' \theta^{p'-1} D \frac{(1 + \log \theta)^2}{\theta^p} d\theta < +\infty.$$

Particularly, $D^*(0, x)$ is integrable for all x , and the existence of a limit (say $\mu(x)$) for $D^*(0, nx)/n$ immediately follows from Kingman's subadditive theorem. The fact that μ is a semi-norm easily follows from the subadditivity property.

Let us now prove the shape theorem: For $x \in \mathbb{R}^d$, note $\hat{x} = x/\|x\|_2$. Fix $\epsilon \in (0, 1)$ and consider a finite family $(v_i)_{1 \leq i \leq N}$ of unital vectors such that for each $x \in \mathbb{R}^2$ we can find i with $\|\hat{x} - v_i\| \leq \epsilon$.

Now consider

$$S_n = \sum_{i=1}^N \sum_{\xi \in B_{(n+2)\epsilon}^{nv_i}} \sum_{\xi' \in B_{(n+2)\epsilon}^{nv_i}} \left(\frac{d(\xi, \xi')}{n\epsilon} \right)^p$$

As previously $B_{(n+2)\epsilon}^{nv_i}$ can be covered with $O(n^2)$ copies of $[0, 1] \times [0, 1]$, so using translation invariance, we get

$$\mathbb{E}[S_n] \leq K_{\epsilon, p} n^{4-p}.$$

If $p > 5$, then the Borel–Cantelli lemma ensures that $S_n \leq 1$ for n large enough.

By the Borel–Cantelli lemma, we have $\|x - x^*\| < \sqrt{2, 5 \log \|x\|}$ for large $x \in \mathbb{Z}^d$. It follows that $\|x - x^*\| < \sqrt{3 \log \|x\|}$ holds for large $x \in \mathbb{R}^d$.

Finally, we have a random M such that

- $\|x - x^*\| < \sqrt{3 \log \|x\|}$ for $\|x\|_2 \geq M$.
- $S_n \leq 1$ for $n \geq M$

- $|D^*(0, nv_i) - n\mu(v_i)| \leq n\epsilon$ for $n \geq M$.

Let x with $\|x\|_2 \geq M$. Take i such that $\|\hat{x} - v_i\|_2 \leq \epsilon$ and let n be smallest integer larger than $\|x\|_2$.

$(nv_i)^*$ and x^* are both inside $B_{(n+1)\epsilon}^{nv_i}$, so $D^*(nv_i, x) \leq n\epsilon$. Then,

$$\begin{aligned}
|D^*(0, x) - \mu(x)| &= |D^*(0, x) - \|x\|_2 \mu(\hat{x})| \\
&\leq |D^*(0, x) - D^*(0, nv_i)| + |D^*(0, nv_i) - n\mu(v_i)| \\
&\quad + |n\mu(v_i) - \|x\|_2 \mu(v_i)| + \|\|x\|_2 \mu(v_i) - \|x\|_2 \mu(\hat{x})\| \\
&\leq n\epsilon + n\epsilon + \mu(v_i) + \|x\| \mu(v_i - \hat{x}) \\
&\leq (2\epsilon + M_1) + \|x\| (M_1 + 2)\epsilon
\end{aligned}$$

Then,

$$\limsup_{\|x\| \rightarrow +\infty} \frac{|D^*(0, x) - \mu(x)|}{\|x\|} \leq (M_1 + 2)\epsilon \text{ a.s.}$$

Since ϵ is arbitrary, we have

$$\lim_{\|x\| \rightarrow +\infty} \frac{D^*(0, x) - \mu(x)}{\|x\|} = 0 \text{ a.s.}$$

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