A Markov Chain Financial Market

Ragnar Norberg
Univ. Copenhagen/London School of Economics/Univ. Melbourne

Summary: We consider a financial market driven by a continuous time homogeneous Markov chain. Conditions for absence of arbitrage and for completeness are spelled out, non-arbitrage pricing of derivatives is discussed, and details are worked out for a number of cases. Closed form expressions are obtained for interest rate derivatives. Computations typically amount to solving a set of first order partial differential equations. With a view to insurance applications, an excursion is made into risk minimization in the incomplete case.

Key-words: Continuous time Markov chains, martingale analysis, arbitrage pricing theory, risk minimization, insurance derivatives, interest rate guarantees.

1 Introduction

A. Prospectus.
The theory of diffusion processes, with its wealth of powerful theorems and model variations, is an indispensable toolkit in modern financial mathematics. The seminal papers of Black and Scholes [3] and Merton [13] were crafted with Brownian motion, and so were most of the almost countless papers on arbitrage pricing theory and its bifurcations that followed over the past quarter of a century.

A main course of current research, initiated by the martingale approach to arbitrage pricing ([10] and [11]), aims at generalization and unification. Today the core of the matter is well understood in a general semimartingale setting, see e.g. [5]. Another course of research investigates special models, in particular various Levy motion alternatives to the Brownian driving process, see e.g. [6] and [17]. Pure jump processes have been widely used in finance, ranging from plain Poisson processes introduced in [14] to quite general marked point processes, see e.g. [1]. And, as a pedagogical exercise, the market driven by a binomial process has been intensively studied since it was launched in [4].

The present paper undertakes to study a financial market driven by a continuous time homogeneous Markov chain. The idea was launched in [16] and reappeared in [7], the context being limited to modelling of the spot rate of interest. The purpose of the present study is two-fold: In the first place, it is instructive to see how well established theory turns out in the framework of a general Markov chain market. In the second place, it is worthwhile investigating the feasibility of the model from a theoretical as well as from a practical point of view. Poisson driven markets are accommodated as special cases.

B. Contents of the paper.
We set out in Section 2 by recapitulating basic definitions and results for the continuous time homogeneous Markov chain. Then we present a market featuring this process as the driving mechanism and spell out conditions for absence of arbitrage and for completeness. Section 3 carries through the program of arbitrage pricing of derivatives in the Markov chain market and works out the details for a number of cases. Special attention is devoted to interest rate derivatives, for which closed form expressions are obtained. Section 4 touches computational aspects. A simulation procedure is sketched. Focus is on numerical procedures for solving the set of first order partial differential equations that typically arise, a slightly complicating circumstance being that they involve function values at different states. Section 5 addresses the Föllmer-Sondermann-Schweizer theory of risk minimization in the incomplete case. Its particulars for the Markov chain market are worked out, and an application to unit-linked life insurance is sketched. Risk minimization is pursued in Section 6 where "degrees of completeness" of finite bond markets is discussed in connection with pricing and hedging of interest rate derivatives under constrained investment opportunities. The theory is applied to an interest rate guarantee product in life insurance.

C. Preliminaries: Notation and some useful results.
Vectors and matrices are denoted by in bold letters, lower and upper case, respectively. They may be equipped with topscripts indicating dimensions, e.g. $\mathbf{A}^{n\times m}$ has n rows and m columns. We may write $\mathbf{A} = (a^{jk})_{j\in\mathcal{J}}^{k\in\mathcal{K}}$ to emphasize the ranges of the row index j and the column index k. The transpose of $\mathbf{A}$ is denoted by $\mathbf{A}'$. Vectors are invariably taken to be of column type, hence row vectors appear as transposed. The identity matrix is denoted by $\mathbf{I}$, the vector with all entries equal to 1 is denoted by $\mathbf{1}$, and the vector with all entries equal to 0 is denoted by $\mathbf{0}$. By $\mathbf{D}_{j=1,\ldots,n}(a^j)$, or just $\mathbf{D}(\boldsymbol{a})$, is meant the diagonal matrix with the entries of $\boldsymbol{a} = (a^1, \ldots, a^n)'$ down the principal diagonal. The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$, and the linear subspace spanned by the columns of $\mathbf{A}^{n\times m}$ is denoted

by $\mathbb{R}(\mathbf{A})$.

A diagonalizable square matrix $\mathbf{A}^{n\times n}$ can be represented as

$$\mathbf{A} = \boldsymbol{\Phi}\, \mathbf{D}_{j=1,\ldots,n}(\rho_j)\, \boldsymbol{\Phi}^{-1} = \sum_{j=1}^{n} \rho_j \boldsymbol{\phi}_j \boldsymbol{\psi}_j' \,, \tag{1.1}$$

where the $\boldsymbol{\phi}_j$ are the columns of $\boldsymbol{\Phi}^{n\times n}$ and the $\boldsymbol{\psi}_j'$ are the rows of $\boldsymbol{\Phi}^{-1}$. The ρ_j are the eigenvalues of $\mathbf{A}$, and $\boldsymbol{\phi}_j$ and $\boldsymbol{\psi}_j'$ are the corresponding right and left eigenvectors, respectively. Eigenvectors (right or left) corresponding to eigenvalues that are distinguishable and non-null are mutually orthogonal. These results can be looked up in e.g. [12].

The exponential function of $\mathbf{A}^{n\times n}$ is the $n \times n$ matrix defined by

$$\exp(\mathbf{A}) = \sum_{p=0}^{\infty} \frac{1}{p!}\mathbf{A}^p = \boldsymbol{\Phi}\, \mathbf{D}_{j=1,\ldots,n}(e^{\rho_j})\, \boldsymbol{\Phi}^{-1} = \sum_{j=1}^{n} e^{\rho_j} \boldsymbol{\phi}_j \boldsymbol{\psi}_j' \,, \tag{1.2}$$

where the last two expressions follow from (1.1). The matrix $\exp(\mathbf{A})$ has full rank.

If $\boldsymbol{\Lambda}^{n\times n}$ is positive definite symmetric, then $\langle \boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2 \rangle_{\boldsymbol{\Lambda}} = \boldsymbol{\zeta}_1' \boldsymbol{\Lambda} \boldsymbol{\zeta}_2$ defines an inner product on $\mathbb{R}^n$. The corresponding norm is given by $\|\boldsymbol{\zeta}\|_{\boldsymbol{\Lambda}} = \langle \boldsymbol{\zeta}, \boldsymbol{\zeta} \rangle_{\boldsymbol{\Lambda}}^{1/2}$. If $\mathbf{F}^{n\times m}$ has full rank m $(\leq n)$, then the $\langle \cdot\,, \cdot \rangle_{\boldsymbol{\Lambda}}$-projection of $\boldsymbol{\zeta}$ onto $\mathbb{R}(\mathbf{F})$ is

$$\boldsymbol{\zeta}_{\mathbf{F}} = \mathbf{P}_{\mathbf{F}} \boldsymbol{\zeta} \,, \tag{1.3}$$

where the projection matrix (or projector) $\mathbf{P}_{\mathbf{F}}$ is

$$\mathbf{P}_{\mathbf{F}} = \mathbf{F}(\mathbf{F}'\boldsymbol{\Lambda}\mathbf{F})^{-1}\mathbf{F}'\boldsymbol{\Lambda} \,. \tag{1.4}$$

The projection of $\boldsymbol{\zeta}$ onto the orthogonal complement $\mathbb{R}(\mathbf{F})^{\perp}$ is

$$\boldsymbol{\zeta}_{\mathbf{F}^{\perp}} = \boldsymbol{\zeta} - \boldsymbol{\zeta}_{\mathbf{F}} = (\mathbf{I} - \mathbf{P}_{\mathbf{F}})\boldsymbol{\zeta} \,.$$

Its squared length, which is the squared $\langle \cdot\,, \cdot \rangle_{\boldsymbol{\Lambda}}$-distance from $\boldsymbol{\zeta}$ to $\mathbb{R}(\mathbf{F})$, is

$$\|\boldsymbol{\zeta}_{\mathbf{F}^{\perp}}\|_{\boldsymbol{\Lambda}}^2 = \|\boldsymbol{\zeta}\|_{\boldsymbol{\Lambda}}^2 - \|\boldsymbol{\zeta}_{\mathbf{F}}\|_{\boldsymbol{\Lambda}}^2 = \boldsymbol{\zeta}'\boldsymbol{\Lambda}(\mathbf{I} - \mathbf{P}_{\mathbf{F}})\boldsymbol{\zeta} \,. \tag{1.5}$$

The cardinality of a set $\mathcal{Y}$ is denoted by $|\mathcal{Y}|$. For a finite set it is just its number of elements.

2 The Markov chain market

A. The continuous time Markov chain.
At the base of everything (although slumbering in the background) is some probability space $(\ , \mathcal{F}, \mathbb{P})$.

Let $\{Y_t\}_{t \geq 0}$ be a continuous time Markov chain with finite state space $\mathcal{Y} = \{1, \ldots, n\}$. We assume that it is time homogeneous so that the transition probabilities

$$p_t^{jk} = \mathbb{P}[Y_{s+t} = k \mid Y_s = j]$$

depend only on the length of the transition period. This implies that the transition intensities

$$\lambda^{jk} = \lim_{t \searrow 0} \frac{p_t^{jk}}{t}\,, \tag{2.1}$$

$j \neq k$, exist and are constant. To avoid repetitious reminders of the type "$j, k \in \mathcal{Y}$", we reserve the indices j and k for states in $\mathcal{Y}$ throughout. We will frequently refer to

$$\mathcal{Y}^j = \{k; \lambda^{jk} > 0\}\,,$$

the set of states that are directly accessible from state j, and denote the number of such states by

$$n^j = |\mathcal{Y}^j|\,.$$

Put

$$\lambda^{jj} = -\lambda^{j\cdot} = -\sum_{k; k \in \mathcal{Y}^j} \lambda^{jk}$$

(minus the total intensity of transition out of state j). We assume that all states intercommunicate so that $p_t^{jk} > 0$ for all j, k (and $t > 0$). This implies that $n^j > 0$ for all j (no absorbing states). The matrix of transition probabilities,

$$\mathbf{P}_t = (p_t^{jk})\,,$$

and the infinitesimal matrix,

$$\mathbf{\Lambda} = (\lambda^{jk})\,,$$

are related by (2.1), which in matrix form reads $\mathbf{\Lambda} = \lim_{t \searrow 0} \frac{1}{t}(\mathbf{P}_t - \mathbf{I})$, and by the backward and forward Kolmogorov differential equations,

$$\frac{d}{dt}\mathbf{P}_t = \mathbf{P}_t \mathbf{\Lambda} = \mathbf{\Lambda} \mathbf{P}_t\,. \tag{2.2}$$

Under the side condition $\mathbf{P}_0 = \mathbf{I}$, (2.2) integrates to

$$\mathbf{P}_t = \exp(\mathbf{\Lambda} t)\,. \tag{2.3}$$

In the representation (1.2),

$$\mathbf{P}_t = \mathbf{\Phi}\,\mathbf{D}_{j=1,\ldots,n}(e^{\rho_j t})\,\mathbf{\Phi}^{-1} = \sum_{j=1}^{n} e^{\rho_j t}\boldsymbol{\phi}_j\boldsymbol{\psi}'_j\,, \tag{2.4}$$

the first (say) eigenvalue is $\rho_1 = 0$, and corresponding eigenvectors are $\boldsymbol{\phi}_1 = \mathbf{1}$ and $\boldsymbol{\psi}'_1 = (p^1, \ldots, p^n) = \lim_{t\nearrow\infty}(p_t^{j1}, \ldots, p_t^{jn})$, the stationary distribution of Y. The remaining eigenvalues, $\rho_2, \ldots, \rho_n$, are all strictly negative so that, by (2.4), the transition probabilities converge exponentially to the stationary distribution as t increases.

Introduce

$$I_t^j = 1[Y_t = j]\,, \tag{2.5}$$

the indicator of the event that Y is in state j at time t, and

$$N_t^{jk} = |\{s;\, 0 < s \le t,\, Y_{s-} = j\,,\, Y_s = k\}|\,, \tag{2.6}$$

the number of direct transitions of Y from state j to state $k \in \mathcal{Y}^j$ in the time interval $(0, t]$. For $k \notin \mathcal{Y}^j$ we define $N_t^{jk} \equiv 0$. Taking Y to be right-continuous, the same goes for the indicator processes I^j and the counting processes N^{jk}. As is seen from (2.5), (2.6), and the obvious relationships

$$Y_t = \sum_j j I_t^j\,, \qquad I_t^j = I_0^j + \sum_{k;k\neq j}(N_t^{kj} - N_t^{jk})\,,$$

the state process, the indicator processes, and the counting processes carry the same information, which at any time t is represented by the sigma-algebra $\mathcal{F}_t^Y = \sigma\{Y_s; 0 \le s \le t\}$. The corresponding filtration, denoted by $\mathbf{F}^Y = \{\mathcal{F}_t^Y\}_{t\ge0}$, is taken to satisfy the usual conditions of right-continuity ($\mathcal{F}_t = \cap_{u>t}\mathcal{F}_u$) and completeness ($\mathcal{F}_0$ contains all subsets of $\mathbb{P}$-nullsets), and $\mathcal{F}_0$ is assumed to be the trivial $(\emptyset, \ )$. This means, essentially, that Y is right-continuous (hence the same goes for the I^j and the N^{jk}) and that Y_0 deterministic.

The compensated counting processes M^{jk}, $j \neq k$, defined by

$$dM_t^{jk} = dN_t^{jk} - I_t^j \lambda^{jk}\,dt \tag{2.7}$$

and $M_0^{jk} = 0$, are zero mean, square integrable, mutually orthogonal martingales w.r.t. $(\mathbf{F}^Y, \mathbb{P})$.

We now turn to the subject matter of our study and, referring to introductory texts like [2] and [19], take basic notions and results from arbitrage pricing theory as prerequisites.

B. The continuous time Markov chain market.
We consider a financial market driven by the Markov chain described above. Thus, Y_t represents the state of the economy at time t, $\mathcal{F}_t^Y$ represents the information available about the economic history by time t, and $\mathbf{F}^Y$ represents the flow of such information over time.

In the market there are $m+1$ basic assets, which can be traded freely and frictionlessly (short sales are allowed, and there are no transaction costs). A special role is played by asset No. 0, which is a "locally risk-free" *bank account* with state-dependent interest rate

$$r_t = r^{Y_t} = \sum_j I_t^j r^j \,,$$

where the state-wise interest rates r^j, $j = 1, \ldots, n$, are constants. Thus, its price process is

$$B_t = \exp\left(\int_0^t r_s \, ds\right) = \exp\left(\sum_j r^j \int_0^t I_s^j \, ds\right) \,,$$

with dynamics

$$dB_t = B_t \, r_t \, dt = B_t \sum_j r^j I_t^j \, dt \,.$$

(Setting $B_0 = 1$ a just a matter of convention.)

The remaining m assets, henceforth referred to as *stocks*, are risky, with price processes of the form

$$S_t^i = \exp\left(\sum_j \alpha^{ij} \int_0^t I_s^j \, ds + \sum_j \sum_{k \in \mathcal{Y}^j} \beta^{ijk} N_t^{jk}\right) \,, \tag{2.8}$$

$i = 1, \ldots, m$, where the α^{ij} and β^{ijk} are constants and, for each i, at least one of the β^{ijk} is non-null. Thus, in addition to yielding state-dependent returns of the same form as the bank account, stock No. i makes a price jump of relative size

$$\gamma^{ijk} = \exp\left(\beta^{ijk}\right) - 1$$

upon any transition of the economy from state j to state k. By the general Itô's formula, its dynamics is given by

$$dS_t^i = S_{t-}^i \left(\sum_j \alpha^{ij} I_t^j \, dt + \sum_j \sum_{k \in \mathcal{Y}^j} \gamma^{ijk} dN_t^{jk} \right) . \tag{2.9}$$

Taking the bank account as numeraire, we introduce the discounted stock prices $\tilde{S}_t^i = S_t^i / B_t$, $i = 0, \ldots, m$. (The discounted price of the bank account is $\tilde{B}_t \equiv 1$, which is certainly a martingale under any measure). The discounted stock prices are

$$\tilde{S}_t^i = \exp \left(\sum_j (\alpha^{ij} - r^j) \int_0^t I_s^j \, ds + \sum_j \sum_{k \in \mathcal{Y}^j} \beta^{ijk} N_t^{jk} \right) , \tag{2.10}$$

with dynamics

$$d\tilde{S}_t^i = \tilde{S}_{t-}^i \left(\sum_j (\alpha^{ij} - r^j) I_t^j \, dt + \sum_j \sum_{k \in \mathcal{Y}^j} \gamma^{ijk} dN_t^{jk} \right) , \tag{2.11}$$

$i = 1, \ldots, m$.

C. Portfolios.
A dynamic *portfolio* or *investment strategy* is an $m+1$-dimensional stochastic process

$$\boldsymbol{\theta}_t' = (\eta_t, \boldsymbol{\xi}_t') ,$$

where η_t represents the number of units of the bank account held at time t, and the i-th entry in

$$\boldsymbol{\xi}_t = (\xi_t^1, \ldots, \xi_t^m)'$$

represents the number of units of stock No. i held at time t. (As it will turn out, the bank account and the stocks will appear to play different parts in the show, the latter being the more visible. It is, therefore, convenient to costume the two types of assets and their corresponding portfolio entries accordingly.) The portfolio $\boldsymbol{\theta}$ is adapted to $\mathbf{F}^Y$, and the shares of stocks, $\boldsymbol{\xi}$, must also be $\mathbf{F}^Y$-predictable.

The *value* of the portfolio at time t is

$$V_t^\theta = \eta_t B_t + \boldsymbol{\xi}_t' \mathbf{S}_t = \eta_t B_t + \sum_{i=0}^m \xi_t^i S_t^i .$$

where

$$\tilde{\mathbf{S}}_t = (\tilde{S}_t^1, \ldots, \tilde{S}_t^m)'\,.$$

Henceforth we will mainly work with discounted prices and values and, in accordance with (2.10), equip their symbols with a tilde. The discounted value of the portfolio at time t is

$$\tilde{V}_t^{\theta} = \eta_t + \boldsymbol{\xi}_t'\,\tilde{\mathbf{S}}_t\,. \tag{2.12}$$

The strategy $\boldsymbol{\theta}$ is *self-financing* (SF) if $dV_t^{\theta} = \eta_t\,dB_t + \boldsymbol{\xi}_t'\,d\mathbf{S}_t$ or, equivalently,

$$d\tilde{V}_t^{\theta} = \boldsymbol{\xi}_t'\,d\tilde{\mathbf{S}}_t = \sum_{i=1}^{m} \xi_t^i\,d\tilde{S}_t^i\,. \tag{2.13}$$

D. Absence of arbitrage.
Let

$$\tilde{\boldsymbol{\Lambda}} = (\tilde{\lambda}^{jk})$$

be an infinitesimal matrix that is equivalent to $\boldsymbol{\Lambda}$ in the sense that $\tilde{\lambda}^{jk} = 0$ if and only if $\lambda^{jk} = 0$. By Girsanov's theorem, there exists a measure $\tilde{\mathbb{P}}$, equivalent to $\mathbb{P}$, under which Y is a Markov chain with infinitesimal matrix $\tilde{\boldsymbol{\Lambda}}$. Consequently, the processes $\tilde{M}^{jk}$, $j = 1, \ldots, n$, $k \in \mathcal{Y}^j$, defined by

$$d\tilde{M}_t^{jk} = dN_t^{jk} - I_t^j\,\tilde{\lambda}^{jk}\,dt\,, \tag{2.14}$$

and $\tilde{M}_0^{jk} = 0$, are zero mean, mutually orthogonal martingales w.r.t. $(\mathbf{F}^Y, \tilde{\mathbb{P}})$. Rewrite (2.11) as

$$d\tilde{S}_t^i = \tilde{S}_{t-}^i \left[\sum_j \left(\alpha^{ij} - r^j + \sum_{k\in\mathcal{Y}^j} \gamma^{ijk}\tilde{\lambda}^{jk} \right) I_t^j\,dt + \sum_j \sum_{k\in\mathcal{Y}^j} \gamma^{ijk} d\tilde{M}_t^{jk} \right], \tag{2.15}$$

$i = 1, \ldots, m$. The discounted stock prices are martingales w.r.t. $(\mathbf{F}^Y, \tilde{\mathbb{P}})$ if and only if the drift terms on the right vanish, that is,

$$\alpha^{ij} - r^j + \sum_{k\in\mathcal{Y}^j} \gamma^{ijk}\tilde{\lambda}^{jk} = 0\,, \tag{2.16}$$

$j = 1, \ldots, n$, $i = 1, \ldots, m$. From general theory it is known that the existence of such an equivalent martingale measure $\tilde{\mathbb{P}}$ implies absence of arbitrage.

The relation (2.16) can be cast in matrix form as

$$r^j \mathbf{1} - \boldsymbol{\alpha}^j = \boldsymbol{\Gamma}^j \tilde{\boldsymbol{\lambda}}^j \,, \tag{2.17}$$

$j = 1, \ldots, n$, where $\mathbf{1}$ is $m \times 1$ and

$$\boldsymbol{\alpha}^j = \left(\alpha^{ij}\right)_{i=1,\ldots,m} \,, \quad \boldsymbol{\Gamma}^j = \left(\gamma^{ijk}\right)_{i=1,\ldots,m}^{k\in\mathcal{Y}^j} \,, \quad \tilde{\boldsymbol{\lambda}}^j = \left(\tilde{\lambda}^{jk}\right)_{k\in\mathcal{Y}^j} \,.$$

The existence of an equivalent martingale measure is equivalent to the existence of a solution $\tilde{\boldsymbol{\lambda}}^j$ to (2.17) with all entries strictly positive. Thus, the market is arbitrage-free if (and we can show only if) for each j, $r^j\mathbf{1} - \boldsymbol{\alpha}^j$ is in the interior of the convex cone of the columns of $\boldsymbol{\Gamma}^j$.

Assume henceforth that the market is arbitrage-free so that (2.15) reduces to

$$d\tilde{S}^i_t \;=\; \tilde{S}^i_{t-} \sum_j \sum_{k\in\mathcal{Y}^j} \gamma^{ijk} d\tilde{M}^{jk}_t \,, \tag{2.18}$$

where the $\tilde{M}^{jk}$ defined by (2.14) are martingales w.r.t. $(\mathbf{F}^Y, \tilde{\mathbb{P}})$ for some measure $\tilde{\mathbb{P}}$ that is equivalent to $\mathbb{P}$.

Inserting (2.18) into (2.13), we find that $\boldsymbol{\theta}$ is SF if and only if

$$d\tilde{V}^\theta_t = \sum_j \sum_{k\in\mathcal{Y}^j} \sum_{i=1}^m \xi^i_t \tilde{S}^i_{t-} \gamma^{ijk} d\tilde{M}^{jk}_t \,, \tag{2.19}$$

implying that $\tilde{V}^\theta$ is a martingale w.r.t. $(\mathbf{F}^Y, \tilde{\mathbb{P}})$ and, in particular,

$$\tilde{V}^\theta_t = \tilde{\mathbb{E}}[\tilde{V}^\theta_T \,|\, \mathcal{F}_t] \,. \tag{2.20}$$

Here $\tilde{\mathbb{E}}$ denotes expectation under $\tilde{\mathbb{P}}$. (Note that the tilde, which in the first place was introduced to distinguish discounted values from the nominal ones, is also attached to the equivalent martingale measure and certain related entities. This usage is motivated by the fact that the martingale measure arises from the discounted basic price processes, roughly speaking.)

E. Attainability.
A T-claim is a contractual payment due at time T. Formally, it is an $\mathcal{F}^Y_T$-measurable random variable H with finite expected value. The claim is *attainable* if it can be perfectly duplicated by some SF portfolio $\boldsymbol{\theta}$, that is,

$$\tilde{V}^\theta_T = \tilde{H} \,. \tag{2.21}$$

If an attainable claim should be traded in the market, then its price must at any time be equal to the value of the duplicating portfolio in order to avoid arbitrage. Thus, denoting the price process by π_t and, recalling (2.20) and (2.21), we have

$$\tilde{\pi}_t = \tilde{V}_t^{\theta} = \tilde{\mathbb{E}}[\tilde{H} \mid \mathcal{F}_t]\,, \tag{2.22}$$

or

$$\pi_t = \tilde{\mathbb{E}}\left[e^{-\int_t^T r} H \,\middle|\, \mathcal{F}_t \right] . \tag{2.23}$$

(We use the short-hand $e^{-\int_t^T r}$ for $\exp\left(-\int_t^T r_u\,du\right)$.)

By (2.22) and (2.19), the dynamics of the discounted price process of an attainable claim is

$$d\tilde{\pi}_t = \sum_j \sum_{k\in\mathcal{Y}^j} \sum_{i=1}^m \xi_t^i \tilde{S}_{t-}^i \gamma^{ijk} d\tilde{M}_t^{jk} . \tag{2.24}$$

F. Completeness.

Any T-claim H as defined above can be represented as

$$\tilde{H} = \tilde{\mathbb{E}}[\tilde{H}] + \int_0^T \sum_j \sum_{k\in\mathcal{Y}^j} \zeta_t^{jk} d\tilde{M}_t^{jk} , \tag{2.25}$$

where the ζ_t^{jk} are $\mathbf{F}^Y$-predictable and integrable processes. Conversely, any random variable of the form (2.25) is, of course, a T-claim. By virtue of (2.21), and (2.19), attainability of H means that

$$\begin{aligned} \tilde{H} &= \tilde{V}_0^{\theta} + \int_0^T d\tilde{V}_t^{\theta} \\ &= \tilde{V}_0^{\theta} + \int_0^T \sum_j \sum_{k\in\mathcal{Y}^j} \sum_i \xi_t^i \tilde{S}_{t-}^i \gamma^{ijk} d\tilde{M}_t^{jk} . \end{aligned} \tag{2.26}$$

Comparing (2.25) and (2.26), we see that H is attainable iff there exist predictable processes $\xi_t^1, \ldots, \xi_t^m$ such that

$$\sum_{i=1}^m \xi_t^i \tilde{S}_{t-}^i \gamma^{ijk} = \zeta_t^{jk} ,$$

for all j and $k \in \mathcal{Y}^j$. This means that the n^j-vector

$$\boldsymbol{\zeta}_t^j = (\zeta_t^{jk})_{k\in\mathcal{Y}^j}$$

is in $\mathbb{R}(\mathbf{\Gamma}^{j\prime})$.

The market is *complete* if every T-claim is attainable, that is, if every n^j-vector is in $\mathbb{R}(\mathbf{\Gamma}^{j\prime})$. This is the case if and only if $\mathrm{rank}(\mathbf{\Gamma}^j) = n^j$, which can be fulfilled for each j only if $m \geq \max_j n_j$.

3 Arbitrage-pricing of derivatives in a complete market

A. Differential equations for the arbitrage-free price.
Assume that the market is arbitrage-free and complete so that prices of T-claims are uniquely given by (2.22) or (2.23).

Let us for the time being consider a T-claim of the form

$$H = h(Y_T, S_T^\ell)\,. \tag{3.1}$$

Examples are a European call option on stock No. ℓ defined by $H = (S_T^\ell - K)^+$, a caplet defined by $H = (r_T - g)^+ = (r^{Y_T} - g)^+$, and a zero coupon T-bond defined by $H = 1$.

For any claim of the form (3.1) the relevant state variables involved in the conditional expectation (2.23) are t, Y_t, S_t^ℓ, hence π_t is of the form

$$\pi_t = \sum_{j=1}^{n} I_t^j f^j(t, S_t^\ell)\,, \tag{3.2}$$

where the

$$f^j(t,s) = \tilde{\mathbb{E}}\left[e^{-\int_t^T r} H \,\middle|\, Y_t = j,\, S_t^\ell = s \right] \tag{3.3}$$

are the state-wise price functions.

The discounted price (2.22) is a martingale w.r.t. $(\mathbf{F}^Y, \tilde{\mathbb{P}})$. Assume that the functions $f^j(t,s)$ are continuously diferentiable. Using Itô on

$$\tilde{\pi}_t = e^{-\int_0^t r} \sum_{j=1}^{n} I_t^j f^j(t, S_t^\ell)\,, \tag{3.4}$$

we find

$$\begin{aligned} d\tilde{\pi}_t \;&=\; e^{-\int_0^t r} \sum_j I_t^j \left(-r^j\, f^j(t, S_t^\ell) + \frac{\partial}{\partial t} f^j(t, S_t^\ell) + \frac{\partial}{\partial s} f^j(t, S_t^\ell) S_t^\ell \alpha^{\ell j} \right) dt \\ &\quad + e^{-\int_0^t r} \sum_j \sum_{k \in \mathcal{Y}^j} \left(f^k(t, S_{t-}^\ell (1 + \gamma^{\ell jk})) - f^j(t, S_{t-}^\ell) \right) dN_t^{jk} \end{aligned}$$

$$
\begin{aligned}
= \quad & e^{-\int_0^t r} \sum_j I_t^j \Big(-r^j\, f^j(t, S_t^\ell) + \frac{\partial}{\partial t} f^j(t, S_t^\ell) + \frac{\partial}{\partial s} f^j(t, S_t^\ell) S_t^\ell \alpha^{\ell j} \\
& + \sum_{k \in \mathcal{Y}^j} \{ f^k(t, S_{t-}^\ell (1 + \gamma^{\ell j k})) - f^j(t, S_{t-}^\ell) \} \tilde{\lambda}^{jk} \Big)\, dt \\
& + e^{-\int_0^t r} \sum_j \sum_{k \in \mathcal{Y}^j} \Big(f^k(t, S_{t-}^\ell (1 + \gamma^{\ell j k})) - f^j(t, S_{t-}^\ell) \Big)\, d\tilde{M}_t^{jk} . \qquad (3.5)
\end{aligned}
$$

By the martingale property, the drift term must vanish, and we arrive at the non-stochastic partial differential equations

$$
\begin{aligned}
& -r^j\, f^j(t, s) + \frac{\partial}{\partial t} f^j(t, s) + \frac{\partial}{\partial s} f^j(t, s) s \alpha^{\ell j} \\
& \quad + \sum_{k \in \mathcal{Y}^j} \Big(f^k(t, s(1 + \gamma^{\ell j k})) - f^j(t, s) \Big)\, \tilde{\lambda}^{jk} = 0 , \qquad (3.6)
\end{aligned}
$$

$j = 1, \ldots, n$, which are to be solved subject to the side conditions

$$
f^j(T, s) = h(j, s) , \qquad (3.7)
$$

$j = 1, \ldots, n$.

In matrix form, with

$$
\mathbf{R} = \mathbf{D}_{j=1,\ldots,n}(r^j) , \qquad \mathbf{A}^\ell = \mathbf{D}_{j=1,\ldots,n}(\alpha^{\ell j}) ,
$$

and other symbols (hopefully) self-explaining, the differential equations and the side conditions are

$$
-\mathbf{R}\mathbf{f}(t, s) + \frac{\partial}{\partial t} \mathbf{f}(t, s) + s \mathbf{A}^\ell \frac{\partial}{\partial s} \mathbf{f}(t, s) + \tilde{\mathbf{\Lambda}} \mathbf{f}(t, s(1 + \gamma)) = \mathbf{0} , \qquad (3.8)
$$

$$
\mathbf{f}(T, s) = \mathbf{h}(s) . \qquad (3.9)
$$

B. Identifying the strategy.
Once we have determined the solution $f^j(t, s)$, $j = 1, \ldots, n$, the price process is known and given by (3.2).

The duplicating SF strategy can be obtained as follows. Setting the drift term to 0 in (3.5), we find the dynamics of the discounted price;

$$
d\tilde{\pi}_t = e^{-\int_0^t r} \sum_j \sum_{k \in \mathcal{Y}^j} \Big(f^k(t, S_{t-}^\ell (1 + \gamma^{\ell j k})) - f^j(t, S_{t-}^\ell) \Big)\, d\tilde{M}_t^{jk} . \qquad (3.10)
$$

Identifying the coefficients in (3.10) with those in (2.24), we obtain, for each state j, the equations

$$\sum_{i=1}^{m} \xi_t^i S_{t-}^i \gamma^{ijk} = f^k(t, S_{t-}^\ell(1+\gamma^{\ell jk})) - f^j(t, S_{t-}^\ell)\,, \tag{3.11}$$

$k \in \mathcal{Y}^j$. The solution $\boldsymbol{\xi}_t^j = (\xi_t^{i,j})'_{i=1,\ldots,m}$ (say) certainly exists since $\operatorname{rank}(\boldsymbol{\Gamma}^j) \leq m$, and it is unique iff $\operatorname{rank}(\boldsymbol{\Gamma}^j) = m$. Furthermore, it is a function of t and $\mathbf{S}_{t-}$ and is thus predictable. This simplistic argument works on the open intervals between the jumps of the process Y, where $d\tilde{M}_t^{jk} = -I_t^j \tilde{\lambda}^{jk}\,dt$. For the dynamics (3.10) and (2.24) to be the same also at jump times, the coefficients must clearly be left-continuous. We conclude that

$$\boldsymbol{\xi}_t = \sum_{j=1}^{n} I_{t-}^j \boldsymbol{\xi}_t\,,$$

which is predictable.

Finally, η is determined upon combining (2.12), (2.22), and (3.4):

$$\eta_t = e^{-\int_0^t r} \sum_{j=1}^{n} \left(I_t^j f^j(t, S_t^\ell) - I_{t-}^j \sum_{i=1}^{m} \xi_t^{i,j} S_t^i \right)\,.$$

C. The Asian option.

As an example of a path-dependent claim, let us consider an Asian option, which essentially is a T-claim of the form $H = \left(\int_0^T S_\tau^\ell\,d\tau - K\right)^+$, where $K \geq 0$. The price process is

$$\begin{aligned}
\pi_t &= \tilde{\mathbb{E}}\left[e^{-\int_t^T r} \left(\int_0^T S_\tau^\ell\,d\tau - K \right)^+ \middle| \mathcal{F}_t^Y \right] \\
&= \sum_{j=1}^{n} I_t^j f^j \left(t, S_t^\ell, \int_0^t S_\tau^\ell\,d\tau \right)\,,
\end{aligned}$$

where

$$f^j(t,s,u) = \tilde{\mathbb{E}}\left[e^{-\int_t^T r} \left(\int_t^T S_\tau^\ell + u - K \right)^+ \middle| Y_t = j,\, S_t^\ell = s \right]\,.$$

The discounted price process is

$$\tilde{\pi}_t = e^{-\int_0^t r} \sum_{j=1}^{n} I_t^j\, f^j \left(t, S_t^\ell, \int_0^t S_s^\ell \right)\,.$$

We obtain partial differential equations in three variables.

The special case $K = 0$ is simpler, with only two state variables.

D. Interest rate derivatives.

A particularly simple, but still important, class of claims are those of the form $H = h(Y_T)$. Interest rate derivatives of the form $H = h(r_T)$ are included since $r_T = r^{Y_T}$. For such claims the only relevant state variables are t and Y_t, so that the function in (3.3) depends only on t and j. The equation (3.6) reduces to

$$\frac{d}{dt} f_t^j = r^j f_t^j - \sum_{k \in \mathcal{Y}^j} (f_t^k - f_t^j) \tilde{\lambda}^{jk} \,, \tag{3.12}$$

and the side condition is (put $h(j) = h^j$)

$$f_T^j = h^j \,. \tag{3.13}$$

In matrix form,

$$\frac{d}{dt} \mathbf{f}_t = (\tilde{\mathbf{R}} - \tilde{\mathbf{\Lambda}}) \mathbf{f}_t \,,$$

subject to

$$\mathbf{f}_T = \mathbf{h} \,.$$

The solution is

$$\mathbf{f}_t = \exp\{(\tilde{\mathbf{\Lambda}} - \mathbf{R})(T - t)\} \mathbf{h} \,. \tag{3.14}$$

It depends on t and T only through $T - t$.

In particular, the zero coupon bond with maturity T corresponds to $\mathbf{h} = \mathbf{1}$. We will henceforth refer to it as the T-bond in short and denote its price process by $p(t, T)$ and its state-wise price functions by $\mathbf{p}(t, T) = (p^j(t, T))_{j=1,\ldots,n}$;

$$\mathbf{p}(t, T) = \exp\{(\tilde{\mathbf{\Lambda}} - \mathbf{R})(T - t)\} \mathbf{1} \,. \tag{3.15}$$

For a call option on a U-bond, exercised at time T $(< U)$ with price K, $\mathbf{h}$ has entries $h^j = (p^j(T, U) - K)^+$.

In (3.14) – (3.15) it may be useful to employ the representation shown in (1.2),

$$\exp\{(\tilde{\mathbf{\Lambda}} - \mathbf{R})(T - t)\} = \tilde{\mathbf{\Phi}} \, \mathbf{D}_{j=1,\ldots,n}(e^{\tilde{\rho}_j (T-t)}) \, \tilde{\mathbf{\Phi}}^{-1} \,, \tag{3.16}$$

say.

4 Numerical procedures

A. Simulation.
The homogeneous Markov process $\{Y_t\}_{t\in[0,T]}$ is simulated as follows: Let K be the number of transitions between states in $[0,T]$, and let $T_1,\ldots,T_K$ be the successive times of transition. The sequence $\{(T_n, Y_{T_n})\}_{n=0,\ldots,K}$ is generated recursively, starting from the initial state Y_0 at time $T_0 = 0$, as follows. Having arrived at T_n and Y_{T_n}, generate the next waiting time $T_{n+1}-T_n$ as an exponential variate with parameter $\lambda_{Y_n\cdot}$ (e.g. $-\ln(U_n)/\lambda_{Y_n\cdot}$, where U_n has a uniform distribution over $[0,1]$), and let the new state $Y_{T_{n+1}}$ be k with probability $\lambda_{Y_n k}/\lambda_{Y_n\cdot}$. Continue in this manner $K+1$ times until $T_K < T \le T_{K+1}$.

B. Numerical solution of differential equations.
Alternatively, the differential equations must be solved numerically. For interest rate derivatives, which involve only ordinary first order differential equations, a Runge Kutta will do. For stock derivatives, which involve partial first order differential equations, one must employ a suitable finite difference method, see e.g. [20].

5 Risk minimization in incomplete markets

A. Incompleteness.
The notion of incompleteness pertains to situations where a contingent claim cannot be duplicated by an SF portfolio and, consequently, does not receive a unique price from the no arbitrage postulate alone.

In Paragraph 2F we were dealing implicitly with incompleteness arising from a scarcity of traded assets, that is, the discounted basic price processes are incapable of spanning the space of all martingales w.r.t. $(\mathbf{F}^Y, \tilde{\mathbb{P}})$ and, in particular, reproducing the value (2.25) of every financial derivative (function of the basic asset prices).

Incompleteness also arises when the contingent claim is not a purely financial derivative, that is, its value depends also on circumstances external to the financial market. We have in mind insurance claims that are caused by events like death or fire and whose claim amounts are e.g. inflation adjusted or linked to the value of some investment portfolio.

In the latter case we need to work in an extended model specifying a basic probability space with a filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t\ge 0}$ containing $\mathbf{F}^Y$ and satisfying the usual conditions. Typically it will be the natural filtration of Y

and some other process that generates the insurance events. The definitions and conditions laid down in Paragraphs 2C-E are modified accordingly, so that adaptedness of η and predictability of $\boldsymbol{\xi}$ are taken to be w.r.t. $(\mathbf{F}, \mathbb{P})$ (keeping the symbol $\mathbb{P}$ for the basic probability measure), a T-claim H is $\mathcal{F}_T$ measurable, etc.

B. Risk minimization.

Throughout the remainder of the paper we will mainly be working with discounted prices and values without any other mention than the notational tilde. The reason is that the theory of risk minimization rests on certain martingale representation results that apply to discounted prices under a martingale measure. We will be content to give just a sketchy review of some main concepts and results from the seminal paper of Föllmer and Sondermann [8].

Let $\tilde{H}$ be a T-claim that is not attainable. This means that an *admissible* portfolio $\boldsymbol{\theta}$ satisfying

$$\tilde{V}_T^{\boldsymbol{\theta}} = \tilde{H}$$

cannot be SF. The *cost*, $\tilde{C}_t^{\boldsymbol{\theta}}$, of the portfolio by time t is defined as that part of the value that has not been gained from trading:

$$\tilde{C}_t^{\boldsymbol{\theta}} = \tilde{V}_t^{\boldsymbol{\theta}} - \int_0^t \boldsymbol{\xi}_\tau' d\tilde{\mathbf{S}}_\tau \,.$$

The *risk* at time t is defined as the mean squared outstanding cost,

$$\tilde{R}_t = \tilde{\mathbb{E}}\left[(\tilde{C}_T^{\boldsymbol{\theta}} - \tilde{C}_t^{\boldsymbol{\theta}})^2 \middle| \mathcal{F}_t \right] \,. \tag{5.1}$$

By definition, the risk of an admissible portfolio $\boldsymbol{\theta}$ is

$$\tilde{R}_t^{\boldsymbol{\theta}} = \tilde{\mathbb{E}}\left[(\tilde{H} - \tilde{V}_t^{\boldsymbol{\theta}} - \int_t^T \boldsymbol{\xi}_\tau' d\tilde{\mathbf{S}}_\tau)^2 \middle| \mathcal{F}_t \right] \,,$$

which is a measure of how well the current value of the portfolio plus future trading gains approximates the claim. The theory of risk minimization takes this entity as its object function and proves the existence of an optimal admissible portfolio that minimizes the risk (5.1) for all $t \in [0, T]$. The proof is constructive and provides a recipe for how to actually determine the optimal portfolio.

One sets out by defining the *intrinsic value* of $\tilde{H}$ at time t as

$$\tilde{V}_t^H = \tilde{\mathbb{E}}\left[\tilde{H} \,|\, \mathcal{F}_t \right] \,.$$

Thus, the intrinsic value process is the martingale that represents the natural current forecast of the claim under the chosen martingale measure. By the Galchouk-Kunita-Watanabe representation, it decomposes uniquely as

$$\tilde{V}_t^H = \tilde{\mathbb{E}}[\tilde{H}] + \int_0^t \boldsymbol{\xi}_t^{H\prime} \, d\tilde{\mathbf{S}}_t + L_t^H \,,$$

where L^H is a martingale w.r.t. $(\mathbf{F}, \tilde{\mathbb{P}})$ which is orthogonal to $\tilde{S}$. The portfolio $\boldsymbol{\theta}^H$ defined by this decomposition minimizes the risk process among all admissible strategies. The minimum risk is

$$\tilde{R}_t^H = \tilde{\mathbb{E}}\left[\int_t^T d\langle L^H \rangle_\tau \,\middle|\, \mathcal{F}_t \right] .$$

C. Unit-linked insurance.
As the name suggests, a life insurance product is said to be *unit-linked* if the benefit is a certain predetermined number of units of an asset (or portfolio) into which the premiums are currently invested. If the contract stipulates a minimum value of the benefit, disconnected from the asset price, then one speaks of *unit-linked insurance with guarantee.* A risk minimization approach to pricing and hedging of unit-linked insurance claims was first taken by Møller [15], who worked with the Black-Scholes-Merton financial market. We will here sketch how the analysis goes in our Markov chain market, which conforms well with the life history process in that they both are intensity-driven.

Let T_x be the remaining life time of an x years old who purchases an insurance at time 0, say. The conditional probability of survival to age $x+u$, given survival to age $x+t$ $(0 \leq t < u)$, is

$$ {}_{u-t}p_{x+t} = \mathbb{P}[T_x > u \,|\, T_x > t] = e^{-\int_t^u \mu_{x+s}\, ds} \,, \tag{5.2}$$

where μ_y is the mortality intensity at age y. We have

$$d\, {}_{u-t}p_{x+t} = {}_{u-t}p_{x+t}\, \mu_{x+t}\, dt \,. \tag{5.3}$$

Introduce the indicator of survival to age $x+t$,

$$I_t = 1[T_x > t] \,,$$

and the indicator of death before time t,

$$N_t = 1[T_x \leq t] = 1 - I_t \,.$$

The process N_t is a (very simple) counting process with intensity $I_t\, \mu_{x+t}$, that is, M given by

$$dM_t = dN_t - I_t\, \mu_{x+t}\, dt \tag{5.4}$$

is a martingale w.r.t. $(\mathbf{F}, \mathbb{P})$. Assume that the life time T_x is independent of the economy Y. We will work with the martingale measure $\tilde{\mathbb{P}}$ obtained by replacing the intensity matrix $\mathbf{\Lambda}$ of Y with the martingalizing $\tilde{\mathbf{\Lambda}}$ and leaving the rest of the model unaltered.

Consider a unit-linked pure endowment benefit payable at a fixed time T, contingent on survival of the insured, with sum insured equal to one unit of stock No. ℓ, but guaranteed no less than a fixed amount g. This benefit is a contingent T-claim,

$$H = (S_T^\ell \vee g)\, I_T\, .$$

The single premium payable as a lump sum at time 0 is to be determined.

Let us assume that the financial market is complete so that every purely financial derivative has a unique price process. Then the intrinsic value of H at time t is

$$\tilde{V}_t^H = \tilde{\pi}_t\, I_t\, {}_{T-t}p_{x+t}\, ,$$

where $\tilde{\pi}_t$ is the discounted price process of the derivative $S_T^\ell \vee g$.

Using Itô and inserting (5.4), we find

$$\begin{aligned} d\tilde{V}_t^H &= d\tilde{\pi}_t\, I_{t-}\ {}_{T-t}p_{x+t} + \tilde{\pi}_t\, I_{t-}\ {}_{T-t}p_{x+t}\, \mu_{x+t}\, dt + (0 - \tilde{\pi}_{t-}\ {}_{T-t}p_{x+t})\, dN_t \\ &= d\tilde{\pi}_t\, I_{t-}\ {}_{T-t}p_{x+t} - \tilde{\pi}_{t-}\ {}_{T-t}p_{x+t}\, dM_t\, . \end{aligned}$$

It is seen that the optimal trading strategy is that of the price process of the sum insured multiplied with the conditional probability that the sum will be paid out, and that

$$dL_t^H = -{}_{T-t}p_{x+t}\, \tilde{\pi}_{t-}\, dM_t\, .$$

Consequently,

$$\begin{aligned} \tilde{R}_t^H &= \int_t^T {}_{T-s}p_{x+s}^2\, \tilde{\mathbb{E}}\left[\tilde{\pi}_s^2 \middle| \mathcal{F}_t \right] {}_{s-t}p_{x+t}\, \mu_{x+s}\, ds \\ &= {}_{T-t}p_{x+t} \int_t^T \tilde{\mathbb{E}}\left[\tilde{\pi}_s^2 \middle| \mathcal{F}_t \right] {}_{T-s}p_{x+s}\, \mu_{x+s}\, ds\, . \end{aligned} \tag{5.5}$$

6 Trading with bonds: How much can be hedged?

A. A finite zero coupon bond market.
Suppose an agent faces a contingent T-claim and is allowed to invest only in the bank account and a finite number m of zero coupon bonds with maturities T_i, $i = 1, \ldots, m$, all post time T. For instance, regulatory constraints may be imposed on the investment strategies of an insurance company. The question is, to what extent can the claim be hedged by self-financed trading in these available assets?

An allowed SF portfolio has discounted value process $\tilde{V}_t^\theta$ of the form

$$d\tilde{V}_t^\theta = \sum_{i=1}^m \xi_t^i \sum_j \sum_{k \in \mathcal{Y}^j} (\tilde{p}^k(t, T_i) - \tilde{p}^j(t, T_i)) d\tilde{M}_t^{jk} = \sum_j d(\tilde{\mathbf{M}}_t^j)' \mathbf{F}_t^j \boldsymbol{\xi}_t \,,$$

where $\boldsymbol{\xi}$ is predictable, $\tilde{\mathbf{M}}_t^{j\,\prime} = (\tilde{M}_t^{jk})^{k \in \mathcal{Y}^j}$ is the n^j-dimensional row vector comprising the non-null entries in the j-th row of $\tilde{\mathbf{M}}_t = (\tilde{M}_t^{jk})$, and

$$\mathbf{F}_t^j = \mathbf{Y}^j \mathbf{F}_t$$

where

$$\mathbf{F}_t = (\tilde{p}^j(t, T_i))_{j=1,\ldots,n}^{i=1,\ldots,m} = (\tilde{\mathbf{p}}(t, T_1), \cdots, \tilde{\mathbf{p}}(t, T_m)) \,, \tag{6.1}$$

and $\mathbf{Y}^j$ is the $n^j \times n$ matrix which maps $\mathbf{F}_t$ to $(\tilde{p}^k(t, T_i) - \tilde{p}^j(t, T_i))_{k \in \mathcal{Y}^j}^{i=1,\ldots,m}$. If e.g. $\mathcal{Y}^n = \{1, \ldots, p\}$, then $\mathbf{Y}^n = (\mathbf{I}^{p \times p}\,,\, \mathbf{0}^{p \times (n-p-1)}\,,\, -\mathbf{1}^{p \times 1})$.

The sub-market consisting of the bank account and the m zero coupon bonds is complete in respect of T-claims iff the discounted bond prices span the space of all martingales w.r.t. $(\mathbf{F}^Y, \tilde{\mathbb{P}})$ over the time interval $[0, T]$. This is the case iff, for each j, $\text{rank}(\mathbf{F}_t^j) = n^j$. Now, since $\mathbf{Y}^j$ obviously has full rank n^j, the rank of $\mathbf{F}_t^j$ is determined by the rank of $\mathbf{F}_t$ in (6.1). We will argue that, typically, $\mathbf{F}_t$ has full rank. Thus, suppose $\mathbf{c} = (c_1, \ldots, c_m)'$ is such that

$$\mathbf{F}_t \mathbf{c} = \mathbf{0}^{n \times 1} \,.$$

Recalling (3.15), this is the same as

$$\sum_{i=1}^m c_i \exp\{(\tilde{\boldsymbol{\Lambda}} - \mathbf{R}) T_i\} \mathbf{1} = \mathbf{0} \,,$$

or, by (3.16) and since $\tilde{\boldsymbol{\Phi}}$ has full rank,

$$\mathbf{D}_{j=1,\ldots,n}\Big(\sum_{i=1}^m c_i e^{\tilde{\rho}^j T_i}\Big) \tilde{\boldsymbol{\Phi}}^{-1} \mathbf{1} = \mathbf{0} \,. \tag{6.2}$$

Since $\tilde{\mathbf{\Phi}}^{-1}$ has full rank, the entries of the vector $\tilde{\mathbf{\Phi}}^{-1}\mathbf{1}$ cannot be all null. Typically all entries are non-null, and we assume this is the case. Then (6.2) is equivalent to

$$\sum_{i=1}^{m} c_i e^{\tilde{\rho}^j T_i} = 0\,, \qquad j = 1, \ldots, n. \tag{6.3}$$

Using the fact that the generalized Vandermonde matrix has full rank, we know that (6.3) has a non-null solution $\mathbf{c}$ if and only if the number of distinct eigenvalues $\tilde{\rho}^j$ is less than m (see [9] and [18]).

In the case where $\mathrm{rank}(\mathbf{F}_t^j) < n^j$ for some j we would like to determine the Galchouk-Kunita-Watanabe decomposition for a given $\mathcal{F}_T^Y$-claim. The intrinsic value process has dynamics

$$d\tilde{H}_t = \sum_j \sum_{k\in\mathcal{Y}^j} \zeta_t^{jk} d\tilde{M}_t^{jk} = \sum_j d(\tilde{\mathbf{M}}_t^j)' \boldsymbol{\zeta}_t^j\,. \tag{6.4}$$

We seek a decomposition of the form

$$\begin{aligned} d\tilde{V}_t &= \sum_i \xi_t^i \, d\tilde{p}(t,T_i) + \sum_j \sum_{k\in\mathcal{Y}^j} \psi_t^{jk} \, d\tilde{M}_t^{jk} \\ &= \sum_j \sum_{j\in\mathcal{Y}^j} \sum_i \xi_t^i \left(\tilde{p}^k(t,T_i) - \tilde{p}^j(t,T_i)\right) d\tilde{M}_t^{jk} + \sum_j \sum_{k\in\mathcal{Y}^j} \psi_t^{jk} d\tilde{M}_t^{jk} \\ &= \sum_j d(\tilde{\mathbf{M}}_t^j)' \mathbf{F}_t^j \boldsymbol{\xi}_t^j + \sum_j d(\tilde{\mathbf{M}}_t^j)' \boldsymbol{\psi}_t^j\,, \end{aligned}$$

such that the two martingales on the right hand side are orthogonal, that is,

$$\sum_j I_{t-}^j \sum_{k\in\mathcal{Y}^j} (\mathbf{F}_t^j \boldsymbol{\xi}_t^j)' \tilde{\mathbf{\Lambda}}^j \boldsymbol{\psi}_t^j = 0\,,$$

where $\tilde{\mathbf{\Lambda}}^j = \mathbf{D}(\tilde{\boldsymbol{\lambda}}^j)$. This means that, for each j, the vector $\boldsymbol{\zeta}_t^j$ in (6.4) is to be decomposed into its $\langle\,,\,\rangle_{\tilde{\mathbf{\Lambda}}^j}$ projections onto $\mathbb{R}(\mathbf{F}_t^j)$ and its orthocomplement. From (1.3) and (1.4) we obtain

$$\mathbf{F}_t^j \boldsymbol{\xi}_t^j = \mathbf{P}_t^j \boldsymbol{\zeta}_t^j\,,$$

where

$$\mathbf{P}_t^j = \mathbf{F}_t^j (\mathbf{F}_t^{j\prime} \tilde{\mathbf{\Lambda}}^j \mathbf{F}_t^j)^{-1} \mathbf{F}_t^{j\prime} \tilde{\mathbf{\Lambda}}^j\,,$$

hence

$$\boldsymbol{\xi}_t^j = (\mathbf{F}_t^{j\prime} \tilde{\mathbf{\Lambda}}^j \mathbf{F}_t^j)^{-1} \mathbf{F}_t^{j\prime} \tilde{\mathbf{\Lambda}}^j \boldsymbol{\zeta}_t^j\,. \tag{6.5}$$

Furthermore,

$$\boldsymbol{\psi}_t^j = (\mathbf{I} - \mathbf{P}_t^j)\boldsymbol{\zeta}_t^j\,, \tag{6.6}$$

and the risk is

$$\int_t^T \sum_j p_{s-t}^{Y_t j} \sum_{k \in \mathcal{Y}^j} \lambda^{jk} (\psi_s^{jk})^2 \, ds\,. \tag{6.7}$$

The computation goes as follows: The coefficients ζ^{jk} involved in the intrinsic value process (6.4) and the state-wise prices $p^j(t, T_i)$ of the T_i-bonds are obtained by simultaneously solving (3.6) and (3.12), starting from (3.9) and (3.12), respectively, and at each step computing the optimal trading strategy $\boldsymbol{\xi}$ by (6.5) and the $\boldsymbol{\psi}$ from (6.6), and adding the step-wise contribution to the variance (6.7) (the step-length times the current value of the integrand).

B. First example: The floorlet.

For a simple example, consider a 'floorlet' $H = (r^* - r_T)^+$, where $T < \min_i T_i$. The motivation could be that at time T the insurance company will ascribe interest to the insured's account at current interest rate, but not less than a prefixed guaranteed rate r^*. Then H is the amount that must be provided per unit on deposit and per time unit at time T.

Computation goes by the scheme described above, with the $\zeta_t^{jk} = \tilde{f}_t^k - \tilde{f}_t^j$ obtained from (3.12) subject to (3.13) with $h^j = (r^* - r^j)^+$.

C. Second example: The interest guarantee in insurance.

A more practically relevant example is an interest rate guarantee on a life insurance policy. Premiums and reserves are calculated on the basis of a prudent so-called *first order* assumption, stating that the interest rate will be at some fixed (low) level r^* throughout the term of the insurance contract. Denote the corresponding first order reserve at time t by V_t^*. The (portfolio-wide) mean surplus created by the first order assumption in the time interval $[t, t+dt)$ is $(r^* - r_t)^+ {}_t p_x^* V_t^* \, dt$. This surplus is currently credited to the account of the insured as *dividend*, and the total amount of dividends is paid out to the insured at the term of the contracts at time T. Negative dividends are not permitted, however, so at time T the insurer must cover

$$H \quad = \quad \int_0^T e^{\int_s^T r} (r^* - r_s)^+ {}_s p_x^* V_s^* \, ds\,.$$

The intrinsic value of this claim is

$$\begin{aligned}\tilde{H}_t &= \tilde{\mathbb{E}}\left[\int_0^T e^{-\int_0^s r}(r^* - r_s)^+ \, {}_sp_x^* V_s^* \, ds \middle| \mathcal{F}_t\right] \\ &= \int_0^t e^{-\int_0^s r}(r^* - r_s)^+ \, {}_sp_x^* V_s^* \, ds + e^{-\int_0^t r} \sum_j I_t^j f_t^j \,,\end{aligned}$$

where the f_t^j are the state-wise expected values of future guarantees, discounted at time t,

$$f_t^j = \tilde{\mathbb{E}}\left[\int_t^T e^{-\int_t^s r}(r^* - r_s)^+ \, {}_sp_x^* V_s^* \, ds \middle| Y_t = j\right] .$$

Working along the lines of Section 3, we determine the f_t^j by solving

$$\frac{d}{dt} f_t^j = -(r^* - r^j)^+ \, {}_tp_x^* V_t^* + r^j f_t^j - \sum_{k \in \mathcal{Y}^j} (f_t^k - f_t^j)\tilde{\lambda}^{jk} \,,$$

subject to

$$f_T^j = 0 \,. \tag{6.8}$$

The intrinsic value has dynamics (6.4) with $\zeta_t^{jk} = \tilde{f}_t^k - \tilde{f}_t^j$.

From here we proceed as described in Paragraph A.

D. Computing the risk.

Constructive differential equations may be put up for the risk. As a simple example, for an interest rate derivative the state-wise risk is

$$\tilde{R}_t^j = \int_t^T \sum_g p_{\tau-t}^{jg} \sum_{k;k\neq g} \lambda^{gk} \left(\psi_\tau^{gk}\right)^2 \, d\tau \,.$$

Differentiating this equation, we find

$$\frac{d}{dt}\tilde{R}_t^j = -\sum_{k;k\neq j} \lambda^{jk} \left(\psi_t^{jk}\right)^2 + \int_t^T \sum_g \frac{d}{dt} p_{\tau-t}^{jg} \sum_{k;k\neq g} \left(\psi_\tau^{gk}\right)^2 \, d\tau \,,$$

and, using the backward version of (2.2),

$$d_t \, p_{s-t}^{jg} = -\sum_{h;h\neq j} \lambda^{jh} p_{s-t}^{hg} + \lambda^{j\cdot} p_{s-t}^{jg} \,,$$

we arrive at

$$\frac{d}{dt}\tilde{R}_t^j = -\sum_{k;k\neq j} \lambda^{jk} \left(\psi_t^{jk}\right)^2 - \sum_{k;k\neq j} \lambda^{jk} \tilde{R}_t^k + \lambda^{j\cdot} \tilde{R}_t^j \, .$$

Acknowledgments
The present paper was drafted during the author's stay as GIO professorial fellow at the Centre for Actuarial Studies, University of Melbourne, December 1998 – January 1999, and finished during a visit at the Department of Mathematics, Stanford University, November 1999 - March 2000. It was invited for presentation to the QMF99 Conference in Sydney, July 1999. The work was partly supported by the Mathematical Finance Network, Grant No. 9800335, under the Danish Social Science Research Council. My thanks are due to all these institutions and, in particular, to the inviting parties: David Dickson in Melbourne, George Papanicolaou at Stanford, and QMF organizers Carl Chiarella and Eckhard Platen at TU Sydney.

References

[1] Björk, T., Kabanov, Y., Runggaldier, W. (1997): Bond market structures in the presence of marked point processes. *Mathematical Finance*, **7**, 211-239.

[2] Björk, T. (1998): *Arbitrage Theory in Continuous Time*, Oxford University Press.

[3] Black, F., Scholes, M. (1973): The pricing of options and corporate liabilities. *J. Polit. Economy*, **81**, 637-654.

[4] Cox, J., Ross, S., Rubinstein, M. (1979): Option pricing: A simplified approach. *J. of Financial Economics*, **7**, 229-263.

[5] Delbaen, F., Schachermayer, W. (1994): A general version of the fundamental theorem on asset pricing. *Mathematische Annalen*, **300**, 463-520.

[6] Eberlein, E., Raible, S. (1999): Term structure models driven by general Lévy processes. *Mathematical Finance*, **9**, 31-53.

[7] Elliott, R.J., Kopp, P.E. (1998): *Mathematics of financial markets*, Springer-Verlag.

[8] Föllmer, H., Sondermann, D. (1986): Hedging of non-redundant claims. In *Contributions to Mathematical Economics in Honor of Gerard Debreu*, 205-223, eds. Hildebrand, W., Mas-Collel, A., North-Holland.

[9] Gantmacher, F.R. (1959): *Matrizenrechnung II*, VEB Deutscher Verlag der Wissenschaften, Berlin.

[10] Harrison, J.M., Kreps, D.M. (1979): Martingales and arbitrage in multiperiod securities markets. *J. Economic Theory*, **20**, 1979, 381-408.

[11] Harrison, J.M., Pliska, S. (1981): Martingales and stochastic integrals in the theory of continuous trading. *J. Stoch. Proc. and Appl.*, **11**, 215-260.

[12] Karlin, S., Taylor, H. (1975): *A first Course in Stochastic Processes*, 2nd. ed., Academic Press.

[13] Merton, R.C. (1973): The theory of rational option pricing. *Bell Journal of Economics and Management Science*, **4**, 141-183.

[14] Merton, R.C. (1976): Option pricing when underlying stock returns are discontinuous. *J. Financial Economics*, **3**, 125-144.

[15] Møller, T. (1998): Risk minimizing hedging strategies for unit-linked life insurance. *ASTIN Bull.*, **28**, 17-47.

[16] Norberg, R. (1995): A time-continuous Markov chain interest model with applications to insurance. *J. Appl. Stoch. Models and Data Anal.*, 245-256.

[17] Norberg, R. (1998): Vasiček beyond the normal. *Working paper No. 152*, Laboratory of Actuarial Math., Univ. Copenhagen.

[18] Norberg, R. (1999): On the Vandermonde matrix and its role in mathematical finance. (Forthcoming.)

[19] Pliska, S.R. (1997): *Introduction to Mathematical Finance*, Blackwell Publishers.

[20] Thomas, J.W. (1995): *Numerical Partial Differential Equations: Finite Difference Methods*, Springer-Verlag.

Address for correspondence:
Ragnar Norberg
Department of Statistics
London School of Economics
Houghton Street
London WC2A 2AE
United Kingdom

e-mail: R.Norberg@lse.ac.uk