On the evolution of random geometric graphs on spaces of negative curvature*

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Abstract

We study a family of random geometric graphs on hyperbolic spaces. In this setting, N points are chosen randomly on the hyperbolic plane of curvature $-\zeta^2$ and any two of them are joined by an edge with probability that depends on their hyperbolic distance, independently of every other pair. In particular, when the positions of the points have been fixed, the distribution over the set of graphs on these points is the Boltzmann distribution, where the Hamiltonian is given by the sum of weighted indicator functions for each pair of points. The weight is proportional to a real parameter $\beta > 0$ (interpreted as the inverse temperature) as well as to the hyperbolic distance between the corresponding points. The N points are distributed according to a quasi-uniform distribution which is a distorted version of the uniform distribution and depends on a parameter $\alpha > 0$ – when $\alpha = \zeta$ it coincides with the uniform distribution. This class of random graphs was proposed by Krioukov et al. [17] as a model for complex networks.

We provide a rigorous analysis of aspects of this model for all values of the ratio ζ/α, focusing on its dependence on the parameter β. We show that a phase transition occurs around $\beta = 1$. More specifically, let us assume that $0 < \zeta/\alpha < 2$. In this case, we show that when $\beta > 1$ the degree of a typical vertex is bounded in probability (in fact it follows a distribution which for large values exhibits a power-law tail whose exponent depends only on ζ and α), whereas for $\beta < 1$ the degree is a random variable whose expected value grows polynomially in N. When $\beta = 1$, we establish logarithmic growth.

For the case $\beta > 1$, we establish a connection with a class of inhomogeneous random graphs known as the *Chung-Lu* model. We show that the probability that two given points are adjacent is expressed through the kernel of this inhomogeneous random graph. We also consider the case $\zeta/\alpha \geq 2$.

1 Introduction

The theory of geometric random graphs was initiated by Gilbert [13] already in 1961 in the context of what is called *continuum percolation*. There, a random infinite graph is formed whose vertex set is the set of points of a stationary Poisson point process in the 2-dimensional Euclidean space and two vertices/points are joined when their distance is smaller than a

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given threshold. The parameter explored there is the probability that a given vertex is contained in an infinite component. About a decade later, in 1972, Hafner [15] focused on the typical properties of large but finite random geometric graphs. Here N points are sampled within a certain region of $\mathbb{R}^d$ following a certain distribution (most usually this is the uniform distribution or the distribution of the point-set of a Poisson point process) and any two of them are joined when their Euclidean distance is smaller than some threshold which, in general, is a function of N. In the last two decades, this kind of random graphs was studied extensively by several researchers – see the monograph of Penrose [21] and the references therein. Numerous typical properties of such random graphs have been investigated, such as the chromatic number [18], Hamiltonicity [3] etc.

From the point of view of applications, random geometric graphs on Euclidean spaces have been considered as models for wireless communication networks. Though the above model might seem slightly simplistic, variations of this have been developed which incorporate parameters of actual wireless networks; see for example [13] or [2].

However, what structural characteristics emerge when one considers these points distributed on a curved space where distances are measured through some (non-Euclidean) metric? This question has been studied in the context of percolation theory by Benjamini and Schramm [4]. However, a finite-scale model was introduced only recently by Krioukov et al. [17] and typical properties of the resulting random graphs were studied with the use of non-rigorous methods.

1.1 Random geometric graphs on a hyperbolic space

The most common representations of the hyperbolic space is the upper-half plane representation $\{x+iy \ : \ y>0\}$ as well as the Poincaré unit disc which is simply the open disc of radius one, that is, $\{(u,v) \in \mathbb{R}^2 \ : \ 1-u^2-v^2>0\}$. Both spaces are equipped with the hyperbolic metric; in the former case this is $\frac{1}{(\zeta y)^2}dy^2$ whereas in the latter this is $\frac{4}{\zeta^2}\ \frac{du^2+dv^2}{(1-u^2-v^2)^2}$, where ζ is some positive real number. It can be shown that the (Gaussian) curvature in both cases is equal to $-\zeta^2$ and the two spaces are isometric, that is, there exists a bijection between the two spaces which preserves (hyperbolic) distances. In fact, there are more representations of the 2-dimensional hyperbolic space of curvature $-\zeta^2$ which are isometrically equivalent to the above two. We will denote by $\mathbb{H}^2_\zeta$ the class of these spaces.

In this paper, following the setting in [17], we shall be using the native representation of $\mathbb{H}^2_\zeta$. Under this representation, the ground space is $\mathbb{R}^2$ and every point $x \in \mathbb{R}^2$ whose polar coordinates are (r,θ) has hyperbolic distance from the origin equal to r. It can be shown that a circle of radius r around the origin has length equal to $\frac{2\pi}{\zeta}\ \sinh \zeta r$ and area equal to $\frac{2\pi}{\zeta^2}(\cosh \zeta r - 1)$.

We are now ready to give the definitions of the two basic models introduced in [17]. Consider the native representation of the hyperbolic space of curvature $K=-\zeta^2$, for some $\zeta>0$. For a real number $\nu>0$, let $N=\nu e^{\zeta R/2}$ – thus R is a function of N which grows logarithmically in N. The parameter ν controls the average degree of the random graph. We create a random graph by selecting randomly N points from the disc of radius R centred at the origin O, which we denote by $\mathcal{D}_R$. The distribution of these points is as follows. Assume that a random point u has polar coordinates (r, θ). Then θ is uniformly distributed in $(0, 2\pi]$, whereas the probability density function of r, which we denote by $\rho(r)$, is determined by a parameter $\alpha > 0$ and is equal to

$$\rho(r) = \alpha \frac{\sinh \alpha r}{\cosh \alpha R - 1}. \tag{1.1}$$

When $\alpha = \zeta$, then this is the uniform distribution. This set of points will be the vertex set of the random graph and we will be denoting this random vertex set by V_N. We will be also treating the vertices as points in the hyperbolic space indistinguishably.

1. *The disc model* This model is the most commonly studied in the theory of random geometric graphs on Euclidean spaces. We join two vertices if they are within (hyperbolic) distance R from each other.

2. *The binomial model* We join any two distinct vertices u, v with probability

   $$p_{u,v} = \frac{1}{\exp\left(\beta\, \frac{\zeta}{2}(d(u,v) - R)\right) + 1},$$

   independently of every other pair, where $\beta > 0$ is fixed and $d(u, v)$ is the hyperbolic distance between u and v. We denote the resulting random graph by $\mathcal{G}(N; \nu, \zeta, \alpha, \beta)$.

The parameter $\beta > 0$ is interpreted as the inverse of the temperature of a fermionic system where particles correspond to edges. The distance between two points determines the field that is incurred by the pair. In particular, the field that is incurred by the pair $\{u, v\}$ is $\omega_{u,v} = \beta\, \frac{\zeta}{2}\, (d(u, v) - R)$.

An edge between two points corresponds to a particle that "occupies" the pair. In turn, the Hamiltonian of a graph G on the N points, assuming that their positions on $\mathcal{D}_R$ have been realized, is $H(G) = \sum_{u,v} \omega_{u,v} e_{u,v}$, where $e_{u,v}$ is the indicator that is equal to 1 if and only if the edge between u and v is present. (Here the sum is over all distinct unordered pairs of points.) Each graph G has probability weight that is equal to $e^{-H(G)}/Z$, where $Z = \prod_{u,v} (1 + e^{-\omega_{u,v}})$ is the normalizing factor also known as the partition function. It can be shown (cf. [20] for example) that in this distribution the probability that u is adjacent to v is equal to $1/(e^{\omega_{u,v}} + 1)$. See also [17] for a more detailed description.

In this paper, our focus will be on the binomial model. As we shall see, the central structural features of the resulting random graph heavily depend on the value of β. In particular, when $0 < \zeta/\alpha < 2$ we shall distinguish between three regimes:

1. $\beta > 1$ (*cold regime*) the random graph $\mathcal{G}(N; \nu, \zeta, \alpha, \beta)$ has constant average degree depending on β and ζ.

2. $\beta = 1$ (*critical regime*) the average degree grows logarithmically in N.

3. $\beta < 1$ (*hot regime*) the average degree of $\mathcal{G}(N; \nu, \zeta, \alpha, \beta)$ grows polynomially in N.

We establish analogous behaviour for $\zeta/\alpha \geq 2$. We shall make these findings precise immediately.

1.2 Results

The main results of this paper describe the degree of a typical vertex of $\mathcal{G}(N;\nu,\zeta,\alpha,\beta)$. For a vertex $u \in V_N$ we let D_u denote the degree of u in $\mathcal{G}(N;\nu,\zeta,\alpha,\beta)$. Our first theorem regards the cold regime.

Theorem 1.1. *If $\beta > 1$ and $0 < \zeta/\alpha < 2$, then*

$$D_u \xrightarrow{d} \mathsf{MP}(F),$$

where $\mathsf{MP}(F)$ denotes a random variable that follows the mixed Poisson distribution with mixing distribution F such that

$$F(t) = 1 - \left(\frac{K}{t}\right)^{2\alpha/\zeta},$$

for any $t \geq K$, where $K = \frac{4\alpha\nu}{2\alpha-\zeta}\ \frac{1}{\beta}\ \sin^{-1}\left(\frac{\pi}{\beta}\right)$ and $F(t) = 0$ otherwise.

In particular, as k and N grow

$$\Pr\Big[\,D_u = k\,\Big] = \frac{2\alpha}{\zeta}\ K^{2\alpha/\zeta}\ k^{-(2\alpha/\zeta+1)} + o(1),$$

that is, the degree of vertex u follows a power law with exponent $2\alpha/\zeta + 1$.

We also show a law of large numbers for the fraction of vertices of any given degree in $\mathcal{G}(N;\nu,\zeta,\alpha,\beta)$.

Theorem 1.2. *For any k let N_k denote the number of vertices of degree k in $\mathcal{G}(N;\nu,\zeta,\alpha,\beta)$. Let $\beta > 1$ and $0 < \zeta/\alpha < 2$. For any fixed integer $k \geq 0$, we have*

$$\frac{N_k}{N} \xrightarrow{p} \Pr\Big[\,\mathsf{MP}(F) = k\,\Big],$$

as $N \to \infty$.

The distribution of the degree of a given vertex changes abruptly when $\beta \leq 1$. For a vertex $v \in V_N$, if r_v is the distance of v from the origin, that is, the radius of v, then we set $t_v = R - r_v$ – we call this quantity the *type* of vertex v. As we shall see, with probability that tends to 1 as $N \to \infty$ (*asymptotically almost surely* – a.a.s.) all vertices have their types less than $\zeta R/(2\alpha) + \omega(N)$, where $\omega(N) \to \infty$ as $N \to \infty$ (cf. Corollary 2.2). When $\beta = 1$, the degree of any vertex conditional on its type is binomially distributed with expected value proportional to R. Let $\mathsf{Bin}(n,p)$ denote the binomial distribution with parameters n, p. For a random variable X_n we write $X_n \overset{\Delta}{\sim} \mathsf{Bin}(n,p)$ to denote the fact that for any $\varepsilon > 0$ and any n sufficiently large the distribution of X_n lies stochastically between two random variables distributed as $\mathsf{Bin}(n,(1-\varepsilon)p)$ and $\mathsf{Bin}(n,(1+\varepsilon)p)$, respectively.

Theorem 1.3. *Let $0 < \zeta/\alpha < 2$ and let $\omega(N)$ be a non-negative function such that $\omega(N) \to \infty$ as $N \to \infty$. Let $u \in V_N$ be a vertex such that $t_u \leq \zeta R/(2\alpha) + \omega(N)$. For any $\varepsilon > 0$ the following hold.*

If $\beta = 1$, *then*

$$D_u \stackrel{\Delta}{\sim} \mathsf{Bin}\left(N-1, (R-t_u)\ \frac{Ke^{\zeta t_u/2}}{N}\right),$$

where $K = \frac{1}{\pi}\ \frac{2\alpha\zeta\nu}{2\alpha-\zeta}$.
If $\beta < 1$, *then*

$$D_u \stackrel{\Delta}{\sim} \mathsf{Bin}\left(N-1, K\left(\frac{e^{\zeta t_u/2}}{N}\right)^{\beta}\right),$$

where $K = \frac{1}{\sqrt{\pi}}\ \frac{2\alpha\nu^\beta}{2\alpha-\beta\zeta}\ \frac{\Gamma\left(\frac{1-\beta}{2}\right)}{\Gamma\left(1-\frac{\beta}{2}\right)}$.

It is not hard to see that when $\beta < 1$, the random graph $\mathcal{G}(N;\nu,\zeta,\alpha,\beta)$ stochastically contains a $\mathcal{G}(N,\nu^\beta/2N^\beta)$ Erdős-Rényi random graph. Indeed, for any two distinct points $u, v \in V_N$ if r_u, r_v denote their radii in $\mathcal{D}_R$, then $d(u,v) \leq r_u + r_v$ – this follows from the triangle inequality. Thereby, we have

$$\beta\ \frac{\zeta}{2}\ (d(u,v) - R) \leq \beta\ \frac{\zeta}{2}\ (r_u + r_v - R) = \beta\ \frac{\zeta}{2}(R - t_u - t_v).$$

Hence, if $R - t_u - t_v \geq 0$, then

$$p_{u,v} \geq \frac{1}{\exp\left(\beta\ \frac{\zeta}{2}(R-t_u-t_v)\right)+1} \geq \frac{\nu^\beta}{2}\ \left(\frac{e^{\frac{\zeta}{2}(t_u+t_v)}}{N}\right)^{\beta}.$$

If $R - t_u - t_v < 0$, then $p_{u,v} \geq 1/2$. Hence for any two distinct $u, v \in V_N$, the probability that they are joined is

$$p_{u,v} \geq \frac{1}{2}\ \left\{\left(\nu\ \frac{e^{\frac{\zeta}{2}(t_u+t_v)}}{N}\right)^{\beta} \wedge 1\right\} \geq \frac{\nu^\beta}{2N^\beta},$$

for N that is large enough. Hence, if $\beta < 1$, the random graph $\mathcal{G}(N;\nu,\zeta,\alpha,\beta)$ is a.a.s. connected and, moreover, it has bounded diameter (cf. Theorem 10.10 in [5]).

When $\zeta/\alpha \geq 2$, the degree of a vertex is with high probability a growing function of N. This is due to the fact that there is a large number of vertices that are located close to the centre of $\mathcal{D}_R$. These have typically degree that is a polynomial function of N and typically any vertex is adjacent to a large number of those. For a random variable X_n we write that $X_n \stackrel{\Delta}{\asymp} \mathsf{Bin}(n,p)$, where $p = p(n) \in (0,1)$, to denote the fact that there are positive real numbers $c_1 < c_2$ such that for any n large enough the random variable X_n lies stochastically between two random variables distributed as $\mathsf{Bin}(n, c_1 p)$ and $\mathsf{Bin}(n, c_2 p)$, respectively. The following theorem describes the distribution of the degree of a vertex with a given type. For given real numbers x and y, we denote by $\delta_{x,y}$ the indicator function that is equal to 1 precisely when $x = y$.

Theorem 1.4. *Let* $\zeta/\alpha \geq 2$ *and let* $u \in V_N$ *be a vertex. Then uniformly over* $t_u < R$ *the following hold.*

If $\beta > 2\alpha/\zeta$, *then*

$$D_u \overset{\Delta}{\asymp} \mathsf{Bin}\left(N-1, (R-t_u+1)^{\delta_{2,\zeta/\alpha}} \left(\frac{e^{\zeta t_u/2}}{N}\right)^{2\alpha/\zeta}\right).$$

If $\beta = 2\alpha/\zeta$, *then*

$$D_u \overset{\Delta}{\asymp} \mathsf{Bin}\left(N-1, \left((R-t_u)^{1+\delta_{2,\zeta/\alpha}}+1\right) \left(\frac{e^{\zeta t_u/2}}{N}\right)^{2\alpha/\zeta}\right).$$

If $\beta < 2\alpha/\zeta$, *then*

$$D_u \overset{\Delta}{\asymp} \mathsf{Bin}\left(N-1, \left(\frac{e^{\zeta t_u/2}}{N}\right)^{\beta}\right).$$

Note that when $t_u \geq R - o(1)$, then for any point $v \neq u$, we have $d(u,v) \leq R + o(1)$, which in turn implies that u is joined to v with probability that asymptotically as $N \to \infty$ is bounded from below by $1/2$. When $\zeta/\alpha \geq 2$, the vertices that are close to the origin play a significant role to the degree of a vertex. In particular, in the case $\beta > 1$, the degree of a vertex u is essentially determined by the number of vertices that are within hyperbolic distance R from u and have radius at most $R/2$.

1.3 The disc model vs the binomial model

The disc model can be viewed as a limiting case of the binomial model as $\beta \to \infty$. Assume that the positions of the vertices in $\mathcal{D}_R$ have been realized. If $u, v \in V_N$ are such that $d(u,v) < R$, then when $\beta \to \infty$ the probability that u and v are adjacent tends to 1; however, if $d(u,v) > R$, then this probability converges to 0 as β grows. In other words, the disc model is the "frozen" version ($T = 0$) of the binomial model. Rigorous results for the disc model were obtained by Gugelmann et al. [14], regarding their degree sequence as well as the clustering coefficient.

1.4 Hyperbolic random graphs as a model for complex networks

We will now discuss the potential of using random graphs on spaces of negative curvature as a model for the so-called complex networks. The typical features of complex networks can be summarised as follows:

1. they are *large*, that is, they contain thousands or millions of nodes;
2. they are *sparse*, that is, the number of their edges is proportional to the number of nodes;
3. they exhibit the *small world phenomenon*; most pairs of vertices are within a short distance from each other;

4. a significant amount of *clustering* is present. The latter means that two nodes of the network that have a common neighbour are somewhat more likely to be connected with each other;

5. their degree sequence follows a *power-law* distribution.

See the book of Chung and Lu [12] for a detailed discussion of these properties.

Over the last decade a number of models have been developed whose aim was to capture these features. Among the first such models is the *preferential attachment model*. This is a class of models of randomly growing graphs whose aim is to capture a basic feature of such networks: nodes which are already popular tend to become more popular as the network grows. These models were first defined by Albert and Barabási [1] and subsequently defined and studied rigorously by Bollobás and Riordan (see for example [8], [7]).

Another extensively studied model was defined by Chung and Lu [10], [11]. Here every vertex has a weight which effectively corresponds to its expected degree and every two vertices are joined independently of every other pair with probability that is proportional to the product of their weights. When the weights are set suitably, then the resulting random graph has power-law degree distribution.

All these models are nonetheless insufficient in the sense that none of them succeeds in incorporating all the above features. For example the Chung–Lu model although it exhibits a power law degree distribution and average distance of order $O(\log \log N)$, when the exponent of the power law is between 2 and 3 (see [10]) (with N being the number of nodes of the random network) it is locally tree-like around a typical vertex. Thus, for the majority of the vertices their neighbourhoods form an independent set. This is also the situation regarding the Barabási-Albert model. Thus, it seems that there is a "missing link" in these models which is a key ingredient to the process of creating a social network. It seems plausible that the factor which is missing in these models is the *hierarchical structure* of a social network. To be more precise, the hierarchies are not among nodes, but more importantly on the level of groups of nodes.

Real-world networks consist of heterogeneous nodes, which can be classified into groups. In turn, these groups can be classified into larger groups which consist of smaller subgroups and so on. For example, if we consider the network of citations, whose set of nodes is the set of research papers and there is a link from one paper to another if one cites the other, there is a natural classification of the nodes according to the scientific fields each paper belongs to (see for example [9]). In the case of the network of web pages, a similar classification can be considered in terms of the similarity between two web pages. That is, the more similar two web pages are, the more likely it is that there exists a hyperlink between them (see [19]).

This classification can be approximated by tree-like structures representing the hidden hierarchy of the network. The tree-likeness suggests that in fact the geometry of this hierarchy is *hyperbolic*. As we have already seen in the above definitions, the volume growth in the hyperbolic space is exponential which is also the case, for example, when one considers a k-ary tree, that is, a rooted tree where every vertex has k children. Let us consider the Poincaré disc model. If we place the root of an infinite k-ary tree at the centre of the disc, then the hyperbolic metric provides the necessary room to embed the tree into the disc so that every edge has unit length in the embedding. See also [17] for a related discussion.

In this contribution, however, we have only explored the degree sequence of random graphs on a hyperbolic space showing that in the sparse regime, they exhibit a power-law degree distribution with exponent that depends on the curvature of the underlying space. It remains to show that they also exhibit the small world phenomenon, proving that the typical distance between two randomly chosen vertices is small, for example $O(\log\log N)$. Regarding the presence of clustering, Krioukov et al. [17] have shown with the use of non-rigorous arguments that these random graphs do exhibit clustering which can be adjusted through the parameter β. This dependence also remains to be quantified rigorously. For the disc model, this existence of clustering has been verified by Gugelmann et al. [14].

1.5 Outline

We begin our analysis with some preliminary results which will be used throughout our proofs. In Sections 3 and 4, we prove Theorems 1.1, 1.3 and 1.4 and we continue in Section 5 with the analysis of the asymptotic correlation of the degrees of finite collections of vertices for $\beta > 1$ and $0 < \zeta/\alpha < 2$ and the proof of Theorem 5.1. Its proof immediately yields Theorem 1.2 with the use of Chebyschev's inequality.

2 Preliminaries

The next lemma shows that the type of a vertex is essentially exponentially distributed.

Lemma 2.1. *For any $v \in V_N$ we have*

$$\Pr\Big[\, t_v \leq x \,\Big] = 1 - e^{-\alpha x} + O\left(N^{-2\alpha/\zeta}\right),$$

uniformly for all $0 \leq x \leq R$.

Proof. We use the definition of $\rho(r)$ and write

$$\Pr\Big[\, t_v \leq x \,\Big] = \alpha \int_{R-x}^{R} \frac{\sinh(\alpha r)}{\cosh(\alpha R) - 1} dr = \frac{1}{\cosh(\alpha R) - 1} \left[\cosh(\alpha R) - \cosh(\alpha(R-x))\right].$$

Now, note that $\cosh(\alpha R) = \frac{1}{2}\, e^{\alpha R}(1 + o(1)) = \frac{1}{2}\, \left(\frac{N}{\nu}\right)^{2\alpha/\zeta} (1 + o(1))$ and therefore

$$\frac{\cosh(\alpha R)}{\cosh(\alpha R) - 1} = 1 + O(N^{-2\alpha/\zeta}).$$

Also,

$$\begin{aligned}\frac{\cosh(\alpha(R-x))}{\cosh(\alpha R)-1} &= \frac{\cosh(\alpha(R-x))}{\cosh(\alpha R)}\left(1+O(N^{-2\alpha/\zeta})\right)\\ &= \frac{e^{\alpha(R-x)}}{e^{\alpha R}}\ \frac{1+e^{-2\alpha(R-x)}}{1+e^{-2\alpha R}}\ \left(1+O(N^{-2\alpha/\zeta})\right)\\ &= e^{-\alpha x}\left(1+e^{-2\alpha(R-x)}\right)\ \left(1-e^{-2\alpha R}+O(e^{-4\alpha R})\right)\ \left(1+O(N^{-2\alpha/\zeta})\right).\end{aligned}$$

But $e^{-x-2(R-x)} = e^{-2R+x}$ and $x \leq R$. Thus $e^{-\alpha x-2\alpha(R-x)} \leq e^{-\alpha R} = (\nu/N)^{2\alpha/\zeta}$, which implies the statement of the lemma. □

Now, let us set

$$x_0 = \begin{cases} \frac{\zeta}{2\alpha}R+\omega(N), & \text{if } 0<\frac{\zeta}{\alpha}<2 \\ R, & \text{if } \frac{\zeta}{\alpha}\geq 2 \end{cases},$$

where $\omega(N) \to \infty$ as $N \to \infty$ but $\omega(N) = o(R)$. The above lemma immediately yields the following corollary.

Corollary 2.2. *If $0 < \zeta/\alpha < 2$, then a.a.s all vertices $v \in V_N$ have $t_v \leq x_0$.*

Proof. Note that $x_0 = \frac{1}{\alpha}\ \log N - \frac{1}{\alpha}\ \log\nu + \omega(N)$. For a vertex $v \in V_N$ applying Lemma 2.1, we have

$$\Pr\Big[\,t_v > x_0\,\Big] = e^{-\alpha x_0} + O\left(N^{-2\alpha/\zeta}\right) = o\left(\frac{1}{N}\right),$$

since $\zeta/\alpha < 2$. The corollary follows from Markov's inequality. □

We will also need an estimate on the distance between two points in the case their relative angle is not too small (this is the typical case).

Lemma 2.3. *Assume that $0 < \zeta/\alpha < 2$ and let $h : \mathbb{N} \to \mathbb{R}^+$ such that $h(N) \to \infty$ as $N \to \infty$. Let u, v be two distinct points in $\mathcal{D}_R$ such that $t_u, t_v \leq x_0$, with $\theta_{u,v}$ denoting their relative radius. Let also $\hat{\theta}_{u,v} := \left(e^{-2\zeta(R-t_u)} + e^{-2\zeta(R-t_v)}\right)^{1/2}$. If*

$$h(N)\hat{\theta}_{u,v} \leq \theta_{u,v} \leq \pi,$$

then

$$d(u,v) = 2R - (t_u+t_v) + \frac{2}{\zeta}\ \log\sin\left(\frac{\theta_{u,v}}{2}\right) + O\left(\left(\frac{\hat{\theta}_{u,v}}{\theta_{u,v}}\right)^2\right),$$

uniformly for all u, v with $R - t_u - t_v \geq h(N)$.

Proof. We begin with the hyperbolic law of cosines:

$$\cosh(\zeta d(u,v)) = \cosh(\zeta(R-t_u))\cosh(\zeta(R-t_v)) - \sinh(\zeta(R-t_u))\sinh(\zeta(R-t_v))\cos(\theta_{u,v}).$$

Since $R - t_u - t_v \geq h(N)$, it follows that both $R - t_u, R - t_v \to \infty$ as $N \to \infty$. Thus the right-hand side of the above becomes:

$$\cosh(\zeta(R-t_u))\cosh(\zeta(R-t_v)) - \sinh(\zeta(R-t_u))\sinh(\zeta(R-t_v))\cos(\theta_{u,v}) =$$
$$\frac{e^{\zeta(2R-(t_u+t_v))}}{4}\left(\left(1+e^{-2\zeta(R-t_u)}\right)\left(1+e^{-2\zeta(R-t_v)}\right) - \left(1-e^{-2\zeta(R-t_u)}\right)\left(1-e^{-2\zeta(R-t_v)}\right)\cos(\theta_{u,v})\right)$$
$$= \frac{e^{\zeta(2R-(t_u+t_v))}}{4}\left(1-\cos(\theta_{u,v}) + (1+\cos(\theta_{u,v}))\left(e^{-2\zeta(R-t_u)}+e^{-2\zeta(R-t_v)}\right) + O\left(e^{-2\zeta(2R-(t_u+t_v))}\right)\right).$$

By the convexity of the function $e^{-2\zeta x}$, we have

$$e^{-\zeta(2R-(t_u+t_v))} = e^{-2\zeta\frac{2R-(t_u+t_v)}{2}} \leq \frac{1}{2}\left(e^{-2\zeta(R-t_u)}+e^{-2\zeta(R-t_v)}\right) \leq \hat{\theta}^2_{u,v}. \tag{2.1}$$

Thus, the previous estimate can be written as

$$\cosh(\zeta d(u,v)) = \frac{e^{\zeta(2R-(t_u+t_v))}}{4}\left(1-\cos(\theta_{u,v}) + (1+\cos(\theta_{u,v}))\,\hat{\theta}^2_{u,v} + O\left(\hat{\theta}^4_{u,v}\right)\right).$$

If $\theta_{u,v}$ is bounded away from 0, then clearly $1-\cos(\theta_{u,v})$ dominates the expression in brackets. Now assume that $\theta_{u,v} = o(1)$. It is a basic trigonometric identity that $1 - \cos(\theta_{u,v}) = 2\sin^2\left(\frac{\theta_{u,v}}{2}\right)$. Then $1-\cos(\theta_{u,v}) = \frac{\theta^2_{u,v}}{2}(1-o(1))$. But the assumption that $\theta_{u,v} \gg \hat{\theta}_{u,v}$ again implies that also in this case $1-\cos(\theta_{u,v})$ dominates expression in brackets. Thus

$$\begin{aligned}\cosh(\zeta d(u,v)) &= \frac{e^{\zeta(2R-(t_u+t_v))}}{4}\ (1-\cos(\theta_{u,v}))\ \left(1+O\left(\left(\frac{\hat{\theta}_{u,v}}{\theta_{u,v}}\right)^2\right)\right) \\ &= \frac{e^{\zeta(2R-(t_u+t_v))}}{2}\ \sin^2\left(\frac{\theta_{u,v}}{2}\right)\ \left(1+O\left(\left(\frac{\hat{\theta}_{u,v}}{\theta_{u,v}}\right)^2\right)\right).\end{aligned} \tag{2.2}$$

Now, we take logarithms in (2.2) and divide both sides by ζ thus obtaining:

$$d(u,v) + \frac{1}{\zeta}\log\left(1-e^{-2\zeta d(u,v)}\right) = 2R - t_u - t_v + \frac{2}{\zeta}\log\sin\left(\frac{\theta_{u,v}}{2}\right) + O\left(\left(\frac{\hat{\theta}_{u,v}}{\theta_{u,v}}\right)^2\right). \tag{2.3}$$

We now need to give an asymptotic estimate on $e^{-2\zeta d(u,v)}$. We derive this from (2.2) as well. For N large enough, we have

$$e^{\zeta d(u,v)} \geq \frac{e^{\zeta(2R-(t_u+t_v))}}{4}\ \sin^2\left(\frac{\theta_{u,v}}{2}\right) \geq \frac{1}{32}\ e^{\zeta(2R-(t_u+t_v))}\ \theta^2_{u,v} \overset{(2.1)}{\geq} \frac{1}{32}\ \left(\frac{\theta_{u,v}}{\hat{\theta}_{u,v}}\right)^2,$$

from which it follows that

$$\left|\log\left(1-e^{-2\zeta d(u,v)}\right)\right| = O\left(\left(\frac{\hat{\theta}_{u,v}}{\theta_{u,v}}\right)^4\right).$$

Substituting this into (2.3) completes the proof of the lemma. □

Let $\hat{p}_{u,v} = \frac{1}{\pi}\int_0^\pi p_{u,v} d\theta$ - this is the probability that two points u and v are connected by an edge, conditional on their types. For an arbitrarily slowly growing function $\omega : \mathbb{N} \to \mathbb{N}$ we define

$$\mathcal{D}_{R,\omega}^{(2)} = \left\{ \{u,v\} \in \binom{\mathcal{D}_R}{2} \ : t_v, t_u \leq x_0, \ R - t_u - t_v \geq \omega(N) \right\}.$$

Also, we let $\bar{\mathcal{D}}_{R,\omega}^{(2)}$ denote the complement of this set in $\mathcal{D}_R$. For two points $u, v \in \mathcal{D}_R$ we set $A(t_u, t_v) = \exp\left(\frac{\zeta}{2}\left(R - (t_u + t_v)\right)\right)$. The following lemma gives an asymptotic estimate on $\hat{p}_{u,v}$, for all values of β, in terms of $A(t_u, t_v)$, whenever $\{u, v\} \in \mathcal{D}_{R,\omega}^{(2)}$.

Lemma 2.4. *Let $\beta > 0$. There exists a constant $C_\beta > 0$ such that uniformly for all $u, v \in \mathcal{D}_{R,\omega}^{(2)}$ we have*

$$\hat{p}_{uv} = \begin{cases} (1 + o(1))\frac{C_\beta}{A_{u,v}} \ , & \text{if } \beta > 1 \\ \\ (1 + o(1))\frac{C_\beta \ln A_{u,v}}{A_{u,v}}, & \text{if } \beta = 1 \\ \\ (1 + o(1))\frac{C_\beta}{A_{u,v}^\beta}, & \text{if } \beta < 1 \end{cases}.$$

In particular,

$$C_\beta = \begin{cases} \frac{2}{\beta} \ \sin^{-1}\left(\frac{\pi}{\beta}\right), & \text{if } \beta > 1 \\ \frac{2}{\pi}, & \text{if } \beta = 1 \\ \frac{1}{\sqrt{\pi}} \ \frac{\Gamma\left(\frac{1-\beta}{2}\right)}{\Gamma\left(1 - \frac{\beta}{2}\right)}, & \text{if } \beta < 1 \end{cases}.$$

Proof. Throughout this proof we write $A_{u,v}$ for $A(t_u, t_v)$. Recall that

$$p_{u,v} = \frac{1}{\exp\left(\beta \ \frac{\zeta}{2}(d(u,v) - R)\right) + 1}.$$

We will estimate the integral of $p_{u,v}$ over $\theta_{u,v}$. When $\theta_{u,v}$ is within the range given in Lemma 2.3 we will use the estimate given there. In particular, we shall define $\tilde{\theta}_{u,v} \gg \hat{\theta}_{u,v}$ and split the integral into two parts, namely when $0 \leq \theta_{u,v} < \tilde{\theta}_{u,v}$ and when $\tilde{\theta}_{u,v} \leq \theta_{u,v} \leq \pi$. The parameter $\tilde{\theta}_{u,v}$ is close to $A_{u,v}^{-1}$. Note that the following holds.

Claim 2.5. *If $R - t_u - t_v \geq \omega(N)$, then*

$$A_{u,v}^{-1} \gg \hat{\theta}_{u,v}.$$

We postpone the proof of this claim until later.

Let $\omega(N)$ be slowly enough growing so that in the following definition of $\tilde{\theta}_{u,v}$, we have

$\tilde{\theta}_{u,v} \gg \hat{\theta}_{u,v}$, when $\beta \geq 1$, and $\tilde{\theta}_{u,v} = o(A_{u,v}^{-\beta})$, when $\beta < 1$. We set

$$\tilde{\theta}_{u,v} = \begin{cases} \frac{A_{u,v}^{-1}}{\omega(N)}, & \text{if } \beta \geq 1 \\ \omega(N) A_{u,v}^{-1}, & \text{if } \beta < 1 \end{cases}.$$

Thus when $\tilde{\theta}_{u,v} \leq \theta_{u,v} \leq \pi$, we use Lemma 2.3 and write

$$\exp\left(\beta \ \frac{\zeta}{2}(d(u,v) - R)\right) = C \ e^{\beta \frac{\zeta}{2}(R-(t_u+t_v)) + \beta \ \log \sin(\theta_{u,v}/2)}, \tag{2.4}$$

where $C = 1 + O\left(\left(\frac{\hat{\theta}_{u,v}}{\tilde{\theta}_{u,v}}\right)^2\right)$. By Claim 2.5 and the choice of the function $\omega(N)$, we have that $C = 1 + o(1)$.

We decompose the integral that gives $\hat{p}_{u,v}$ into two parts which we bound separately.

$$\hat{p}_{u,v} = \frac{1}{\pi} \int_0^\pi p_{u,v} d\theta = \frac{1}{\pi} \int_0^{\tilde{\theta}_{u,v}} p_{u,v} d\theta + \frac{1}{\pi} \int_{\tilde{\theta}_{u,v}}^\pi p_{u,v} d\theta. \tag{2.5}$$

The first integral can bounded trivially as follows:

$$\int_0^{\tilde{\theta}_{u,v}} p_{u,v} d\theta \leq \tilde{\theta}_{u,v} = \begin{cases} o\left(A_{u,v}^{-1}\right), & \text{if } \beta \geq 1 \\ o\left(A_{u,v}^{-\beta}\right), & \text{if } \beta < 1 \end{cases}. \tag{2.6}$$

We now focus on the second integral in (2.5). We will treat the cases $\beta < 1$ and $\beta \geq 1$ separately, starting with the former one.

$\beta < 1$

Recall that $\tilde{\theta}_{u,v}$ is such that $\tilde{\theta}_{u,v} \gg A_{u,v}^{-1}$. Thus, we write

$$\begin{aligned} \int_{\tilde{\theta}_{u,v}}^\pi p_{u,v} d\theta &= \int_{\tilde{\theta}_{u,v}}^\pi \frac{1}{C A_{u,v}^\beta \sin^\beta\left(\frac{\theta}{2}\right) + 1} d\theta = \frac{1 + o(1)}{A_{u,v}^\beta} \int_{\tilde{\theta}_{u,v}}^\pi \frac{1}{\sin^\beta\left(\frac{\theta}{2}\right) + \Theta\left(A_{u,v}^{-\beta}\right)} d\theta \\ &= \frac{1 + o(1)}{A_{u,v}^\beta} \int_{\tilde{\theta}_{u,v}}^\pi \frac{1}{\sin^\beta\left(\frac{\theta}{2}\right)} d\theta = \frac{1 + o(1)}{A_{u,v}^\beta} \int_0^\pi \frac{1}{\sin^\beta\left(\frac{\theta}{2}\right)} d\theta. \end{aligned} \tag{2.7}$$

Substituting the estimates of (2.6) and (2.7) into (2.5) we obtain

$$\hat{p}_{u,v} = (1 + o(1)) \ \frac{1}{\pi} \ \left(\int_0^\pi \frac{1}{\sin^\beta\left(\frac{\theta}{2}\right)} d\theta\right) \ \frac{1}{A_{u,v}^\beta}, \tag{2.8}$$

uniformly for all $u, v \in \mathcal{D}_{R,\omega}^{(2)}$. Finally, note that

$$\int_0^\pi \frac{1}{\sin^\beta\left(\frac{\theta}{2}\right)} d\theta = 2 \int_0^{\pi/2} \frac{1}{\sin^\beta(\theta)} d\theta = \sqrt{\pi} \ \frac{\Gamma\left(\frac{1-\beta}{2}\right)}{\Gamma\left(1 - \frac{\beta}{2}\right)}.$$

$\beta \geq 1$
We use the inequality $\sin\theta \leq \theta$, which holds for all $\theta \in [0,\pi]$, and obtain an upper bound on the right-hand side of (2.4).

$$\exp\left(\beta\ \frac{\zeta}{2}(d(u,v)-R)\right) \leq C\ e^{\beta\frac{\zeta}{2}(R-(t_u+t_v))}\ \left(\frac{\theta_{u,v}}{2}\right)^{\beta} = C\ A_{u,v}^{\beta}\ \left(\frac{\theta_{u,v}}{2}\right)^{\beta}.$$

Using this bound we can bound the second integral in (2.5) from below as follows.

$$\int_{\tilde{\theta}_{u,v}}^{\pi} p_{u,v} d\theta \geq \int_{\tilde{\theta}_{u,v}}^{\pi} \frac{1}{C\ A_{u,v}^{\beta}\ \left(\frac{\theta}{2}\right)^{\beta}+1} d\theta. \tag{2.9}$$

We perform a change of variable setting $z = C^{1/\beta}\ A_{u,v}\ \frac{\theta}{2}$. Thus with $C' = C^{1/\beta}/2$ the integral on the right-hand side of (2.9) becomes

$$\int_{\tilde{\theta}_{u,v}}^{\pi} \frac{1}{C\ A_{u,v}^{\beta}\ \left(\frac{\theta}{2}\right)^{\beta}+1} d\theta = \frac{1}{C'}\ \frac{1}{A_{u,v}}\ \int_{C' A_{u,v}\tilde{\theta}_{u,v}}^{C'\pi A_{u,v}} \frac{1}{z^{\beta}+1} dz. \tag{2.10}$$

We now provide an estimate for the integral on the right-hand side of (2.10) for any $\beta \geq 1$ - its proof is elementary and we omit it.

Claim 2.6. *Let $g_1(N)$ and $g_2(N)$ be non-negative real-valued functions on the set of natural numbers, such that $g_1(N) \to 0$ and $g_2(N) \to \infty$ as $N \to \infty$. We have*

$$\int_{g_1(N)}^{g_2(N)} \frac{1}{z^{\beta}+1} dz = \begin{cases} (1+o(1)) \int_0^{\infty} \frac{1}{z^{\beta}+1} dz, & \text{if } \beta > 1 \\ \\ (1+o(1)) \ln g_2(N), & \text{if } \beta = 1 \end{cases}.$$

We take $g_1(N) = C' A_{u,v}\tilde{\theta}_{u,v} = C'/\omega(N) \to 0$ and $g_2(N) = C'\pi A_{u,v} \to \infty$, since $u,v \in \mathcal{D}_{R,\omega}^{(2)}$, and we obtain through (2.9):

$$\int_{\tilde{\theta}_{u,v}}^{\pi} p_{u,v} d\theta \geq \begin{cases} (1+o(1))\ \frac{2}{C^{1/\beta}}\ \frac{1}{A_{u,v}}\ \int_0^{\infty} \frac{1}{z^{\beta}+1} dz, & \text{if } \beta > 1 \\ \\ (1+o(1))\ \frac{2}{C^{1/\beta}}\ \frac{\ln A_{u,v}}{A_{u,v}}, & \text{if } \beta = 1 \end{cases}. \tag{2.11}$$

To deduce the upper bound we will split the integral into two parts. For an $\varepsilon \in (0,\pi)$, we write

$$\int_{\tilde{\theta}_{u,v}}^{\pi} p_{u,v} d\theta = \int_{\tilde{\theta}_{u,v}}^{\varepsilon} p_{u,v} d\theta + \int_{\varepsilon}^{\pi} p_{u,v} d\theta. \tag{2.12}$$

We will bound each one of the integrals on the right-hand side separately.

For the first integral we will use (2.4) together with the bound $\sin\theta \geq \theta - \theta^3$, which holds for any $\theta \leq \varepsilon$, provided that the latter is sufficiently small. More specifically, we let

$\varepsilon = \varepsilon(N) = 1/\omega(N)$; so $A_{u,v}\varepsilon(N) \to \infty$ as $N \to \infty$. We shall also use $(1-\theta^2)^\beta \geq 1 - \beta\theta^2$. Thus (after a change of variable where we replace $\theta/2$ by θ) for sufficiently large N, we have

$$\int_{\tilde{\theta}_{u,v}}^{\varepsilon} p_{u,v} d\theta \leq 2\int_{\tilde{\theta}_{u,v}/2}^{\varepsilon/2} \frac{1}{CA_{u,v}^\beta(\theta-\theta^3)^\beta+1} d\theta \leq 2\int_{\tilde{\theta}_{u,v}/2}^{\varepsilon/2} \frac{1}{CA_{u,v}^\beta \theta^\beta(1-\beta\theta^2)+1} d\theta$$
$$\leq 2\int_{\tilde{\theta}_{u,v}/2}^{\varepsilon/2} \frac{1}{CA_{u,v}^\beta \theta^\beta(1-\beta\varepsilon^2/4)+1} d\theta.$$

We change the variable in the last integral setting $z = \left[C(1-\beta\varepsilon^2/4)\right]^{1/\beta} A_{u,v}\theta$. Thus, we obtain:

$$\int_{\tilde{\theta}_{u,v}}^{\hat{\theta}'_{u,v}} p_{u,v} d\theta \leq \frac{2}{\left[C(1-\beta\varepsilon^2/4)\right]^{1/\beta} A_{u,v}} \int_{B_1}^{B_2} \frac{1}{z^\beta+1} dz,$$

where $B_1 = \left[C(1-\beta\varepsilon^2/4)\right]^{1/\beta} A_{u,v}\tilde{\theta}_{u,v}/2$ and $B_2 = \left[C(1-\beta\varepsilon^2/4)\right]^{1/\beta} A_{u,v}\varepsilon/2$. We have $B_1 = o(1)$ whereas, $B_2 = \left[C(1-\beta\varepsilon^2/4)\right]^{1/\beta} A_{u,v}\varepsilon/2 \to \infty$. So, Claim 2.6 yields:

$$\int_{\tilde{\theta}_{u,v}}^{\varepsilon} p_{u,v} d\theta \leq \begin{cases} (1+o(1)) \; \frac{2}{A_{u,v}} \; \int_0^\infty \frac{1}{z^\beta+1} dz, & \text{if } \beta > 1 \\ \\ (1+o(1)) \; \frac{2\ln A_{u,v}}{A_{u,v}}, & \text{if } \beta = 1 \end{cases}, \tag{2.13}$$

uniformly for all $u, v \in \mathcal{D}_{R,\omega}^{(2)}$. The second integral in (2.12) can be bounded easily.

$$\int_\varepsilon^\pi p_{u,v} d\theta = \int_\varepsilon^\pi \frac{1}{CA_{u,v}^\beta \; \sin^\beta(\theta/2)+1} d\theta = O\left(\frac{1}{(A_{u,v}\varepsilon)^\beta}\right) = o(A_{u,v}^{-1}).$$

Hence, uniformly for all $u, v \in \mathcal{D}_{R,\omega}^{(2)}$ we have

$$\int_{\tilde{\theta}_{u,v}}^{\pi} p_{u,v} d\theta \leq \begin{cases} (1+o(1)) \; \frac{2}{A_{u,v}} \; \int_0^\infty \frac{1}{z^\beta+1} dz, & \text{if } \beta > 1 \\ \\ (1+o(1)) \; \frac{2\ln A_{u,v}}{A_{u,v}}, & \text{if } \beta = 1 \end{cases}. \tag{2.14}$$

Thereby, (2.5) together with (2.6) and (2.11),(2.14) yield the lemma for $\beta \geq 1$. Finally, note that when $\beta > 1$ we have $\int_0^\infty \frac{1}{z^\beta+1} dz = \frac{\pi}{\beta} \sin^{-1}\left(\frac{\pi}{\beta}\right)$.

We now conclude the proof of the lemma with the proof of Claim 2.5.

Proof of Claim 2.5. We will show that $A_{u,v}^{-1} \gg \hat{\theta}_{u,v}$. We will show that

$$\frac{\zeta}{2}\left(R - (t_u + t_v)\right) + \frac{1}{2}\log\left(e^{-2\zeta(R-t_u)} + e^{-2\zeta(R-t_v)}\right) \leq \frac{-\zeta\omega(N)+1}{2}. \tag{2.15}$$

Adding and subtracting R inside the brackets in the first summand, we obtain:

$$\frac{\zeta}{2}(R-(t_u+t_v))+\frac{1}{2}\log\left(e^{-2\zeta(R-t_u)}+e^{-2\zeta(R-t_v)}\right)$$
$$=-\frac{\zeta R}{2}+\frac{1}{2}\left(\zeta(R-t_u)+\zeta(R-t_v)\right)+\frac{1}{2}\log\left(e^{-2\zeta(R-t_u)}+e^{-2\zeta(R-t_v)}\right).$$

For notational convenience, we write $a=\zeta(R-t_u)$ and $b=\zeta(R-t_v)$. Without loss of generality, assume that $a\le b$. Thus, the above expression is now written as

$$-\frac{\zeta R}{2}+\frac{1}{2}(a+b)+\frac{1}{2}\log\left(e^{-2a}+e^{-2b}\right)=-\frac{\zeta R}{2}+\frac{1}{2}\left[a+b+\log e^{-2a}+\log(1+e^{-2(b-a)})\right]$$
$$=-\frac{\zeta R}{2}+\frac{1}{2}\left[b-a+\log(1+e^{-2(b-a)})\right]\le-\frac{\zeta R}{2}+\frac{1}{2}\left[b-a+e^{-2(b-a)}\right].$$

Note that $b-a=\zeta(t_u-t_v)\le\zeta\left(R-t_v-\omega(N)-t_v\right)\le\zeta\left(R-\omega(N)\right)$. Therefore,

$$-\frac{\zeta R}{2}+\frac{1}{2}(a+b)+\frac{1}{2}\log\left(e^{-2a}+e^{-2b}\right)\le\frac{-\zeta\omega(N)+1}{2}\to-\infty.$$

□

□

The following lemma provides bounds for pairs of points that (essentially) belong to the complement of $\mathcal{D}^{(2)}_{R,\omega}$. In fact, we will give estimates on $\hat{p}_{u,v}$ for pairs u,v satisfying $t_v>R-t_u-\omega(N)$. These bounds will be particularly useful for the case $\zeta/\alpha\ge 2$. For a point $v\in\mathcal{D}_R$, recall that r_v denotes its radius in $\mathcal{D}_R$; thus $r_v=R-t_v$. Note also that the assumption $t_v>R-t_u-\omega(N)$ implies that $r_v<t_u+\omega(N)$.

Lemma 2.7. *Let $\beta>0$. For all $u,v\in\mathcal{D}_R$ such that $t_v>R-t_u-\omega(N)$,*

$$\text{if } r_v\le t_u, \text{ then } \hat{p}_{u,v}\ge\frac{1}{2}, \text{ and}$$

if $t_u<r_v\le R$, then with $x=r_v-t_u$ we have

$$\frac{1}{2\pi}e^{-\frac{\zeta}{2}x}\int_2^{2\pi e^{\frac{\zeta}{2}x}}\frac{1}{z^\beta+1}dz\le\hat{p}_{u,v}\le\frac{4}{\pi}e^{-\frac{\zeta}{2}x}\int_{1/4}^{\frac{\pi}{4}e^{\frac{\zeta}{2}x}}\frac{1}{z^\beta+1}dz+\frac{e^{-\zeta x/2}}{\pi}.$$

Proof. Here, we need to give cruder bounds on $d(u,v)$, as we may no longer be able to apply Lemma 2.3. Let us consider first the case where $r_v\le t_u$. Since $t_u=R-r_u$, we have $r_v+r_u\le R$. Also, the triangle inequality implies that $d(u,v)\le r_v+r_u$, which, in turn, yields that $d(u,v)\le R$. Hence

$$\exp\left(\beta\ \frac{\zeta}{2}\ (d(u,v)-R)\right)\le 1$$

and therefore

$$p_{u,v}\ge\frac{1}{2}.$$

This implies the statement of the lemma in the case where $r_v \leq t_u$.

Assume now that $r_v > t_u$. Let $r_v = t_u + x$, where $0 < x \leq R - t_u$. The hyperbolic law of cosines can be written as follows:

$$\cosh\left(\zeta d(u,v)\right) = \frac{1}{2}\left(\cosh\left(\zeta(r_u + r_v)\right)\left(1 - \cos(\theta_{u,v})\right) + \cosh\left(\zeta|r_u - r_v|\right)\left(1 + \cos(\theta_{u,v})\right)\right), \tag{2.16}$$

where $\theta_{u,v}$ is the relative angle between u and v. We will use this to provide bounds on $e^{\zeta d(u,v)}$. Note that the following inequalities hold for every $y > 0$:

$$\frac{1}{2}e^{y} \leq \cosh(y) \leq \frac{1}{2}\left(e^{y} + 1\right). \tag{2.17}$$

Hence applying the above bounds on (2.16) we obtain bounds on $e^{\zeta d(u,v)}$

$$\begin{aligned}
e^{\zeta d(u,v)} + 1 &\geq \frac{1}{2}\ e^{\zeta(r_u + r_v)}\left(1 - \cos(\theta_{u,v})\right) \ \text{ and} \\
\frac{1}{2}e^{\zeta d(u,v)} &\leq \frac{1}{4}\left(e^{\zeta(r_u+r_v)} + 1\right)\left(1 - \cos(\theta_{u,v})\right) + \frac{1}{4}\ \left(e^{\zeta|r_u - r_v|} + 1\right)\left(1 - \cos(\theta_{u,v})\right) \\
&\leq \frac{1}{4}e^{\zeta(r_u+r_v)}(1 - \cos(\theta_{u,v})) + \frac{1}{2}\ e^{\zeta|r_u - r_v|} + \frac{1}{2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
e^{\zeta(d(u,v)-R)} &\geq \frac{1}{2}\ e^{\zeta(r_u+r_v-R)}\left(1 - \cos(\theta_{u,v})\right) - e^{-\zeta R} \\
&\overset{r_v = t_u + x}{=} \frac{1}{2}\ e^{\zeta(r_u + t_u + x - R)}\left(1 - \cos(\theta_{u,v})\right) - e^{-\zeta R} \\
&\overset{r_u + t_u = R}{=} \frac{1}{2}\ e^{\zeta x}\left(1 - \cos(\theta_{u,v})\right) - e^{-\zeta R}, \ \text{ and}
\end{aligned}$$

$$\begin{aligned}
e^{\zeta(d(u,v)-R)} &\leq \frac{1}{2}\ e^{\zeta(r_u+r_v-R)}\left(1 - \cos(\theta_{u,v})\right) + e^{\zeta(|r_u - r_v| - R)} + e^{-\zeta R} \\
&\overset{|r_u - r_v| \leq R}{\leq} \frac{1}{2}\ e^{\zeta(r_u+r_v-R)}\left(1 - \cos(\theta_{u,v})\right) + 2 \\
&= \frac{1}{2}\ e^{\zeta x}\left(1 - \cos(\theta_{u,v})\right) + 2.
\end{aligned}$$

Note also that

$$\frac{\theta_{u,v}^2}{5} \leq 1 - \cos(\theta_{u,v}) \leq \theta_{u,v}^2$$

whereby

$$\frac{1}{2}\ e^{\zeta x}\ \frac{\theta_{u,v}^2}{5} - e^{-\zeta R} \leq e^{\zeta(d(u,v)-R)} \leq \frac{1}{2}\ e^{\zeta x}\ \theta_{u,v}^2 + 2. \tag{2.18}$$

But if $e^{-\zeta x/2} \leq \theta_{u,v} \leq \pi$, then

$$\frac{1}{2}\ e^{\zeta x}\ \theta_{u,v}^2 + 2 = \frac{1}{2}\ e^{\zeta x}\ \theta_{u,v}^2(1 + 4e^{-\zeta x}\ \theta_{u,v}^{-2}) \leq 4\ e^{\zeta x}\ \theta_{u,v}^2, \text{ and}$$
$$\frac{1}{2}\ e^{\zeta x}\ \frac{\theta_{u,v}^2}{5} - e^{-\zeta R} \geq e^{\zeta x}\ \frac{\theta_{u,v}^2}{4^2},$$

for N sufficiently large. Hence (2.18) becomes

$$\frac{1}{4^2}\ e^{\zeta x}\ \theta_{u,v}^2 \leq e^{\zeta (d(u,v)-R)} \leq 4\ e^{\zeta x}\ \theta_{u,v}^2, \text{for } e^{-\zeta x/2} \leq \theta_{u,v} \leq \pi,$$

which, raised to the $\beta/2$, yields

$$\frac{1}{4^\beta}\ e^{\beta\frac{\zeta}{2}x}\ \theta_{u,v}^\beta \leq e^{\beta\frac{\zeta}{2}(d(u,v)-R)} \leq 2^\beta\ e^{\beta\frac{\zeta}{2}x}\ \theta_{u,v}^\beta, \text{ for } e^{-\zeta x/2} \leq \theta_{u,v} \leq \pi. \tag{2.19}$$

In turn, this implies that

$$\frac{1}{2^\beta\ e^{\beta\frac{\zeta}{2}x}\ \theta_{u,v}^\beta + 1} \leq p_{u,v} \leq \frac{1}{4^{-\beta}\ e^{\beta\frac{\zeta}{2}x}\ \theta_{u,v}^\beta + 1}, \text{ if } e^{-\zeta x/2} \leq \theta_{u,v} \leq \pi. \tag{2.20}$$

Integrating over $\theta_{u,v}$ we obtain:

$$\frac{1}{\pi}\int_{e^{-\zeta x/2}}^{\pi} \frac{1}{2^\beta\ e^{\beta\frac{\zeta}{2}x}\ \theta^\beta + 1} d\theta \leq \hat{p}_{u,v} \leq \frac{1}{\pi}\int_{e^{-\zeta x/2}}^{\pi} \frac{1}{4^{-\beta}\ e^{\beta\frac{\zeta}{2}x}\ \theta^\beta + 1} d\theta + \frac{e^{-\zeta x/2}}{\pi}. \tag{2.21}$$

The integral that appears in the above inequalities can be written as follows:

$$\int_{e^{-\zeta x/2}}^{\pi} \frac{1}{c^\beta\ e^{\beta\frac{\zeta}{2}x}\ \theta^\beta + 1} d\theta \overset{z=ce^{\frac{\zeta}{2}x}\theta}{=} c^{-1}e^{-\frac{\zeta}{2}x} \int_{c}^{c\pi e^{\frac{\zeta}{2}x}} \frac{1}{z^\beta + 1} dz,$$

which implies that

$$\frac{1}{2\pi}e^{-\frac{\zeta}{2}x} \int_{2}^{2\pi e^{\frac{\zeta}{2}x}} \frac{1}{z^\beta + 1} dz \leq \hat{p}_{u,v} \leq \frac{4}{\pi}e^{-\frac{\zeta}{2}x} \int_{1/4}^{\frac{\pi}{4}e^{\frac{\zeta}{2}x}} \frac{1}{z^\beta + 1} dz + \frac{e^{-\zeta x/2}}{\pi}.$$

The lemma now follows. □

2.1 Interlude: hyperbolic random graphs as inhomogeneous random graphs

The notion of *inhomogeneous random graphs*, was introduced Söderberg [22] but was defined more generally and studied in great detail by Bollobás, Janson and Riordan in [6]. In its most general setting, there is an underlying compact metric space $\mathcal{S}$ equipped with a measure μ on its Borel σ-algebra. This is the space of *types* of the vertices. A *kernel* κ is a bounded real-valued, non-negative function on $\mathcal{S} \times \mathcal{S}$, which is symmetric and measurable. It is assumed that the vertices of the random graph are points in $\mathcal{S}$. If $x, y \in \mathcal{S}$, then the corresponding vertices are joined with probability that is equal to $\frac{\kappa(x,y)}{N} \wedge 1$, where N is the total number of vertices, independently of every other pair. The points that are the vertices of the graph are approximately distributed according to μ. More specifically, the empirical distribution function on the N points converges weakly to μ as $N \to \infty$.

When $\beta > 1$, the above lemma gives an expression for the probability that two vertices u and v having types t_u and t_v, respectively, are adjacent. This expression is proportional to $A_{u,v}^{-1} = \frac{e^{\zeta t_u/2} e^{\zeta t_v/2}}{N}$. Thus a hyperbolic random graph at the cold regime may be viewed as an inhomogeneous random graph on N vertices with kernel function which is equal (up to a multiplicative constant) to $(1/x)^{\zeta/2}(1/y)^{\zeta/2}$, where $x, y \in (0,1]$. (Here, we have applied the transformation $e^{t_u} = 1/T_u$, where $T_u \in (0,1]$.) In fact, this kernel corresponds to the *Chung-Lu* model of random graphs with given expected degrees - see [10], [11] as well as (6.1.20) on page 124 in [16]. When $\beta < 1$, the corresponding kernel is that of a not-too-sparse inhomogeneous random graph.

However, the analogy does not go beyond the marginal probabilities of the edges, since if we condition on the types of the vertices the edges do not appear independently. As we shall see later in our analysis (cf. Section 5), if we condition on the event that v_1 and v_2 are both adjacent to a vertex u, then this increases the probability that v_1 is adjacent to v_2.

3 The distribution of the degree of a vertex: case $0 < \zeta/\alpha < 2$

In this section, we prove Theorems 1.1 and 1.3. Let us fix some $u \in V_N$. We will condition on the position of u in $\mathcal{D}_R$; in particular, we will condition on t_u and the angle θ_u. In this section as well as later, we will be writing $\rho(t)$ for $\alpha \sinh(\alpha(R-t))/(\cosh(\alpha R) - 1)$ – this is the density function of the type of a vertex. The re-use of notation at this point should be causing no confusion.

Let us fix first t_u and θ_u such that $t_u \le x_0$ and $\theta_u \in (0, 2\pi]$. (Recall that by Corollary 2.2 we have that a.a.s. $t_u \le x_0 = \zeta R/(2\alpha) + \omega(N)$ for all $u \in V_N$.) Denoting by V_N^u the set $V_N \setminus \{u\}$, we write $D_u = \sum_{v \in V_N^u} I_{uv}$, where I_{uv} is the indicator random variable that is equal to 1 if and only if the edge $\{u, v\}$ is present in $\mathcal{G}(N; \nu, \zeta, \alpha, \beta)$. Note that conditional on t_u and θ_u the family $\{I_{uv}\}_{v \in V_N^u}$ is a family of independent and identically distributed random variables.

We begin with the estimation of the expectation of I_{uv} for an arbitrary $v \in V_N^u$ conditional on t_u and θ_u.

Lemma 3.1. *Let $\beta > 0$ and $0 < \zeta/\alpha < 2$. Let also $u, v \in V_N$ be two distinct vertices. There exists a constant $K = K(\zeta, \alpha, \beta, \nu) > 0$ such that uniformly for all $t_u \le x_0$ and $\theta_u \in (0, 2\pi]$, we have*

$$\Pr\Big[I_{uv} = 1 \mid t_u, \theta_u \Big] = \begin{cases} (1+o(1))\ K\ \frac{e^{\zeta t_u/2}}{N}, & \text{if } \beta > 1 \\ (1+o(1))\ K\ (R - t_u)\ \frac{e^{\zeta t_u/2}}{N}, & \text{if } \beta = 1 \\ (1+o(1))\ K\ \left(\frac{e^{\zeta t_u/2}}{N}\right)^{\beta}, & \text{if } \beta < 1 \end{cases}.$$

In particular, we have

$$K = \begin{cases} \frac{4\alpha\nu}{2\alpha-\zeta}\ \frac{1}{\beta}\ \sin^{-1}\left(\frac{\pi}{\beta}\right), & \text{if } \beta > 1 \\ \\ \frac{1}{\pi}\ \frac{2\alpha\zeta\nu}{2\alpha-\zeta}, & \text{if } \beta = 1 \\ \\ \frac{1}{\sqrt{\pi}}\ \frac{2\alpha\nu^{\beta}}{2\alpha-\beta\zeta}\ \frac{\Gamma\left(\frac{1-\beta}{2}\right)}{\Gamma\left(1-\frac{\beta}{2}\right)}, & \text{if } \beta < 1 \end{cases}.$$

Proof. We write $\hat{p}(t_u, t_v)$ for $\hat{p}_{u,v}$ as the latter depends only on t_u and t_v. Hence, we have

$$\begin{aligned} \Pr\Big[I_{uv} = 1 \mid t_u, \theta_u \Big] &= \int_0^R \hat{p}(t_u, t_v)\rho(t_v)dt_v \\ &= \int_0^{R-t_u-\omega(N)} \hat{p}(t_u, t_v)\rho(t_v)dt_v + \int_{R-t_u-\omega(N)}^R \hat{p}(t_u, t_v)\rho(t_v)dt_v. \end{aligned} \tag{3.1}$$

The second integral can be bounded as follows.

$$\begin{aligned} &\int_{R-t_u-\omega(N)}^R \hat{p}(t_u, t_v)\rho(t_v)dt_v \le \int_{R-t_u-\omega(N)}^R \rho(t_v)dt_v = \frac{\cosh(\alpha(t_u+\omega(N))) - \cosh(0)}{\cosh(\alpha R) - 1} \\ &= \frac{e^{\alpha(t_u+\omega(N))}}{\cosh(\alpha R) - 1}(1-o(1)) = \nu^{2\alpha/\zeta}\ \frac{e^{\alpha(t_u+\omega(N))}}{N^{2\alpha/\zeta}}(1-o(1)) = e^{\alpha\omega(N)}\left(\nu\ \frac{e^{\zeta t_u/2}}{N}\right)^{2\alpha/\zeta}(1-o(1)). \end{aligned} \tag{3.2}$$

For the first integral we use the estimates obtained in Lemma 2.4. We set $K_1 := (1+o(1))C_\beta$. We treat each one of the three cases separately. Here as well as in the next section, where we treat the case $\zeta/\alpha \ge 2$, we will be using the following elementary integrals.

Lemma 3.2. *Let $0 \le A < B$ and $\gamma, c > 0$ be real numbers. Then the following hold:*

$$\begin{aligned} \mathfrak{I}_1(\gamma, c; A, B) &:= \int_A^B e^{-\gamma x}\sinh(\alpha x + c)dx \\ &= \begin{cases} \frac{1}{2}\left(\frac{e^{-(\gamma-\alpha)A} - e^{-(\gamma-\alpha)B}}{\gamma-\alpha}\ e^{c} - \frac{e^{-(\gamma+\alpha)A} - e^{-(\gamma+\alpha)B}}{\gamma+\alpha}\ e^{-c}\right), & \text{if } \gamma \ne \alpha \\ \\ \frac{1}{2}\left(e^{c}(B-A) - \frac{e^{-c}}{\gamma+\alpha}\left(e^{-2\alpha A} - e^{-2\alpha B}\right)\right), & \text{if } \gamma = \alpha \end{cases}. \end{aligned}$$

$$\begin{aligned} \mathfrak{I}_2(\gamma, c; A, B) &:= \int_A^B x e^{-\gamma x}\sinh(\alpha x + c)dx \\ &= \begin{cases} \frac{1}{2}\ \left(\frac{e^{c}}{\gamma-\alpha}\ \left(\left(A + \frac{1}{\gamma-\alpha}\right)e^{-(\gamma-\alpha)A} - \left(B + \frac{1}{\gamma-\alpha}\right)e^{-(\gamma-\alpha)B}\right) \right. \\ \left. -\frac{e^{-c}}{\gamma+\alpha}\ \left(\left(A + \frac{1}{\gamma+\alpha}\right)e^{-(\gamma+\alpha)A} - \left(B + \frac{1}{\gamma+\alpha}\right)e^{-(\gamma+\alpha)B}\right)\right), & \text{if } \gamma \ne \alpha \\ \\ \frac{1}{2}\ \left(\frac{e^{c}}{2}\ (B^2 - A^2) - \frac{e^{-c}}{2\alpha}\ \left(\left(A + \frac{1}{2\alpha}\right)e^{-2\alpha A} - \left(B + \frac{1}{2\alpha}\right)e^{-2\alpha B}\right)\right), & \text{if } \gamma = \alpha \end{cases}. \end{aligned}$$

We now proceed with the three different cases according to the value of β.

$\beta > 1$

In this case, we have:

$$\begin{aligned}
&\int_0^{R-t_u-\omega(N)} \hat{p}(t_u,t_v)\rho(t_v)dt_v = K_1 \int_0^{R-t_u-\omega(N)} \frac{1}{A(t_u,t_v)}\ \rho(t_v)dt_v \\
&= K_1 e^{\zeta t_u/2} \int_0^{R-t_u-\omega(N)} e^{-\frac{\zeta}{2}(R-t_v)}\ \rho(t_v)dt_v \\
&= \alpha K_1 e^{\zeta t_u/2} \int_0^{R-t_u-\omega(N)} e^{-\frac{\zeta}{2}(R-t_v)}\ \frac{\sinh(\alpha(R-t_v))}{\cosh(\alpha R)-1} dt_v \\
&= \alpha K_1 \frac{e^{\zeta t_u/2}}{\cosh(\alpha R)-1} \int_{t_u+\omega(N)}^{R} e^{-\frac{\zeta t_v}{2}}\ \sinh(\alpha t_v)dt_v \\
&= \alpha K_1 \frac{e^{\zeta t_u/2}}{\cosh(\alpha R)-1} \mathfrak{I}_1(\zeta/2, 0; t_u+\omega(N), R) \\
&\overset{Lemma\ 3.2, t_u\le x_0}{=} (1+o(1))\ \frac{\alpha K_1}{2\alpha-\zeta}\ \frac{e^{\zeta t_u/2}}{\cosh(\alpha R)-1}\ e^{R\left(\alpha-\frac{\zeta}{2}\right)} \\
&= (1+o(1))\ \frac{2\alpha K_1}{2\alpha-\zeta}\ \frac{e^{\zeta t_u/2}}{e^{\zeta R/2}}\ \frac{e^{\alpha R}}{2(\cosh(\alpha R)-1)} = (1+o(1))\ K\ \nu\ \frac{e^{\zeta t_u/2}}{N}.
\end{aligned} \tag{3.3}$$

$\beta = 1$

We perform a similar but slightly more involved calculation.

$$\begin{aligned}
&\int_0^{R-t_u-\omega(N)} \hat{p}(t_u,t_v)\rho(t_v)dt_v = K_1 \int_0^{R-t_u-\omega(N)} \frac{\ln A(t_u,t_v)}{A(t_u,t_v)}\ \rho(t_v)dt_v \\
&= K_1 e^{\zeta t_u/2} \int_0^{R-t_u-\omega(N)} \left(\frac{\zeta}{2}(R-t_u-t_v)\right) e^{-\frac{\zeta}{2}(R-t_v)}\ \rho(t_v)dt_v = \\
&= K_1 \frac{\zeta e^{\zeta t_u/2}}{2} \times \\
&\left[\int_0^{R-t_u-\omega(N)} (R-t_v)\, e^{-\frac{\zeta}{2}(R-t_v)}\ \rho(t_v)dt_v - t_u \int_0^{R-t_u-\omega(N)} e^{-\frac{\zeta}{2}(R-t_v)}\ \rho(t_v)dt_v\right].
\end{aligned} \tag{3.4}$$

The second integral is as in (3.3)

$$\int_0^{R-t_u-\omega(N)} e^{-\frac{\zeta}{2}(R-t_v)}\ \rho(t_v)dt_v = (1+o(1))\frac{2\alpha}{2\alpha-\zeta}\ \frac{\nu}{N}.$$

For the first one, we use Lemma 3.2:

$$\int_0^{R-t_u-\omega(N)} (R-t_v)\, e^{-\frac{\zeta}{2}(R-t_v)}\ \rho(t_v)dt_v =$$
$$\frac{\alpha}{\cosh(\alpha R)-1}\int_0^{R-t_u-\omega(N)} (R-t_v)\, e^{-\frac{\zeta}{2}(R-t_v)}\ \sinh(\alpha(R-t_v))dt_v$$
$$= \frac{\alpha}{\cosh(\alpha R)-1}\Im_2(\zeta/2, 0; t_u+\omega(N), R)$$
$$\stackrel{Lemma\ 3.2, t_u\le x_0}{=} \frac{1-o(1)}{2\alpha-\zeta}\ \frac{\alpha}{\cosh(\alpha R)-1} R e^{\left(\alpha-\frac{\zeta}{2}\right)R}.$$

Substituting this into (3.4), we finally obtain

$$\int_0^{R-t_u-\omega(N)} \hat{p}(t_u,t_v)\rho(t_v)dt_v = (1+o(1))K_1\frac{2\alpha\zeta\nu}{2\alpha-\zeta}\ (R-t_u)\ \frac{e^{\zeta t_u/2}}{N}. \tag{3.5}$$

$\beta < 1$

This case is similar to (3.3)

$$\begin{aligned}
&\int_0^{R-t_u-\omega(N)} \hat{p}(t_u,t_v)\rho(t_v)dt_v = K_1\int_0^{R-t_u-\omega(N)} \frac{1}{A(t_u,t_v)^{\beta}}\ \rho(t_v)dt_v \\
&= K_1 e^{\beta\zeta t_u/2}\int_0^{R-t_u-\omega(N)} e^{-\beta\frac{\zeta}{2}(R-t_v)}\ \rho(t_v)dt_v \\
&= \alpha K_1 e^{\beta\zeta t_u/2}\int_0^{R-t_u-\omega(N)} e^{-\frac{\beta\zeta}{2}(R-t_v)}\ \frac{\sinh(\alpha(R-t_v))}{\cosh(\alpha R)-1}dt_v \\
&= \alpha K_1\frac{e^{\beta\zeta t_u/2}}{\cosh(\alpha R)-1}\int_{t_u+\omega(N)}^{R} e^{-\frac{\beta\zeta t_v}{2}}\ \sinh(\alpha t_v)dt_v \\
&= \alpha K_1\frac{e^{\beta\zeta t_u/2}}{\cosh(\alpha R)-1}\Im_1(\beta\zeta/2, 0; t_u+\omega(N), R) \\
&\stackrel{Lemma\ 3.2, t_u\le x_0}{=} (1+o(1))\ \frac{\alpha K_1}{2\alpha-\beta\zeta}\ \frac{e^{\beta\zeta t_u/2}}{\cosh(\alpha R)-1}\ e^{R\left(\alpha-\beta\frac{\zeta}{2}\right)} \\
&= (1+o(1))\ \frac{2\alpha K_1}{2\alpha-\beta\zeta}\ \frac{e^{\beta\zeta t_u/2}}{e^{\beta\zeta R/2}}\ \frac{e^{\alpha R}}{2(\cosh(\alpha R)-1)} = (1+o(1))\ K\ \left(\frac{e^{\zeta t_u/2}}{N}\right)^{\beta}.
\end{aligned} \tag{3.6}$$

Thus combining (3.2) together (3.3), (3.5) and (3.6) the lemma follows. □

It is clear that the proof of the above lemma yields also the following corollary.

Corollary 3.3. *Let $0 < \zeta/\alpha < 2$. There exists a constant $K = K(\zeta,\alpha,\beta,\nu) > 0$ such that* *uniformly for all* $t_u \leq x_0$*, we have*

$$\Pr\Big[I_{uv} = 1 \mid t_u \Big] = \begin{cases} (1+o(1))\ K\ \frac{e^{\zeta t_u/2}}{N}, & \text{if } \beta > 1 \\ (1+o(1))\ K\ (R - t_u)\ \frac{e^{\zeta t_u/2}}{N}, & \text{if } \beta = 1 \\ (1+o(1))\ K\ \left(\frac{e^{\zeta t_u/2}}{N}\right)^{\beta}, & \text{if } \beta < 1 \end{cases}.$$

Now, Theorem 1.3 follows immediately from Corollary 3.3. For the cold regime we have to work slightly more.

3.1 The cold regime $\beta > 1$

Let $\hat{D}_u$ be a Poisson random variable with parameter equal to $T_u := \sum_{v \in V_N^u} \Pr\Big[I_{uv} = 1 \mid t_u \Big] = (1+o(1))Ke^{\zeta t_u/2}$. With $d_{TV}(\cdot,\cdot)$ denoting the total variation distance between two random variables, we deduce the following lemma.

Lemma 3.4. *Conditional on the value of* t_u*, we have*

$$d_{TV}\left(D_u, \hat{D}_u\right) = o(1),$$

uniformly for all $t_u \leq R/2 - \omega(N)$*.*

Proof. Recall that $D_u = \sum_{v \in V_N^u} I_{uv}$ and conditional on t_u and θ_u this is a sum of independent and identically distributed indicator random variables. Keeping t_u fixed, this is also the case when we average over $\theta_u \in (0, 2\pi]$. Thus conditioning only on t_u, the family $\{I_{uv}\}_{v \in V_N^u}$ is still a family of i.i.d. indicator random variables, whose expected values are given by Lemma 3.1.

By Theorem 2.9 in [16] and Corollary 3.3 we have for N sufficiently large

$$d_{TV}(D_u, \hat{D}_u) \leq \sum_{v \in V_N^u} \Pr\Big[I_{uv} = 1 \mid t_u \Big]^2 \leq 2KN \left(\frac{e^{\zeta R/4 - \zeta\omega(N)/2}}{N} \right)^2 = 2Ne^{-\zeta\omega(N)}\ \frac{1}{N} = o(1).$$

□

For any integer $k \geq 0$ we have

$$\begin{aligned} \Pr\Big[D_u = k \Big] &= \frac{1}{2\pi} \int_0^R \int_0^{2\pi} \Pr\Big[D_u = k \mid t_u, \theta_u \Big] \rho(t_u) dt_u d\theta_u \\ &= \frac{1}{2\pi} \int_{t_u \leq R/2 - \omega(N)} \int_0^{2\pi} \Pr\Big[D_u = k \mid t_u, \theta_u \Big] \rho(t_u) dt_u d\theta_u \\ &+ \frac{1}{2\pi} \int_{t_u > R/2 - \omega(N)} \int_0^{2\pi} \Pr\Big[D_u = k \mid t_u, \theta_u \Big] \rho(t_u) dt_u d\theta_u. \end{aligned} \tag{3.7}$$

We bound the second integral as follows.

$$\int_{R/2-\omega(N)<t_u} \Pr\Big[\, D_u = k \mid t_u \,\Big]\rho(t_u)dt_u \le \int_{R/2-\omega(N)<t_u} \rho(t_u)dt_u = o(1). \tag{3.8}$$

We will use Lemma 3.4 to approximate the first integral.

$$\begin{aligned}\int_{t_u\le R/2-\omega(N)} \Pr\Big[\, D_u = k \mid t_u \,\Big]\rho(t_u)dt_u &= \int_{t_u\le R/2-\omega(N)} \Pr\Big[\, \hat{D}_u = k \,\Big]\rho(t_u)dt_u + o(1)\\ = \int_{t_u\le R} \Pr\Big[\, \hat{D}_u = k \,\Big]\rho(t_u)dt_u + o(1). \end{aligned}\tag{3.9}$$

But recall that $\Pr\Big[\, \hat{D}_u = k \,\Big] = \Pr\Big[\, \mathsf{Po}\,(T_u) = k \,\Big]$. Let $K_N = (1+o(1))K$ denote the factor of $e^{\zeta t_u/2}$ in the expression of T_u. If $t < K$, then $\Pr\Big[\, T_u \le t \,\Big] \to 0$ as $N \to \infty$. However, for any $t \ge K$ we have

$$\begin{aligned}\Pr\Big[\, T_u \le t \,\Big] &= \Pr\Big[\, t_u \le \frac{2}{\zeta}\,\ln\frac{t}{K_N} \,\Big] = \alpha \int_0^{\frac{2}{\zeta}\,\ln\frac{t}{K_N}} \frac{\sinh(\alpha(R-x))}{\cosh(\alpha R)-1}dx\\ &= \alpha \int_{R-\frac{2}{\zeta}\,\ln\frac{t}{K_N}}^{R} \frac{\sinh(\alpha x)}{\cosh(\alpha R)-1}dx\\ &= \frac{1}{\cosh(\alpha R)-1}\left[\cosh(\alpha R) - \cosh\left(\alpha R - \frac{2\alpha}{\zeta}\,\ln\frac{t}{K_N}\right)\right] = 1 - \left(\frac{K}{t}\right)^{\frac{2\alpha}{\zeta}} + o(1).\end{aligned}$$

In other words,

$$\Pr\Big[\, T_u \le t \,\Big] \to F(t), \text{ as } N \to \infty.$$

Thus,

$$\int_{t_u\le R/2-\omega(N)} \Pr\Big[\, D_u = k \mid t_u \,\Big]\rho(t_u)dt_u \to \Pr\Big[\, \mathsf{MP}(F) = k \,\Big], \text{ as } N \to \infty.$$

The above together with (3.8) and (3.7) complete the proof of Theorem 1.1.

3.1.1 Power laws

We close this section with a simple calculation proving that $\Pr\Big[\, \mathsf{MP}(F) = k \,\Big]$ has power-law behaviour with exponent $2\alpha/\zeta + 1$ as k grows.

Lemma 3.5. *We have*

$$\Pr\Big[\, \mathsf{MP}(F) = k \,\Big] \to \frac{2\alpha}{\zeta} K^{2\alpha/\zeta}\; k^{-2\alpha/\zeta-1} \text{ as } k, N \to \infty.$$

Proof. The pdf of $F(t)$ for $t > K$ is equal to $\frac{2\alpha}{\zeta}\; K^{2\alpha/\zeta}/t^{-2\alpha/\zeta-1}$ and equal to 0 otherwise.

Thus we have

$$\begin{aligned}\Pr\Big[\,\mathsf{MP}(F)=k\,\Big] &= \frac{2\alpha}{\zeta}\; K^{2\alpha/\zeta}\; \int_K^\infty e^{-t}\;\frac{t^k}{k!}\; t^{-2\alpha/\zeta-1}dt \\ &= \frac{2\alpha}{\zeta}\; K^{2\alpha/\zeta}\; \frac{1}{k!}\; \int_K^\infty e^{-t}\; t^{k-2\alpha/\zeta-1}dt \\ &= \frac{2\alpha}{\zeta}\; K^{2\alpha/\zeta}\; \frac{1}{k!}\; \left(\Gamma(k-2\alpha/\zeta)-\int_0^K e^{-t}\; t^{k-2\alpha/\zeta-1}dt\right).\end{aligned} \tag{3.10}$$

Note now that the last integral is $O(K^k)$ and therefore, as $k\to\infty$, we have

$$\frac{\int_0^K e^{-t}\; t^{k+2\alpha/\zeta+1}dt}{k!} = O\left(\left(\frac{Ke}{k}\right)^k\right).$$

Now using the standard asymptotics for the Gamma function we have

$$\Gamma(k-2\alpha/\zeta) = (1+o(1))\sqrt{2\pi(k-2\alpha/\zeta-1)}\; e^{-k+2\alpha/\zeta+1}\;\; (k-2\alpha/\zeta-1)^{k-2\alpha/\zeta-1},$$

and also $k! = (1+o(1))\sqrt{2\pi k}e^{-k}k^k$. Thus,

$$\begin{aligned}\frac{\Gamma(k-2\alpha/\zeta)}{k!} &= (1+o(1))e^{2\alpha/\zeta+1}\left(1-\frac{2\alpha/\zeta+1}{k}\right)^{k-2\alpha/\zeta-1}\;\; k^{-(2\alpha/\zeta+1)} \\ &= (1+o(1))\; k^{-(2\alpha/\zeta+1)}.\end{aligned}$$

Thus, (3.10) now yields as $k\to\infty$

$$\Pr\Big[\,\mathsf{MP}(F)=k\,\Big] = \frac{2\alpha}{\zeta}\; K^{2\alpha/\zeta}\; k^{-(2\alpha/\zeta+1)}(1+o(1)).$$

□

4 The distribution of the degree of a vertex: case $\zeta/\alpha \geq 2$

We now deal with the case $\zeta/\alpha \geq 2$. In this case, the degree of a vertex is heavily controlled by vertices that are close to the centre of $\mathcal{D}_R$. To quantify this contribution, we will use Lemma 2.7. The following lemma gives the expected value of I_{uv} for an arbitrary $v \in V_N^u$ conditional on the value of t_u. Recall that for any two real numbers x and y, we let $\delta_{x,y}$ be the indicator function that is equal to 1 if and only if $x=y$.

Lemma 4.1. *Let $\beta > 0$ and $\zeta/\alpha \geq 2$. Let $u, v \in V_N$ be two distinct vertices. Then uniformly* *for all $t_u < R$, we have*

$$\Pr\Big[I_{uv} = 1 \mid t_u \Big] = \begin{cases} \Theta\left(\left((R-t_u)^{\delta_{2\alpha/\zeta,1}} + 1\right) \left(\frac{e^{\zeta t_u/2}}{N}\right)^{\frac{2\alpha}{\zeta}} \right), & \text{if } \beta > 2\alpha/\zeta \\ \Theta\left(\left((R-t_u)^{1+\delta_{2\alpha/\zeta,1}} + 1\right) \left(\frac{e^{\zeta t_u/2}}{N}\right)^{\frac{2\alpha}{\zeta}} \right), & \text{if } \beta = 2\alpha/\zeta \\ \Theta\left(\left(\frac{e^{\zeta t_u/2}}{N}\right)^{\beta} \right), & \text{if } \beta < 2\alpha/\zeta \end{cases}.$$

Proof. As before, we write $\hat{p}(t_u, t_v)$ for $\hat{p}_{u,v}$. We have

$$\begin{aligned} \Pr\Big[I_{uv} = 1 \mid t_u \Big] &= \int_0^R \hat{p}(t_u, t_v)\rho(t_v)dt_v = \int_0^R \hat{p}(t_u, R-r_v)\rho(r_v)dr_v \\ &= \int_0^{t_u} \hat{p}(t_u, R-r_v)\rho(r_v)dr_v + \int_{t_u}^R \hat{p}(t_u, R-r_v)\rho(r_v)dr_v =: I_1 + I_2. \end{aligned} \tag{4.1}$$

Here $\rho(r_v)$ is the density function of the radius of vertex v. We will provide bounds on I_1 and I_2 using Lemma 2.7 as well as some estimates that follow immediately from Lemma 3.2. These will be used in bounding I_2.

Corollary 4.2. *Let $\gamma > 0$ be a real number. Then uniformly over $t_u < R$, the following hold:*

$$\frac{\mathfrak{I}_1(\gamma, \alpha t_u; 0, R-t_u)}{\cosh(\alpha R) - 1} = \begin{cases} \frac{1}{\gamma-\alpha}\left(\frac{1}{2}\,\frac{e^{\alpha t_u}}{\cosh(\alpha R)-1} - e^{-\gamma(R-t_u)}(1+o(1)) \right) - O\left(\frac{e^{-\alpha t_u}}{\cosh(\alpha R)}\right), & \text{if } \gamma \neq \alpha \\ \frac{1}{2}\,\frac{e^{\alpha t_u}(R-t_u)}{\cosh(\alpha R)-1} - O\left(\frac{e^{-\alpha t_u}}{\cosh(\alpha R)}\right), & \text{if } \gamma = \alpha \end{cases},$$

$$\frac{\mathfrak{I}_2(\gamma, \alpha t_u; 0, R-t_u)}{\cosh(\alpha R) - 1} = \begin{cases} \frac{1}{2}\,\frac{1}{(\gamma-\alpha)^2}\frac{e^{\alpha t_u}}{\cosh(\alpha R)-1}\left(1 - ((R-t_u)(\gamma-\alpha)+1)e^{-(\gamma-\alpha)(R-t_u)}\right) + O\left(\frac{e^{-\alpha t_u}}{\cosh(\alpha R)}\right), & \text{if } \gamma > \alpha \\ \frac{1}{2}\,\frac{e^{\alpha t_u}(R-t_u)^2}{\cosh(\alpha R)-1} - O\left(\frac{e^{-\alpha t_u}}{\cosh(\alpha R)}\right), & \text{if } \gamma = \alpha \end{cases}.$$

The first bound in Lemma 2.7 implies that

$$\frac{1}{2}\int_0^{t_u} \rho(r_v)dr_v \leq I_1 \leq \int_0^{t_u} \rho(r_v)dr_v, \tag{4.2}$$

and

$$\int_0^{t_u} \rho(r_v)dr_v = \frac{\cosh(\alpha t_u) - 1}{\cosh(\alpha R) - 1} \leq \frac{1}{2}\,\frac{e^{\alpha t_u}}{\cosh(\alpha R) - 1}. \tag{4.3}$$

We estimate I_2 using the second bound in Lemma 2.7. In particular, we have

$$\begin{aligned} I_2 \leq & \frac{4}{\pi} \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v - t_u)} \left(\int_{1/4}^{\frac{\pi}{4} e^{\frac{\zeta}{2}(r_v - t_u)}} \frac{1}{z^\beta + 1} dz \right) \rho(r_v) dr_v \\ & + \frac{1}{\pi} \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v - t_u)} \rho(r_v) dr_v, \end{aligned} \tag{4.4}$$

and

$$I_2 \geq \frac{1}{2\pi} \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v - t_u)} \left(\int_{2}^{2\pi e^{\frac{\zeta}{2}(r_v - t_u)}} \frac{1}{z^\beta + 1} dz \right) \rho(r_v) dr_v. \tag{4.5}$$

The rate of growth of these bounds depends on β, and we will consider a case distinction. We will start however with the second integral in (4.4) whose value does not depend on β. Using Corollary 4.2 we have

$$\begin{aligned} \frac{1}{\pi} \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v - t_u)} \rho(r_v) dr_v &= \frac{1}{\pi} \int_{0}^{R - t_u} e^{-\frac{\zeta}{2} x} \rho(t_u + x) dx \\ &= \frac{\alpha}{\pi(\cosh(\alpha R) - 1)} \int_{0}^{R - t_u} e^{-\frac{\zeta}{2} x} \sinh(\alpha(t_u + x)) dx = \frac{\alpha}{\pi} \frac{\mathfrak{I}_1(\zeta/2, \alpha t_u; 0, R - t_u)}{\cosh(\alpha R) - 1} \\ &= \begin{cases} \frac{\alpha}{\pi} \frac{1}{\zeta - 2\alpha} \frac{e^{\alpha t_u}}{\cosh(\alpha R) - 1} \left(1 - e^{-(\zeta/2 - \alpha)(R - t_u)}(1 + o(1))\right) - O\left(\frac{e^{-\alpha t_u}}{\cosh(\alpha R)}\right), & \text{if } \zeta/\alpha > 2 \\ \frac{\alpha}{2\pi} \frac{e^{\alpha t_u}(R - t_u)}{\cosh(\alpha R)} (1 + o(1)), & \text{if } \zeta/\alpha = 2 \end{cases}. \end{aligned} \tag{4.6}$$

The first integral in (4.4) and that in (4.5) depend on whether $\beta < 1$, $\beta = 1$ or $\beta > 1$. We will consider each one of these cases separately.

$\beta > 1$

The first integral in the upper bound of (4.4) can be further bounded as follows:

$$\begin{aligned} & \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v - t_u)} \left(\int_{1/4}^{\frac{\pi}{4} e^{\frac{\zeta}{2}(r_v - t_u)}} \frac{1}{z^\beta + 1} dz \right) \rho(r_v) dr_v \leq \left(\int_{0}^{\infty} \frac{1}{z^\beta + 1} dz \right) \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v - t_u)} \rho(r_v) dr_v \\ &= \left(\int_{0}^{\infty} \frac{1}{z^\beta + 1} dz \right) \frac{\alpha}{\cosh(\alpha R) - 1} \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v - t_u)} \sinh(\alpha r_v) dr_v \\ &= \left(\int_{0}^{\infty} \frac{1}{z^\beta + 1} dz \right) \frac{\alpha}{\cosh(\alpha R) - 1} \int_{0}^{R - t_u} e^{-\frac{\zeta}{2} x} \sinh(\alpha(x + t_u)) dx \\ &= \left(\int_{0}^{\infty} \frac{1}{z^\beta + 1} dz \right) \frac{\alpha}{\cosh(\alpha R) - 1} \mathfrak{I}_1(\zeta/2, \alpha t_u; 0, R - t_u) \\ &\overset{Corollary\ 4.2}{=} \begin{cases} (1 + o(1)) \left(\int_{0}^{\infty} \frac{1}{z^\beta + 1} dz \right) \frac{\alpha}{\zeta - 2\alpha} \frac{e^{\alpha t_u}}{\cosh(\alpha R)}, & \text{if } \zeta/\alpha > 2 \\ (1 + o(1)) \left(\int_{0}^{\infty} \frac{1}{z^\beta + 1} dz \right) \frac{\alpha}{2} \frac{e^{\alpha t_u}(R - t_u)}{\cosh(\alpha R)} & \text{if } \zeta/\alpha = 2. \end{cases} \end{aligned}$$

Similarly, the integral in the lower bound in (4.5) is

$$\int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v-t_u)} \left(\int_{2}^{2\pi e^{\frac{\zeta}{2}(r_v-t_u)}} \frac{1}{z^\beta+1} dz \right) \rho(r_v) dr_v \geq$$

$$\begin{cases} (1+o(1)) \left(\int_2^{2\pi} \frac{1}{z^\beta+1} dz \right) \frac{\alpha}{\zeta-2\alpha} \frac{e^{\alpha t_u}}{\cosh(\alpha R)}, & \text{if } \zeta/\alpha > 2 \\ (1+o(1)) \left(\int_2^{2\pi} \frac{1}{z^\beta+1} dz \right) \frac{\alpha}{2} \frac{e^{\alpha t_u}(R-t_u)}{\cosh(\alpha R)}, & \text{if } \zeta/\alpha = 2 \end{cases}.$$

Hence the above two inequalities together with (4.2), (4.3), (4.4), (4.5) and (4.6) yield:

$$\int_0^R \hat{p}(t_u, t_v) \rho(t_v) dt_v = \begin{cases} \Theta\left(\frac{e^{\alpha t_u}}{\cosh(\alpha R)} \right), & \text{if } \zeta/\alpha > 2 \\ \Theta\left(\frac{e^{\alpha t_u}(R-t_u+1)}{\cosh(\alpha R)} \right), & \text{if } \zeta/\alpha = 2 \end{cases}. \tag{4.7}$$

$\beta = 1$

In this case, we have

$$\begin{aligned}
&\int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v-t_u)} \left(\int_{1/4}^{\frac{\pi}{4} e^{\frac{\zeta}{2}(r_v-t_u)}} \frac{1}{z+1} dz \right) \rho(r_v) dr_v \\
&\leq \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v-t_u)} \left(\int_{1/4}^{\frac{\pi}{4} e^{\frac{\zeta}{2}(r_v-t_u)}} \frac{1}{z} dz \right) \rho(r_v) dr_v \\
&= \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v-t_u)} \left(\frac{\zeta}{2}(r_v-t_u) + \ln(\pi/4) - \ln(1/4) \right) \rho(r_v) dr_v \\
&= \frac{\zeta}{2} \int_{t_u}^{R} (r_v-t_u) e^{-\frac{\zeta}{2}(r_v-t_u)} \rho(r_v) dr_v + \ln\pi \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v-t_u)} \rho(r_v) dr_v \\
&= \frac{\zeta}{2} \frac{\alpha}{\cosh(\alpha R)-1} \int_0^{R-t_u} x e^{-\frac{\zeta}{2}x} \sinh(\alpha x + \alpha t_u) dr_v \\
&+ \frac{\alpha \ln\pi}{\cosh(\alpha R)-1} \int_0^{R-t_u} e^{-\frac{\zeta}{2}x} \sinh(\alpha x + \alpha t_u) dr_v \\
&= \frac{\zeta}{2} \frac{\alpha}{\cosh(\alpha R)-1} \mathfrak{I}_2(\zeta/2, \alpha t_u; 0, R-t_u) + \frac{\alpha\ln\pi}{\cosh(\alpha R)-1} \mathfrak{I}_1(\zeta/2, \alpha t_u; 0, R-t_u) \\
&\overset{\text{Corollary 4.2}}{\leq} \begin{cases} \left(\frac{\alpha\zeta}{(\zeta-2\alpha)^2} + \frac{\alpha\ln\pi}{\zeta-2\alpha} \right) \frac{e^{\alpha t_u}}{\cosh(\alpha R)-1} + O\left(\frac{e^{-\alpha t_u}}{\cosh(\alpha R)} \right), & \text{if } \zeta/\alpha > 2 \\ (1+o(1)) \frac{e^{\alpha t_u}(R-t_u)^2}{\cosh(\alpha R)} \left(\frac{\zeta\alpha}{4} + \frac{\alpha\ln\pi}{2(R-t_u)} \right), & \text{if } \zeta/\alpha = 2 \end{cases}.
\end{aligned}$$

Hence, substituting this bound in (4.4) together with (4.6) yield

$$I_2 = \begin{cases} O\left(\frac{e^{\alpha t_u}}{\cosh(\alpha R)} \right), & \text{if } \zeta/\alpha > 2 \\ O\left(\frac{e^{\alpha t_u}(R-t_u)^2}{\cosh(\alpha R)} \left(\frac{\zeta\alpha}{2} + \frac{\alpha\ln\pi}{2(R-t_u)} \right) \right), & \text{if } \zeta/\alpha = 2 \end{cases}. \tag{4.8}$$

Similarly, the integral in (4.5) can be bounded as follows:

$$\int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v-t_u)} \left(\int_{2}^{2\pi e^{\frac{\zeta}{2}(r_v-t_u)}} \frac{1}{z+1}dz\right) \rho(r_v)dr_v$$
$$= \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v-t_u)} \left(\ln\left(2\pi e^{\frac{\zeta}{2}(r_v-t_u)}+1\right) - \ln(2+1)\right) \rho(r_v)dr_v$$
$$\geq \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v-t_u)} \left(\ln\left(2\pi e^{\frac{\zeta}{2}(r_v-t_u)}\right) - \ln 3\right) \rho(r_v)dr_v$$
$$= \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v-t_u)} \left(\frac{\zeta}{2}(r_v-t_u) + \ln(2\pi) - \ln 3\right) \rho(r_v)dr_v$$
$$= \frac{\zeta}{2}\int_{t_u}^{R} (r_v-t_u)e^{-\frac{\zeta}{2}(r_v-t_u)}\rho(r_v)dr_v + \ln\left(\frac{2\pi}{3}\right)\int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v-t_u)}\rho(r_v)dr_v$$
$$= \frac{\alpha}{\cosh(\alpha R)-1}\left(\frac{\zeta}{2}\int_0^{R-t_u} xe^{-\frac{\zeta}{2}x}\sinh(\alpha x+\alpha t_u)dx + \ln\left(\frac{2\pi}{3}\right)\int_0^{R-t_u} e^{-\frac{\zeta}{2}x}\sinh(\alpha x+\alpha t_u)dx\right)$$
$$= \frac{\alpha}{\cosh(\alpha R)-1}\left(\frac{\zeta}{2}\,\mathfrak{I}_2(\zeta/2,\alpha t_u;0,R-t_u) + \ln\left(\frac{9\pi}{10}\right)\mathfrak{I}_1(\zeta/2,\alpha t_u;0,R-t_u)\right)$$
$$> \frac{\zeta}{2}\,\frac{\alpha}{\cosh(\alpha R)-1}\,\mathfrak{I}_2(\zeta/2,\alpha t_u;0,R-t_u)$$
$$\overset{Corollary\ 4.2}{=} \begin{cases} \frac{\alpha\zeta}{(\zeta-2\alpha)^2}\,\frac{e^{\alpha t_u}}{\cosh(\alpha R)-1} - O\left(\frac{e^{-\alpha t_u}}{\cosh(\alpha R)}\right), & \text{if } \zeta/\alpha > 2 \\ \frac{\zeta\alpha}{4}\,\frac{e^{\alpha t_u}(R-t_u)^2}{\cosh(\alpha R)-1}, & \text{if } \zeta/\alpha = 2 \end{cases}.$$

The above two inequalities together with (4.2), (4.3), (4.4) and (4.5), (4.6) yield:

$$\int_0^R \hat{p}(t_u,t_v)\rho(t_v)dt_v = \begin{cases} \Theta\left(\frac{e^{\alpha t_u}}{\cosh(\alpha R)}\right), & \text{if } \zeta/\alpha > 2 \\ \Theta\left(\frac{e^{\alpha t_u}((R-t_u)^2+1)}{\cosh(\alpha R)}\right), & \text{if } \zeta/\alpha = 2 \end{cases}. \tag{4.9}$$

$\beta < 1$

In this case, we have

$$\int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v-t_u)} \left(\int_{1/4}^{\frac{\pi}{4}e^{\frac{\zeta}{2}(r_v-t_u)}} \frac{1}{z^\beta+1}dz\right) \rho(r_v)dr_v \leq \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v-t_u)} \left(\int_{1/4}^{\frac{\pi}{4}e^{\frac{\zeta}{2}(r_v-t_u)}} \frac{1}{z^\beta}dz\right) \rho(r_v)dr_v$$
$$\leq \frac{1}{1-\beta}\int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v-t_u)} \left(\frac{\pi}{4}\right)^{1-\beta} e^{(1-\beta)\frac{\zeta}{2}(r_v-t_u)}\rho(r_v)dr_v$$
$$= \frac{1}{1-\beta}\left(\frac{\pi}{4}\right)^{1-\beta}\frac{\alpha}{\cosh(\alpha R)-1}\int_{t_u}^{R} e^{-\beta\frac{\zeta}{2}(r_v-t_u)}\sinh(\alpha r_v)dr_v$$
$$= \frac{1}{1-\beta}\left(\frac{\pi}{4}\right)^{1-\beta}\frac{\alpha}{\cosh(\alpha R)-1}\int_0^{R-t_u} e^{-\beta\frac{\zeta}{2}x}\sinh(\alpha x+\alpha t_u)dx$$
$$= O\left(\frac{\mathfrak{I}_1(\beta\zeta/2,\alpha t_u;0,R-t_u)}{\cosh(\alpha R)-1}\right).$$

Hence by (4.4) and (4.6) we obtain

$$I_2 = O\left(\frac{\mathfrak{I}_1(\beta\zeta/2, \alpha t_u; 0, R - t_u)}{\cosh(\alpha R)} + \frac{\mathfrak{I}_1(\zeta/2, \alpha t_u; 0, R - t_u)}{\cosh(\alpha R)}\right). \tag{4.10}$$

Using that $\frac{1}{z^\beta+1} > \frac{1}{2z^\beta}$, for $z > 1$, the integral in (4.5) can be bounded as follows:

$$\begin{aligned}
&\int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v - t_u)} \left(\int_2^{2\pi e^{\frac{\zeta}{2}(r_v - t_u)}} \frac{1}{z^\beta + 1} dz\right) \rho(r_v) dr_v \\
&\geq \frac{1}{2} \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v - t_u)} \left(\int_2^{2\pi e^{\frac{\zeta}{2}(r_v - t_u)}} \frac{1}{z^\beta} dz\right) \rho(r_v) dr_v \\
&= \frac{1}{2(1-\beta)} \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v - t_u)} \left((2\pi)^{1-\beta}\ e^{(1-\beta)\frac{\zeta}{2}(r_v - t_u)} - 2^{1-\beta}\right) \rho(r_v) dr_v \\
&= \frac{(2\pi)^{1-\beta}}{2(1-\beta)} \int_{t_u}^{R} e^{-\beta\frac{\zeta}{2}(r_v - t_u)} \rho(r_v) dr_v - \frac{2^{1-\beta}}{2(1-\beta)} \int_{t_u}^{R} e^{-\frac{\zeta}{2}(r_v - t_u)} \rho(r_v) dr_v \\
&= \frac{(2\pi)^{1-\beta}}{2(1-\beta)}\ \mathfrak{I}_1(\beta\zeta/2, \alpha t_u; 0, R - t_u) - \frac{2^{-\beta}}{1-\beta}\ \mathfrak{I}_1(\zeta/2, \alpha t_u; 0, R - t_u).
\end{aligned}$$

So (4.5) now gives

$$I_2 \geq \frac{(2\pi)^{\beta-1}}{2(1-\beta)}\ \mathfrak{I}_1(\beta\zeta/2, \alpha t_u; 0, R - t_u) - \frac{1}{2^\beta \pi(1-\beta)}\ \mathfrak{I}_1(\zeta/2, \alpha t_u; 0, R - t_u). \tag{4.11}$$

The asymptotics of the bounds that are given in (4.10) and (4.11) depends up on the value of β in comparison with $2\alpha/\zeta$. More specifically, from Corollary 4.2 together with (4.2), (4.3) and the bounds given in (4.4) and (4.5), (4.6), we have

$$\int_0^R \hat{p}(t_u, R - r_v)\rho(r_v) dr_v = \begin{cases} \Theta\left(\frac{e^{\alpha t_u}}{\cosh(\alpha R)}\right), & \text{if } \beta > 2\alpha/\zeta \\ \Theta\left((R - t_u + 1)\ \frac{e^{\alpha t_u}}{\cosh(\alpha R)}\right), & \text{if } \beta = 2\alpha/\zeta \\ \Theta\left(e^{-\beta\frac{\zeta}{2}(R - t_u)} + \frac{e^{\alpha t_u}}{\cosh(\alpha R)}\right), & \text{if } \beta < 2\alpha/\zeta. \end{cases} \tag{4.12}$$

Note that $\exp\left(-\beta\frac{\zeta}{2}(R - t_u)\right) = \left(\frac{e^{\zeta t_u/2}}{N}\right)^\beta$. Thus, in the case where $\beta\zeta/2 < \alpha$, this term dominates (up to some multiplicative constants) over $e^{\alpha t_u}/\cosh(\alpha R)$.

□

5 Asymptotic correlations of degrees

In this section, we deal with the correlations of the degrees for the case where $\beta > 1$ and $\zeta/\alpha < 2$. We show that the degrees of any finite collection of vertices are asymptotically independent.

Theorem 5.1. *Let $\beta > 1$ and $0 < \zeta/\alpha < 2$. For any integer $m \geq 2$ and for any collection of m pairwise distinct vertices $v_1, \ldots, v_m$ their degrees $D_{v_1}, \ldots, D_{v_m}$ are asymptotically independent in the sense that for any non-negative integers $k_1, \ldots, k_m$ we have*

$$\left| \Pr\Big[D_{v_1} = k_1, \ldots, D_{v_m} = k_m \Big] - \Pr\Big[D_{v_1} = k_1 \Big] \cdots \Pr\Big[D_{v_m} = k_m \Big] \right| = o(1). \tag{5.1}$$

The proof of the above theorem together with Chebyschev's inequality yield the concentration of the number of vertices of any fixed degree and complete the proof of Theorem 1.2.

We proceed with the proof of Theorem 5.1. Let us fix $m \geq 2$ distinct vertices $v_1, \ldots, v_m$ and let $k_1, \ldots, k_m$ be non-negative integers. The proof of Lemma 2.4 suggests that there exists a specific region around each vertex such that if another vertex is located outside it, then the probability that the two vertices are joined becomes much less than the estimate given in Lemma 2.4. In other words, this region is where a vertex is most likely to have its neighbours in – see Figure 1.

Definition 5.2. *For a vertex $v \in V_N$, we let A_v be the set of points $\{w \in \mathcal{D}_R \ : \ t_w \leq x_0, \theta_{vw} \leq \min\{\pi, \tilde{\theta}'_{v,w}\}\}$, where $\tilde{\theta}'_{v,w} := \omega(N) A^{-1}_{v,w}$. We call this the* vital area *of vertex v.*

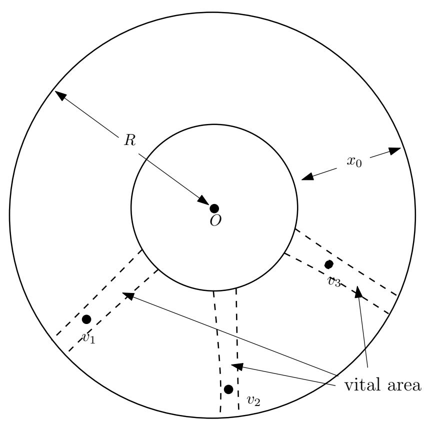

Figure 1: The vital areas of vertices in $\mathcal{D}_R$

We begin our analysis proving that the vital areas A_{v_i}, for $i = 1, \ldots, m$, are mutually disjoint with high probability. We let $\mathcal{E}_1$ be this event. Though m is meant to be fixed, the following claim is also valid for m growing as a function of N.

Claim 5.3. *If $m \leq \frac{\sqrt{N^{1-\zeta/(2\alpha)}}}{\omega(N) e^{\zeta\omega(N)/2}}$, then $\Pr\Big[\mathcal{E}_1\Big] = 1 - o(1)$.*

Proof. Let $i \in \{1, \ldots, m\}$ and assume that $t_{v_i} < R(1 - \zeta/(2\alpha)) - 2\omega(N)$. Considering all points $w \in A_{v_i}$ the parameter $\tilde{\theta}'_{v_i,w}$ is maximised when $t_w = x_0$. So let $\tilde{\theta}'_{v_i}$ be this maximum, that is,

$$\tilde{\theta}'_{v_i} = \nu \ \frac{\omega(N)}{N} \exp\left(\frac{\zeta}{2}\left(t_{v_i} + x_0\right)\right) = \omega(N) \ \left(\frac{\nu}{N}\right)^{1-\zeta/(2\alpha)} \ e^{\zeta\omega(N)/2} \ e^{\zeta t_{v_i}/2}.$$

Observe that our assumption on t_{v_i} implies that

$$e^{\zeta t_{v_i}/2} < e^{\frac{\zeta R}{2}\ \left(1-\frac{\zeta}{2\alpha}\right)-\zeta\omega(N)} = N^{1-\zeta/(2\alpha)} e^{-\zeta\omega(N)},$$

whereby $\tilde{\theta}'_{v_i} = o(1)$. Thus for $i \neq j$, with $t_{v_i}, t_{v_j} < R(1-\zeta/(2\alpha)) - 2\omega(N) =: \hat{x}_0$ we have $A_{v_i} \cap A_{v_j} \neq \emptyset$ if

$$\theta_{v_i v_j} \leq \tilde{\theta}'_{v_i} + \tilde{\theta}'_{v_j} = \omega(N) \left(\frac{\nu}{N}\right)^{1-\zeta/(2\alpha)} \ e^{\zeta\omega(N)/2} \left(e^{\zeta t_{v_i}/2} + e^{\zeta t_{v_j}/2}\right).$$

The probability that this occurs for two given distinct indices i, j is crudely bounded for N large enough using Lemma 2.1 as follows:

$$\begin{aligned}
&\frac{1}{\pi}\ \omega(N) \left(\frac{\nu}{N}\right)^{1-\zeta/(2\alpha)} \ e^{\zeta\omega(N)/2} \int_0^{\hat{x}_0} \int_0^{\hat{x}_0} \left(e^{\zeta t_{v_i}/2} + e^{\zeta t_{v_j}/2}\right) \rho(t_{v_i})\rho(t_{v_j}) dt_{v_i} dt_{v_j} \\
&\leq \frac{2\alpha^2}{\pi}\ \omega(N) \left(\frac{\nu}{N}\right)^{1-\zeta/(2\alpha)} \ e^{\zeta\omega(N)/2} \int_0^{\hat{x}_0} \int_0^{\hat{x}_0} \left(e^{\zeta t_{v_i}/2} + e^{\zeta t_{v_j}/2}\right) e^{-\alpha t_{v_i}} e^{-\alpha t_{v_j}} dt_{v_i} dt_{v_j} \\
&\leq \frac{4\alpha^2}{\pi}\ \omega(N) \left(\frac{\nu}{N}\right)^{1-\zeta/(2\alpha)} \ e^{\zeta\omega(N)/2} \int_0^{\hat{x}_0} e^{-\left(\alpha - \frac{\zeta}{2}\right)x} dx \\
&\leq \frac{4\alpha^2}{\pi}\ \frac{2}{2\alpha-\zeta}\ \omega(N) \left(\frac{\nu}{N}\right)^{1-\zeta/(2\alpha)} \ e^{\zeta\omega(N)/2}.
\end{aligned}$$

Also, by Lemma 2.1 for a vertex v we have

$$\begin{aligned}
\Pr\left[t_v \geq R\left(1-\frac{\zeta}{2\alpha}\right) - 2\omega(N)\right] &= e^{-\alpha R\left(1-\frac{\zeta}{2\alpha}\right)+2\alpha\omega(N)} + O\left(N^{-2\alpha/\zeta}\right) \\
&= \left(\frac{N}{\nu}\right)^{-\frac{2\alpha}{\zeta}\left(1-\frac{\zeta}{2\alpha}\right)} \ e^{2\alpha\omega(N)} + O\left(N^{-2\alpha/\zeta}\right) \\
&= \left(\frac{N}{\nu}\right)^{1-\frac{2\alpha}{\zeta}} \ e^{2\alpha\omega(N)} + O\left(N^{-2\alpha/\zeta}\right).
\end{aligned}$$

Assume now that $m \leq \frac{\sqrt{N^{1-\zeta/(2\alpha)}}}{\omega(N)e^{\zeta\omega(N)/2}}$. Thus the probability that there exists a pair of distinct vertices v_i, v_j with $i, j = 1, \ldots, m$ such that $A_{v_i} \cap A_{v_j} \neq \emptyset$ is bounded by

$$m^2 O\left(\frac{\omega(N)}{N^{1-\zeta/(2\alpha)}}\ e^{\zeta\omega(N)/2}\right) + mO\left(N^{1-\frac{2\alpha}{\zeta}} e^{2\alpha\omega(N)}\right) = o(1).$$

□

We assume that $m \geq 2$ is fixed and we condition on the event that $t_{v_i} \leq x_0$ for all $i = 1, \ldots, m$ (which we denote by $\mathcal{T}_1$) as well as on the event $\mathcal{E}_1$. By Corollary 2.2 and Claim 5.3 both events occur with probability $1 - o(1)$.

For a vertex $w \notin \{v_1, \ldots, v_m\}$ we denote by $\mathcal{A}^w_{v_i}$ the event that w is located within A_{v_i} and it is adjacent to v_i. In what follows, we will be omitting the superscript w, whenever this is clear from the context.

Now, let us consider the event that k_i vertices satisfy the event $\mathcal{A}_{v_i}$, for $i = 1, \ldots, m$, whereas all other vertices do not. We denote this event by $\mathcal{A}(k_1, \ldots, k_m)$. Also, for every $i = 1, \ldots, m$ let $\tilde{\mathcal{A}}_{v_i}^w$ be the event that a certain vertex w is located outside A_{v_i} and is adjacent to v_i. We let $\mathcal{B}_1$ be the event $\cup_{w \in V_N^{v_1,\ldots,v_m}} \cup_{i=1}^m \tilde{\mathcal{A}}_{v_i}^w$, that is, the event that there exists a vertex $w \in V_N \setminus \{v_1, \ldots, v_m\}$ which is adjacent to v_i, for some $i = 1, \ldots, m$, but it is located outside A_{v_i}. Thus conditional on $\mathcal{E}_1 \cap \mathcal{T}_1$, if the event $\mathcal{B}_1$ is *not* realized, then the event that vertex v_i has degree k_i, for all $i = 1, \ldots, m$ is realized if and only if $\mathcal{A}(k_1, \ldots, k_m)$ is realized. Using the union bound, we will show that

$$\Pr\Big[\mathcal{B}_1\Big] = o(1). \tag{5.2}$$

(We will show this without any conditioning.) Thereby, we can deduce the following:

$$\begin{aligned}
&\Pr\Big[D_{v_1} = k_1, \ldots, D_{v_m} = k_m\Big] = \Pr\Big[D_{v_1} = k_1, \ldots, D_{v_m} = k_m \mid \mathcal{E}_1, \mathcal{T}_1\Big] + o(1) \\
&\overset{(5.2)}{=} \Pr\Big[D_{v_1} = k_1, \ldots, D_{v_m} = k_m \mid \mathcal{E}_1, \mathcal{T}_1, \overline{\mathcal{B}_1}\Big] + o(1) \\
&= \Pr\Big[\mathcal{A}(k_1, \ldots, k_m) \mid \mathcal{E}_1, \mathcal{T}_1, \overline{\mathcal{B}_1}\Big] + o(1) \\
&= \Pr\Big[\mathcal{A}(k_1, \ldots, k_m), \overline{\mathcal{B}_1} \mid \mathcal{E}_1, \mathcal{T}_1\Big] / \Pr\Big[\overline{\mathcal{B}_1} \mid \mathcal{E}_1, \mathcal{T}_1\Big] + o(1) \\
&= \Pr\Big[\mathcal{A}(k_1, \ldots, k_m), \overline{\mathcal{B}_1} \mid \mathcal{E}_1, \mathcal{T}_1\Big] + o(1) \\
&= \Pr\Big[\mathcal{A}(k_1, \ldots, k_m) \mid \mathcal{E}_1, \mathcal{T}_1\Big] - \Pr\Big[\mathcal{A}(k_1, \ldots, k_m), \mathcal{B}_1 \mid \mathcal{E}_1, \mathcal{T}_1\Big] + o(1) \\
&= \Pr\Big[\mathcal{A}(k_1, \ldots, k_m) \mid \mathcal{E}_1, \mathcal{T}_1\Big] + o(1).
\end{aligned} \tag{5.3}$$

We will show further that $\Pr\Big[A(k_1, \ldots, k_m) \mid \mathcal{E}_1, \mathcal{T}_1\Big]$ is asymptotically equal to the product of the probabilities that $D_{v_i} = k_i$, over $i = 1, \ldots, m$.

Lemma 5.4. *Let $\beta > 1$ and $0 < \zeta/\alpha < 2$. Assume that $m \geq 2$ and $k_1, \ldots, k_m \geq 0$ are integers. Then we have*

$$\Pr\Big[\mathcal{A}(k_1, \ldots, k_m) \mid \mathcal{E}_1, \mathcal{T}_1\Big] = (1 + o(1)) \prod_{i=1}^m \Pr\Big[D_{v_i} = k_i\Big].$$

Proof. Note that if the positions of $v_1, \ldots, v_m$ have been fixed, then $\cup_{i=1}^m \{\mathcal{A}_{v_i}^w\}_{w \in V_N^{v_1,\ldots,v_m}}$ is an independent family. Thus, assuming that the positions $(t_{v_1}, \theta_{v_1}), \ldots, (t_{v_m}, \theta_{v_m})$ of $v_1, \ldots, v_m$ in $\mathcal{D}_R$ have been exposed so that $\mathcal{E}_1 \cap \mathcal{T}_1$ is realized, we can write

$$\Pr\Big[\mathcal{A}(k_1, \ldots, k_m) \mid (t_{v_1}, \theta_{v_1}), \ldots, (t_{v_m}, \theta_{v_m})\Big] =$$
$$\binom{N - m}{k_1 k_2 \cdots N - \sum_{i=1}^m k_i} \left[\prod_{i=1}^m \Pr\Big[\mathcal{A}_{v_i} \mid t_{v_i}\Big]^{k_i}\right] \left(1 - \sum_{i=1}^m \Pr\Big[\mathcal{A}_{v_i} \mid t_{v_i}\Big]\right)^{N - \sum_{i=1}^m k_i}. \tag{5.4}$$

We now proceed by giving an estimate for $\Pr\Big[\mathcal{A}_{v_i} \mid t_{v_i}\Big]$. That is, we will calculate the probability that a vertex $w \notin \{v_1, \ldots, v_m\}$ is located within A_{v_i} and it is adjacent to v_i. Setting $x_0' = \min\{x_0, R - t_{v_i} - \omega(N)\}$, we have

$$\begin{aligned}
\Pr\Big[\mathcal{A}_{v_i} \mid t_{v_i}\Big] &= \frac{1}{\pi}\int_0^{x_0}\int_0^{\min\{\pi,\tilde{\theta}'_{v_i,w}\}} p_{v_i,w}\rho(t_w)d\theta dt_w \\
&= \frac{1}{\pi}\int_0^{x_0'}\int_0^{\tilde{\theta}'_{v_i,w}} p_{v_i,w}\rho(t_w)d\theta dt_w + \frac{1}{\pi}\int_{x_0'}^{x_0}\int_0^{\min\{\pi,\tilde{\theta}'_{v_i,w}\}} p_{v_i,w}\rho(t_w)d\theta dt_w.
\end{aligned} \tag{5.5}$$

The second integral is bounded as in (3.2). In particular, it is bounded from above by

$$\int_{R-t_{v_i}-\omega(N)}^{R} p_{v_i,w}\rho(t_w)dt_w = O\left(e^{\alpha\omega(N)}\left(\frac{e^{\zeta t_{v_i}/2}}{N}\right)^{2\alpha/\zeta}\right).$$

Regarding the first integral, we argue as in (2.6), (2.13) and (2.10). Recall that for $\beta > 1$, we defined $\tilde{\theta}_{v_i,w} = \frac{1}{\omega(N)}\ A^{-1}_{v_i,w}$.

$$\int_0^{\tilde{\theta}'_{v_i,w}} p_{v_i,w}d\theta = \int_0^{\tilde{\theta}_{v_i,w}} p_{v_i,w}d\theta + \int_{\tilde{\theta}_{v_i,w}}^{\tilde{\theta}'_{v_i,w}} p_{v_i,w}d\theta = \int_{\tilde{\theta}_{v_i,w}}^{\tilde{\theta}'_{v_i,w}} p_{v_i,w}d\theta + o\left(A^{-1}_{v_i,w}\right).$$

For the first integral, we imitate the calculation in (2.10), expressing $p_{v_i,w}$ using Lemma 2.3 and applying the transformation $z = C^{1/\beta}\ A_{v_i,w}\ \frac{\theta}{2}$. We obtain

$$\int_{\tilde{\theta}_{v_i,w}}^{\tilde{\theta}'_{v_i,w}} p_{v_i,w}d\theta = (1+o(1))\ \frac{C_\beta}{A_{v_i,w}},$$

where C_β is as in Lemma 2.4 for $\beta > 1$.

Thereby, as in (3.3), the first integral in (5.5) becomes

$$\frac{1}{\pi}\int_0^{x_0'}\int_0^{\tilde{\theta}'_{v_i,w}} p_{v_i,w}\rho(t_w)d\theta dt_w = (1+o(1))C_\beta\int_0^{x_0'} A^{-1}_{v_i,w}\rho(t_w)dt_w = (1+o(1))\frac{2\alpha\nu C_\beta}{2\alpha-\zeta}\frac{e^{\zeta t_{v_i}/2}}{N}.$$

With $K = 2\alpha\nu C_\beta/(2\alpha - \zeta)$, as it was set in Lemma 3.1 for $\beta > 1$ and substituting the above estimates into (5.5) we obtain

$$\Pr\Big[\mathcal{A}_{v_i} \mid t_{v_i}\Big] = (1+o(1))\ K\ \frac{e^{\zeta t_{v_i}/2}}{N}. \tag{5.6}$$

Under the assumption that $t_{v_i} \le R/2 - \omega(N)$, we have $e^{\zeta t_{v_i}/2}/N = o\left(\frac{1}{N^{1/2}}\right)$. Thus, if $t_{v_i} \le R/2 - \omega(N)$, for all $i = 1, \ldots, m$

$$\left(1 - \sum_{i=1}^{m}\Pr\Big[\mathcal{A}_{v_i} \mid t_{v_i}\Big]\right)^{N-\sum_{i=1}^m k_i} = \exp\left(-(1+o(1))\ K\sum_{i=1}^{m} e^{\zeta t_{v_i}/2}\right). \tag{5.7}$$

Recall now that $T_u = \sum_{v \in V_N^u} \Pr\Big[I_{uv} = 1 \mid t_u \Big]$ and Lemma 3.4 implies that for any integer $k \geq 0$ we have

$$\Pr\Big[D_u = k \mid t_u \Big] = \Pr\Big[\mathrm{Po}(T_u) = k \Big] + o(1).$$

Assume now that the function $\tilde{\omega}(N)$ is such that

i. uniformly for $t_u \leq \tilde{\omega}(N)$ we have $T_u = Ke^{\zeta t_u/2} + o(1)$;

ii. $\tilde{\omega}(N) \leq \min\{R/2 - \omega(N), x_0\}$;

iii. $\tilde{\omega}(N) \to \infty$ as $N \to \infty$.

Substituting the estimates in (5.6) and (5.7) into (5.4) we obtain that uniformly for all $(t_{v_1}, \ldots, t_{v_m}) \in [0, \tilde{\omega}(N)]^m$ and all $\theta_{v_1}, \ldots, \theta_{v_m} \in (0, 2\pi]$ such that $\mathcal{E}_1 \cap \mathcal{T}_1$ is realized:

$$\Pr\Big[\mathcal{A}(k_1, \ldots, k_m) \mid (t_{v_1}, \theta_{v_1}), \ldots, (t_{v_m}, \theta_{v_m}) \Big] =$$
$$(1+o(1)) \prod_{i=1}^{m} \frac{\left(Ke^{\zeta t_{v_i}/2}\right)^{k_i}}{k_i!} \exp\left(-(1+o(1))Ke^{\zeta t_{v_i}/2}\right) = (1+o(1)) \prod_{i=1}^{m} \Pr\Big[D_{v_i} = k_i \mid t_{v_i} \Big], \tag{5.8}$$

by Lemma 3.4. Also, a first moment argument shows that the probability that there exists an index i with $1 \leq i \leq m$ such that $t_{v_i} > \tilde{\omega}(N)$ is $o(1)$. Hence, averaging over all (t_{v_i}, θ_{v_i}), for $i = 1, \ldots, m$, on the measure conditional on $\mathcal{E}_1 \cap \mathcal{T}_1$, the lemma follows. □

We conclude the proof of (5.1) with the proof of (5.2).

Lemma 5.5. *For any $\beta > 1$ and any $0 < \zeta/\alpha < 2$ we have*

$$\Pr\Big[\mathcal{B}_1 \Big] = o(1).$$

Proof. For $i \in [m]$, assume that $t_{v_i} < R - \omega(N)$. Corollary 2.2 implies that this holds for all $i \in [m]$ with probability $1 - o(1)$. Now, for a given $w \in V_N^{v_1, \ldots, v_m}$, the probability of the event $\tilde{A}_{v_i}^w$, conditional on t_{v_i}, can be bounded as follows:

$$\Pr\Big[\tilde{\mathcal{A}}_{v_i}^w \mid t_{v_i} \Big] \leq \frac{1}{\pi} \int_0^{R - t_{v_i} - \omega(N)} \int_{\tilde{\theta}'_{v_i,w}}^{\pi} p_{v_i,w} \rho(t_w) d\theta dt_w + \int_{R - t_{v_i} - \omega(N)}^{R} p_{v_i,w} \rho(t_w) dt_w. \tag{5.9}$$

(Here $\rho(\cdot)$ denotes the density function of the type of a vertex.) The second integral can be bounded as in (3.2) - uniformly for $t_{v_i} < R - \omega(N)$ we have

$$\int_{R - t_{v_i} - \omega(N)}^{R} p_{v_i,w} \rho(t_w) dt_w = O\left(e^{\alpha\omega(N)} \left(\frac{e^{\zeta t_{v_i}/2}}{N} \right)^{2\alpha/\zeta} \right). \tag{5.10}$$

Regarding the first integral, we use the estimate obtained in Lemma 2.3 to bound the inner integral. With C as in the proof of Lemma 2.4, we have

$$\int_{\tilde{\theta}'_{v_i,w}}^{\pi} p_{v_i,w} d\theta = \int_{\tilde{\theta}'_{v_i,w}}^{\pi} \frac{1}{C A_{v_i,w}^{\beta} \sin^{\beta}\left(\frac{\theta}{2}\right) + 1} d\theta \leq \frac{1}{C A_{v_i,w}^{\beta}} \int_{\tilde{\theta}'_{v_i,w}}^{\pi} \frac{1}{\sin^{\beta}\left(\frac{\theta}{2}\right)} d\theta$$
$$\stackrel{\sin\left(\frac{\theta}{2}\right) \geq \frac{\theta}{\pi}}{\leq} \frac{\pi^{\beta}}{C A_{v_i,w}^{\beta}} \int_{\tilde{\theta}'_{v_i,w}}^{\pi} \theta^{-\beta} d\theta = \frac{\pi^{\beta}}{C(\beta-1) A_{v_i,w}^{\beta}} \left(\tilde{\theta}'^{-\beta+1}_{v_i,w} - \pi^{-\beta+1}\right) = O\left(\frac{A_{v_i,w}^{-1}}{\omega(N)^{\beta-1}}\right).$$

Thus, the first integral in (5.9) becomes

$$\frac{1}{\pi} \int_0^{R-t_{v_i}-\omega(N)} \int_{\tilde{\theta}'_{v_i,w}}^{\pi} p_{v_i,w} \rho(t_w) d\theta dt_w = O\left(\frac{1}{\omega(N)^{\beta-1}} \frac{e^{\zeta t_{v_i}/2}}{N}\right) \int_0^{R-t_{v_i}-\omega(N)} e^{\zeta t_w/2} \rho(t_w) dt_w$$
$$\stackrel{(3.3)}{=} O\left(\frac{1}{\omega(N)^{\beta-1}} \frac{e^{\zeta t_{v_i}/2}}{N}\right). \tag{5.11}$$

Now, we take the average of each one of the bounds obtained in (5.11) and (5.10), respectively, over t_{v_i}. To this end, we need the following integral, whose simple calculation we omit. We have

$$\int_0^{R-\omega(N)} e^{\lambda t} \rho(t) dt = \begin{cases} \Theta(R) & \text{if } \lambda = \alpha \\ \Theta(1) & \text{if } \lambda = \zeta/2. \end{cases}$$

Thus, the bound in (5.10) is $O\left(e^{\alpha\omega(N)} \frac{R}{N^{2\alpha/\zeta}}\right)$ and that in (5.11) is $O\left(\frac{1}{\omega(N)^{\beta-1} N}\right)$. Since $\zeta/\alpha < 2$, both terms are $o(N^{-1})$. Therefore, the union bound implies that

$$\Pr\left[\cup_{w \in V_N^{v_1,\ldots,v_m}} \cup_{i=1}^m \tilde{\mathcal{A}}_{v_i}^w\right] = o(1). \tag{5.12}$$

□

Thus, the estimates obtained in Lemmas 5.4 and 5.5 substituted in (5.3) imply (5.1).

6 Conclusions - open questions

This paper initiates a study of random geometric graphs on spaces of negative curvature. We consider a binomial random graph model where the vertices are points that are randomly placed on the hyperbolic plane of a given curvature and each pair of points is included as an edge with probability that depends on their hyperbolic distance, independently of every other pair. This model was introduced recently by Krioukov et al. [17] as an approach to express basic properties of complex networks, such as power law degree distribution and clustering, as a result of the underlying hyperbolic geometry. We develop a rigorous treatment of this approach.

The probability distribution over the set of all possible graphs that can be formed in this model is expressed as the Gibbs distribution associated with a certain Hamiltonian. There is a critical parameter $\beta > 0$, which plays the role of the inverse temperature of the system and it turns out to determine the typical "behaviour" of the resulting random graph. We establish $\beta = 1$ as the critical value around which a transition occurs on the density and the connectivity of the resulting random graph.

For a certain choice of the defining parameters of the model, the regime $\beta > 1$ is associated with some typical features of complex networks such as constant average degree as well as a power law degree distribution. The exponent of the power law can be tuned through the curvature of the underlying space.

This study raises a number of questions regarding the typical structure of these random graphs for various values of β as well as the transition itself when β "crosses" the critical value $\beta = 1$. For example, what is the diameter of the random graph and the typical distance between two vertices that belong to the same component? Is there a unique giant component, wherever the random graph is not connected with high probability, and, if yes, what is the distribution of the smaller components? What is the clustering coefficient of such a random graph and how does it depend on β? Krioukov et al. [17] argue that the clustering coefficient can be tuned through β, but what is the exact dependence? When $\beta = 1$ is the random graph connected with high probability, and if yes, is it Hamiltonian? Can one describe the evolution of the random graph as β approaches 1 from above?

References

[1] R. Albert and A.-L. Barabási, Statistical mechanics of complex networks, *Reviews of Modern Physics* **74** (2002), 47–97.

[2] F. Baccelli, B. Blaszczyszyn and O. Mirsadeghi, Optimal paths on the space-time SINR random graph, *Advances in Applied Probability* **43**(1) (2011), 131–150.

[3] J. Balogh, B. Bollobás, M. Krivelevich, T. Müller and M. Walters, Hamilton cycles in random geometric graphs, *Annals of Applied Probability* **21**(3) (2011), 1053–1072.

[4] I. Benjamini and O. Schramm, Percolation in the hyperbolic plane, *Journal of the American Mathematical Society* **14** (2001), 487–507.

[5] B. Bollobás, *Random graphs*, Cambridge University Press, 2001, xviii+498 pp.

[6] B. Bollobás, S. Janson and O. Riordan, The phase transition in inhomogeneous random graphs, *Random Structures and Algorithms* **31**(2007), 3–122.

[7] B. Bollobás and O. Riordan, The diameter of a scale-free random graph, *Combinatorica* **24** (2004), 5–34.

[8] B. Bollobás, O. Riordan, J. Spencer and G. Tusnády, The degree sequence of a scale-free random graph process, *Random Structures and Algorithms* **18** (2001), 279–290.

[9] K. Börner, J.T. Maru and R.L. Goldstone, Colloquium Paper: Mapping Knowledge Domains: The simultaneous evolution of author and paper networks, *Proc. Natl. Acad. Sci. USA* **101** (2004), 5266–5273.

[10] F. Chung and L. Lu, The average distances in random graphs with given expected degrees, *Proc. Natl. Acad. Sci. USA* **99** (2002), 15879–15882.

[11] F. Chung and L. Lu, Connected components in random graphs with given expected degree sequences, *Annals of Combinatorics* **6** (2002), 125–145.

[12] F. Chung and L. Lu, *Complex Graphs and Networks*, AMS, 2006.

[13] E. N. Gilbert, Random plane networks, *J. Soc. Indust. Appl. Math.* **9** (1961), 533–543.

[14] L. Gugelmann, K. Panagiotou and U. Peter, Random hyperbolic graphs: degree sequence and clustering, In *Proceedings of the 39th International Colloquium on Automata, Languages and Programming (A. Czumaj et al. Eds.)*, Lecture Notes in Computer Science 7392, 2012, pp. 573–585.

[15] R. Hafner, The asymptotic distribution of random clumps, *Computing* **10** (1972), 335–351.

[16] R. van der Hofstad, *Random Graphs and Complex Networks*, preprint available at `http://www.win.tue.nl/~rhofstad/`.

[17] D. Krioukov, F. Papadopoulos, M. Kitsak, A. Vahdat and M. Boguñá, Hyperbolic Geometry of Complex Networks, *Phys. Rev. E* **82** (2010), 036106.

[18] C.J.H. McDiarmid and T. Müller, On the chromatic number of random geometric graphs, *Combinatorica* **31**(4) (2011), 423–488.

[19] F. Menczer, Growing and navigating the small world Web by local content, *Proc. Natl. Acad. Sci. USA* **99** (2002), 14014–14019.

[20] J. Park and M. E. J. Newman, Statistical mechanics of networks, *Phys. Rev. E* **70** (2004), 066117.

[21] M. Penrose, *Random Geometric Graphs*, Oxford University Press, 2003.

[22] B. Söderberg, General formalism for inhomogeneous random graphs, *Phys. Rev. E* **66**(3) (2002), 066121.

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