Multiple random walks in random regular graphs

Colin Cooper* Alan Frieze† Thomas Radzik*

January 18, 2008

Abstract

It was shown in [5] that (**whp**), for $r \geq 3$ the cover time C_G of a random r-regular graph G_r is asymptotic to $\theta_r n \ln n$, where $\theta_r = (r-1)/(r-2)$. In this paper we study problems arising from multiple random walks on random regular graphs, and prove the following (**whp**) results. The time for k independent walks to cover G_r is asymptotic to C_G/k. For most starting positions, the expected number of steps before any of the walks meet is $\theta_r n / \binom{k}{2}$. If the walks can communicate on meeting at a vertex, we show that (for most starting positions) the expected time for k walks to broadcast a single piece of information is asymptotic to $\frac{2\ln k}{k}\theta_r n$, as $k, n \to \infty$.

We also establish properties of walks where particles interact when they meet at a vertex by coalescing or by exploding and destroying each other. As an example, the expected extinction time of k explosive particles (k even) tends to $(2\ln 2)\theta_r n$ as $k \to \infty$.

1 Introduction

Let $G = (V, E)$ be a connected graph, let $|V| = n$, and $|E| = m$. For $v \in V$ let C_v be the expected time taken for a simple random walk W on G starting at v, to visit every vertex of G. The *vertex cover time* C_G of G is defined as $C_G = \max_{v \in V} C_v$. The (vertex) cover time of connected graphs has been extensively studied. It is a classic result of Aleliunas, Karp, Lipton, Lovász and Rackoff [3] that $C_G \leq 2m(n-1)$. It was shown by Feige [9], [10], that for

*Department of Computer Science, King's College, University of London, London WC2R 2LS, UK (colin.cooper@kcl.ac.uk, thomas.radzik@kcl.ac.uk)

†Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213. Email: alan@random.math.cmu.edu. Research supported in part by NSF grant ccf0502793.

any connected graph G, the cover time satisfies $(1-o(1))n\ln n \le C_G \le (1+o(1))\frac{4}{27}n^3$. The complete graph K_n, is an example of a graph achieving the lower bound. The *lollipop* graph consisting of a path of length $n/3$ joined to a clique of size $2n/3$ has cover time asymptotic to the upper bound.

The results in this paper are always asymptotic in n, and the notation $A_n \sim B_n$ means that $\lim_{n\to\infty} A_n/B_n = 1$.

Let $\mathcal{G}_r$ denote the set of r-regular graphs with vertex set $V = \{1, 2, \ldots, n\}$ and the uniform measure. In an earlier paper [5] we studied the cover time of random regular graphs $\mathcal{G}_r$, for $r \ge 3$. In particular the following theorem was proved:

Theorem 1. *Let $r \ge 3$ be constant. Let $\theta_r = \frac{r-1}{r-2}$.*

If $G = G_r$ is chosen randomly from $\mathcal{G}_r$, then **whp**

$$C_G \sim \theta_r n \ln n.$$

In the paper [5] a technique (vertex coalescence) was used to estimate the probability that the random walk had not visited a given set of vertices. This technique can be used to obtain the distribution of the first meeting time of independent random walks.

Suppose there are $k \ge 1$ particles, each making a simple random walk on an random regular graph G. Essentially there are two possibilities. Either the particles are *Oblivious* or *Interactive*.

Oblivious particles act independently of each other, and do not interact on meeting. They may however interact with vertices, possibly in a way determined by previous visits of other particles. Interactive particles, can interact directly in some way on meeting. For example they may exchange information, coalesce, reproduce, destroy each other. We assume that interaction only occurs when meeting at a vertex, and that the random walks made by the particles are otherwise independent. For such models we study various properties of the walks, namely:

1. **Multiple walks**. For k particles walking independently, we establish the cover time of G.
2. **Particles with a finite life**. For k particles walking independently, we study the proportion of the graph covered at extinction.
3. **Talkative particles**. For k particles walking independently, which communicate on meeting at a vertex, we study the time to broadcast a message.

4. **Predator-Prey**. For k predator and ℓ prey particles walking independently, we study the expected time to extinction of the prey particles, when predators eat prey particles on meeting at a vertex.

5. **Sticky particles**. For k particles walking independently, which coalesce on meeting at a vertex, we study time to coalesce to a single particle.

6. **Explosive particles**. For $k = 2\ell$ particles walking independently, which destroy each other on meeting at a vertex, we study the time to extinction and the proportion of the graph covered at extinction.

Results: Oblivious particles

Our first result concerns the speed up in cover time. Let $T(k, v_1, ..., v_k)$ be the time to cover all vertices for k independent walks starting at vertices $v_1, ..., v_k$. Define the k-particle cover time $C_k(G)$ in the natural way as $C_k(G) = \max_{v_1,...,v_k} \mathbf{E}(T(k, v_1, ..., v_k))$ and define the speedup as $S_k = C(G)/C_k(G)$. The value of $C_k(G)$ was studied in [4] on the assumption that all particles start from the same vertex. They found *inter alia* that for expanders the speed up was $\Omega(k)$ for $k \leq n$ particles.

Theorem 2. Multiple particles walking independently.
Let $r \geq 3$ be constant. Let G be chosen randomly from $\mathcal{G}_r$, then **whp**.

(i) For $k = o(n/\ln^2 n)$ the k-particle cover time $C_G(k)$ satisfies

$$C_G(k) \sim \frac{\theta_r}{k} n \ln n,$$

and this result is independent of the initial positions of the particles.

(ii) For any k, and any starting position of the particles $C_G(k) = O\left(\frac{n}{k}\ln n + \ln n\right)$.

Comparing to Theorem 1, we see that $C_G(k) \sim C_G/k$, i.e. the speed up is exactly linear.

We next consider a model of search by particles with a finite life span. In this model, *Breakdown*, particles can stop functioning at any step. Specifically, there is a fixed probability b of the particle stopping at each step. This can be seen as a 'faulty robots' model.

Theorem 3. Breakdown. *Let Π_k be the number of unvisited vertices in a random walk by k faulty particles. Let b/n, $b = \theta(1)$ be the breakdown probability at each step. Then* **whp** *(over choices of G_r)*

(a)

$$\mathbf{E}(\Pi_k) \leq \left(\frac{(1+o(1))b}{b+\theta_r}\right)^k n.$$

(b) *If* $k = O(1)$ *then*

$$\mathbf{E}(\Pi_k) \sim \left(\frac{b}{b+\theta_r}\right)^k n.$$

.

Results: Interacting particles

Let $\widehat{M}(u,v)$ denote the number of steps before two random walks, starting at vertices u and v, first meet. Clearly if $u = v$ then $\widehat{M}(u,v) = 0$, so we need some way to describe the fact that the starting positions of the particles are not too near. We say k particles are in *general position*, if the starting positions $v_1, ..., v_k$ satisfy $d(v_i, v_j) \geq \omega$, where $\omega = \omega(k,n)$ tends to infinity slowly with k and n. After time $T_k = kT_1$, where T_1 is the single particle mixing time, the probability the particles are not distance at least ω apart from each other is $O(k^2 r^\omega/n)$. Thus a general position of

$$\omega(k,n) = \Omega(\max(\ln\ln n,\ \ln k)),$$

is feasible provided $k = n^\epsilon$, where $\epsilon > 0$ is a sufficiently small positive constant; and we choose this value for $\omega(k,n)$.

In the paper [5] a technique (vertex coalescence) was used to estimate the probability that the random walk had not visited a given set of vertices. This technique can be used to obtain the distribution of the first meeting time of independent random walks. For k particles in general position, the result is:

Theorem 4. *Let* $r \geq 3$ *be constant. Let* G *be chosen randomly from* $\mathcal{G}_r$*, then the following holds* **whp***.*
Let $k \leq n^\epsilon$*, for sufficiently small positive constant* ϵ*. For* k *independent random walks starting in a general position* $\boldsymbol{v} = (v_1, v_2, ..., v_k)$*, let* $\widehat{M}(\boldsymbol{v})$ *denote the number of steps before any of the particles meet. Then*

$$\mathbf{E}\widehat{M}(\boldsymbol{v}) \sim \frac{2\theta_r}{k(k-1)} n.$$

For a random walk starting from u, let $H(u,v)$ be the (expected) hitting time of vertex v. It follows directly from [5] that (**whp**) all but $O((\ln n)^A)$ vertices v have hitting time $H(u,v) \sim \theta_r n$, (where $A = O(1)$). Moreover, this is an upper bound for the other $O((\ln n)^A)$ vertices. Thus for most pairs of vertices $\mathbf{E}\widehat{M}(u,v) \sim H(u,v) \sim \theta_r n$.

In [8], Coppersmith, Tetali and Winkler investigate a related quantity $M(u,v)$, in the context of self-stabilizing networks. In their model, two particles starting at u, v make random walks, but only one particle is allowed to move at each step. The particle which moves at a given step is determined by a demon, whose aim is to delay the meeting as long as possible. They establish that $M(u,v)$, the (worst case) expected number of steps before meeting, satisfies

$$H(u,v) \leq M(u,v) \leq H(u,v) + H(v,t) - H(t,v),$$

for some fixed vertex t. The vertex t, which is called *hidden*, is defined by the property that $H(t,v) \leq H(v,t)$ for all $v \in V$.

In our study, there is no demon, and both of the particles walk randomly until they meet. As already remarked, for a pair of particles starting at vertices u, v, in general position we find that $\mathbf{E}\widehat{M}(u,v) \sim \theta_r n$. In fact, for most vertex pairs (u,v) we also have that $M(u,v) \sim \theta_r n$. We briefly outline why this is the case. The results we claim follow from [5], and are summarized in Lemma 13.

Say that a vertex v is *tree-like* if there is no cycle in the neighbourhood $N(v, L_0)$, the set of vertices within distance L_0 of v, where $L_0 = \alpha \ln \ln n$ for some absolute constant $\alpha > 0$. Most vertices v of a random regular graph are *tree-like* **whp**.

If u is not in $N(v, L_0)$ then $H(u,v) \sim \theta_r n$, and moreover $H(u,v) \leq \theta_r n(1 + o(1))$ for all u, v. Let t be hidden and pick a tree-like v, of distance at least L_0 from both u and t. There are $n - O((\ln n)^A)$ such vertices for some constant $A > 0$. It is immediate that

$$H(u,v) \leq M(u,v) \leq H(u,v)(1 + o(1)),$$

and thus for most choices of u and v, $M(u,v) \sim \theta_r n$.

Our next result concerns particles which can communicate on meeting at a vertex (but not in any other manner). We refer to such particles as *agents*, to distinguish them from non-communicating particles. Initially one agent has a message it wants to pass to all the others.

Theorem 5. Broadcast time.
Let $A > 0$, and $k \leq n^\epsilon$ for a sufficiently small positive constant ϵ. Suppose k agents make random walks starting in general position. Let B_k be the time taken for for a given agent to broadcast to all other agents. Then the expected broadcast time $\mathbf{E}(B_k)$ is

$$\mathbf{E}(B_k) \sim \frac{2\theta_r}{k} H_{k-1} n,$$

where H_k is the k-th harmonic number. Thus when $k \to \infty$, $\mathbf{E}(B_k) \sim \frac{2\theta_r \ln k}{k} n$.

Finally, we give some results for particles which interact in a less benign manner.

One variant of interacting particles is a predator-prey model, in which both types of particles make independent random walks. If a predator encounters prey on a vertex it eats them.

Theorem 6. Predator-prey.
Let $A > 0$, and $k, \ell \leq n^\epsilon$ for a sufficiently small positive constant ϵ. Suppose there are k predator and ℓ prey particles walking randomly, starting in general position. The expected extinction time of the prey, $\mathbf{E}(D_{k,\ell})$, satisfies

$$\mathbf{E}(D_{k,\ell}) \sim \frac{\theta_r H_\ell}{k} n.$$

One variant of predator-prey is interacting sticky particles, in which all particles are predatorial, and only one particle survives an encounter.

Theorem 7. Coalescence time: sticky particles.
Let $A > 0$, and $k \leq n^\epsilon$ for a sufficiently small positive constant ϵ. Let S_k be the time to coalesce, when there are originally k sticky particles walking in the graph. Then,

$$\mathbf{E}(S_k) \sim \psi_k n$$

where $\lim_{k\to\infty} \psi_k = 2\theta_r$.

As a last twist on this, we consider particles which destroy each other on meeting at a vertex.

Theorem 8. Extinction time: explosive particles.
Let $A > 0$, and $k \leq n^\epsilon$ for a sufficiently small positive constant ϵ. Let D_k be the time to extinction, when there are originally $k = 2\ell$ explosive particles walking in the graph. Then

$$\mathbf{E}(D_k) \sim 2\psi_k n$$

where $\lim_{k\to\infty} \psi_k = 2\theta_r \ln 2$.

Methodology. For oblivious particles, we use the techniques and results of [5], [7] to establish the probability that a vertex is unvisited by any of the walks at a given time t. For interacting particles, we use the same techniques to derive the probability that a walk on the (suitably defined) product graph G^k has not visited the diagonals (set of vertices $\boldsymbol{v} = (v_1, ..., v_k)$ with repeated vertex entries v_i) at a given time t.

2 Typical r-regular graphs

We say an r-regular graph G is *typical* if it has the properties **P1-P4** listed below: Let $\epsilon_1 > 0$ be a sufficiently small constant, and let

$$L_1 = \lfloor \epsilon_1 \log_r n \rfloor. \tag{1}$$

Let a cycle C be *small* if $|C| \leq L_1$.

P1. G is connected, and not bipartite.

P2. The second eigenvalue of the adjacency matrix of G is at most $2\sqrt{r-1}+\epsilon$, where $\epsilon > 0$ is arbitrarily small.

P3. There are at most $n^{2\epsilon_1}$ vertices on small cycles.

P4. No pair of cycles C_1, C_2 with $|C_1|, |C_2| \leq 100L_1$ are within distance $100L_1$ of each other.

Theorem 9. *Let $\mathcal{G}'_r \subseteq \mathcal{G}_r$ be the set of typical r-regular graphs. Then $|\mathcal{G}'| \sim |\mathcal{G}_r|$.*

P2 is a very difficult result of Friedman [12]. The other properties are easy to check. Note that P3 implies that most vertices of a typical r-regular graph are tree-like.

3 Estimating first visit probabilities

3.1 Convergence of the random walk

Let G be a connected graph with n vertices and m edges. For random walk $\mathcal{W}_u$ starting at a vertex u of G, let $\mathcal{W}_u(t)$ be the vertex reached at step t. Let $P = P(G)$ be the matrix of transition probabilities of the walk and let $P_u^{(t)}(v) = \mathbf{Pr}(\mathcal{W}_u(t) = v)$. Assuming G is not bipartite, the random walk $\mathcal{W}_u$ on G is ergodic with stationary distribution π. Here $\pi(v) = d(v)/(2m)$, where $d(v)$ the degree of vertex v. We often write $\pi(v)$ as π_v.

Let the eigenvalues of $P(G)$ be $\lambda_0 = 1 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq -1$ and let $\lambda_{\max} = \max(\lambda_1, |\lambda_{n-1}|)$. The rate of convergence of the walk is given by

$$|P_u^{(t)}(x) - \pi_x| \leq (\pi_x/\pi_u)^{1/2}\lambda_{\max}^t. \tag{2}$$

For a proof of this, see for example, Lovasz [13].

In this paper we consider the joint convergence of k independent random walks on a graph $G = (V_G, E_G)$. It is convenient to use the following notation. Let $H_k = (V_H, E_H)$ have vertex set $V_H = V^k$ and edge set $E_H = E^k$. If $S \subseteq V_H$, then $\Gamma(S)$ is obtained from H by contracting S to a single vertex $\gamma(S)$. All edges, including loops are retained. Thus $d_\Gamma(\gamma) = d_H(S)$, where d_F denotes vertex degree in graph F. Moreover Γ and H have the same total degree $(nr)^k$, and the degree of any vertex of Γ, except γ, is r^k.

Let $k \geq 1$ be fixed, and let $H = H_k$. For $F = G, H, \Gamma$ let $\mathcal{W}_{u,F}$ be a random walk starting at $u \in V_F$. Thus $\mathcal{W}_{u,G}$ is a single random walk, and $\mathcal{W}_{u,H}$ corresponds to k independent walks in G.

Lemma 10. *Let G be typical. Let $F = G, H, \Gamma$. Let S be such that $d_H(S) \le k^2 n^{k-1} r^k$. Let T_F be such that, for graph $F = (V_F, E_F)$, and $t \ge T_F$, the walk $\mathcal{W}_{u,F}$ satisfies*

$$\max_{x \in V_F} |P_u^{(t)}(x) - \pi_x| \le \frac{1}{n^3},$$

for any $u \in V_F$. Then for $k \le n$,

$$T_G = O(\ln n),\ T_H = O(\ln n) \text{ and } T_\Gamma = O(k \ln n).$$

Proof

Case (i): Single random walk.
We choose $\epsilon = 0.1$ in P1 so that for $r \ge 3$ we have

$$\lambda_{\max} \le 0.977. \tag{3}$$

Let

$$T_G = \frac{3 \ln n}{-\ln \lambda_{\max}}. \tag{4}$$

Using (2) we see that for $t \ge T_G$,

$$\max_{x \in V} |P_{u,G}^{(t)}(x) - \pi_x| \le n^{-3}.$$

Case (ii): k independent random walks.

Let $\mathcal{W}^t_{\boldsymbol{u},H}$ be the corresponding random walk in H. As the k associated walks in G are independent, we have $P^t_{\boldsymbol{u}}(\boldsymbol{x}) = P^t_{u_1}(x_1)...P^t_{u_k}(x_k)$ and $\pi(\boldsymbol{x}) = \pi(x_1)...\pi(x_k)$. At step t, the total variation distance $\Delta_{\boldsymbol{u}}(t, H)$ of the walk is

$$\Delta_{\boldsymbol{u}}(t, H) = \frac{1}{2} \sum_{\boldsymbol{x} \in V_H} |P^t_{\boldsymbol{u}}(\boldsymbol{x}) - \pi(\boldsymbol{x})|.$$

To simplify notation let $P_i = P^t_{u_i}(x_i)$, where $\boldsymbol{u} = (u_1, ..., u_k)$, and let $\pi_i = \pi(x_i)$. Then

$$\begin{aligned} |P^t_{\boldsymbol{u}}(\boldsymbol{x}) - \pi(\boldsymbol{x})| \le & |P_1...P_k - P_1...P_{k-1}\pi_k| + |P_1...P_{k-1}\pi_k - P_1...P_{k-2}\pi_{k-1}\pi_k| + \cdots \\ & + |P_1...P_\ell \pi_{\ell+1}...\pi_k - P_1...P_{\ell-1}\pi_\ell...\pi_k| + \cdots + |P_1\pi_2...\pi_k - \pi_1...\pi_k|. \end{aligned}$$

It follows that

$$\Delta_{\boldsymbol{u}}(t, H) \le \frac{k}{2} \max_{i=1,...,k} \left(\sum_{x \in V(G)} |P^t_{u_i}(x) - \pi(x)| \right) \le k\Delta(t, G),$$

where $\Delta(t,G) = \max_{u \in V(G)} \Delta_u(t,G)$. If we choose

$$T_H = \frac{\ln k + 3 \ln n}{-\ln \lambda_{\max}},$$

then $\Delta(t,G) \le 1/(kn^3)$ and $\Delta(t,H) \le 1/n^3$.

Case (iii): Random walk in Γ.
Let $\lambda_H = \lambda_{\max}(H)$, and let

$$\tau(\epsilon, H) = \min \{ t : \Delta(t,H) \le \epsilon \text{ for all } t' \ge t \},$$

then it is a result of [1] (see also [14]) that

$$\tau(\epsilon, H) \ge \frac{1}{2} \frac{\lambda_H}{1-\lambda_H} \ln \frac{1}{2\epsilon}.$$

Let $\lambda_G = \lambda_{\max} \le 0.977$ from (3). On the assumption that $k \le n$ and using $\epsilon = n^{-3}$ and $\tau(\epsilon, H) \le T_H$, we find that

$$\lambda_H \le \frac{99}{100}.$$

For a simple random walk on a graph G, the conductance Φ is given by

$$\Phi(G) = \min_{\substack{X \subseteq V \\ d(X) \le m(G)}} \frac{e(X : \overline{X})}{d(X)},$$

where $d(X)$ is the degree of set X, and $e(X : \overline{X})$ is the number of edges between X and $V \setminus X$. The second eigenvalue λ_1 of a reversible Markov chain satisfies

$$1 - 2\Phi \le \lambda_1 \le 1 - \frac{\Phi^2}{2}. \tag{5}$$

On the assumption that $\lambda_{\max} = \lambda_1$, we find that

$$\Phi(H) \ge 1/200. \tag{6}$$

The standard way to ensure this is to make the chain *lazy* i.e. the walk only moves to a neighbour with probability 1/2. Otherwise it stays where it is. If we do this until every vertex has been covered, then this will double the cover time. It is simplest therefore to assume that we keep the chain lazy for T_H steps. At this point it is mixed, and we can stop being lazy. All of our walks will be assumed to be lazy until the mixing time.

The quantity we need is $\Phi(\Gamma)$, where Γ is the contraction of H. From the construction of Γ it follows that $\Phi(\Gamma) \ge \Phi(H)$; every set of vertices in V_Γ corresponds to a set in V_H, and edges are preserved on contraction. Thus for any starting starting position $\boldsymbol{u}$ of a walk $\mathcal{W}_{\boldsymbol{u}}(\Gamma)$ we have, from (2) and (5) that, provided $t \geq T_\Gamma = 10^5 k \ln n$,

$$|P_{\boldsymbol{u}}^{(t)}(\boldsymbol{x}) - \pi(\boldsymbol{x})| \leq \left(\frac{d(\gamma)}{r^k}\right)^{1/2} e^{-t\Phi^2/2} \leq \frac{1}{n^3},$$

where $d(\gamma) \leq k^2 n^{k-1} r^k$. □

3.2 Generating function formulation

We use the approach of [5], [7].

Let $d(t) = \max_{u,x \in V} |P_u^{(t)}(x) - \pi_x|$, and let T be such that, for $t \geq T$

$$\max_{u,x \in V} |P_u^{(t)}(x) - \pi_x| \leq n^{-3}. \tag{7}$$

It follows from e.g. Aldous and Fill [2] that $d(s+t) \leq 2d(s)d(t)$ and so for $\ell \geq 1$,

$$\max_{u,x \in V} |P_u^{(\ell T)}(x) - \pi_x| \leq \frac{2^\ell}{n^{3\ell}}. \tag{8}$$

Fix two vertices u, v. Let $h_t = \mathbf{Pr}(\mathcal{W}_u(t) = v)$ be the probability that the walk $\mathcal{W}_u$ visits v at step t. Let

$$H(z) = \sum_{t=T}^{\infty} h_t z^t \tag{9}$$

generate h_t for $t \geq T$. This changes the definition of $H(z)$ from that used in [5], [6] where we included the coefficients $h_0, h_1, \ldots, h_{T-1}$ in the definition of $H(z)$ and gave rise to technical problems.

Next, considering the walk $\mathcal{W}_v$, starting at v, let $r_t = \mathbf{Pr}(\mathcal{W}_v(t) = v)$ be the probability that this walk returns to v at step $t = 0, 1, \ldots$. Let

$$R(z) = \sum_{t=0}^{\infty} r_t z^t$$

generate r_t. Our definition of return involves $r_0 = 1$.

For $t \geq T$ let $f_t = f_t(u \to v)$ be the probability that the first visit of the walk $\mathcal{W}_u$ to v in the period $[T, T+1, \ldots]$ occurs at step t. Let

$$F(z) = \sum_{t=T}^{\infty} f_t z^t$$

generate f_t. Then we have

$$H(z) = F(z)R(z). \tag{10}$$

Finally, for $R(z)$ let

$$R_T(z) = \sum_{j=0}^{T-1} r_j z^j. \tag{11}$$

We remark that (10) is also valid for visits by $\mathcal{W}_u$ to a set S of vertices. By contracting the set S of vertices of G to a single vertex, $\gamma(S)$ we obtain a graph Γ and an equivalent relationship $\widetilde{H}(z) = \widetilde{F}(z)\widetilde{R}(z)$. We see next that for large enough T, if $t > T$ then the first visit probabilities $f_T(t, G, S)$ to S in G, and $f_T(t, \Gamma, \gamma(S))$ to $\gamma(S)$ in Γ, are asymptotically equal. This can be seen as follows.

For a walk $\mathcal{W}_u$ starting at u with t-step transition probabilities $P_{u,t}(v)$ to vertex v,

$$f_T(t) = \sum_{w \notin S} P_{u,T}(w)\phi_{w,S}(t - T),$$

where $\phi_{w,S}(\tau)$ is the probability that a first visit from w to S occurs at step $\tau > 0$.

For the graphs G and Γ, and $w \notin S$, the value of $\phi_{w,S}(\tau, G)$ equals $\phi_{w,S}(\tau, \Gamma)$ as the degrees and neighbourhood structure of $G \setminus S$ and $\Gamma \setminus \{\gamma\}$ are identical. Thus provided we choose T to be a sufficiently large mixing time in both G and Γ we have that $P_{u,T}(w, G) \sim P_{u,T}(w, \Gamma) \sim \frac{1}{n}$ and thus

$$f_T(t, G, S) = (1 + o(1))f_T(t, \Gamma, \gamma(S)).$$

3.3 First visit time lemma: Single vertex v

The following lemma should be viewed in the context that G is an n vertex graph which is part of a sequence of graphs with n growing to infinity. For a proof see [7].

Lemma 11. *Let T be a mixing time such that* (7) *holds. Let $R_T(z)$ be given by* (11)*, let $R_v = R_T(1)$, and let*

$$p_v = \frac{\pi_v}{R_v(1 + O(T\pi_v))}. \tag{12}$$

Suppose the following conditions hold.

(a) *For some constant $0 < \theta < 1$, we have $\min_{|z| \le 1+\lambda} |R_T(z)| \ge \theta$, where $\lambda = \frac{1}{KT}$ for some sufficiently large constant K.*

(b) $T^2\pi_v = o(1)$ *and* $T\pi_v = \Omega(n^{-2})$.

Then for all $t \geq T$,

$$f_t(u \to v) = (1 + O(T\pi_v)) \frac{p_v}{(1+p_v)^{t+1}} + o(e^{-\lambda t/2}). \tag{13}$$

Let v be a (possibly contracted) vertex, and for $t \geq T$ let $\boldsymbol{A}_t(v)$ be the event that $\mathcal{W}_u$ does not visit v during steps $T, T+1, \ldots, t$. Then

$$\mathbf{Pr}(\boldsymbol{A}_t(v)) = \sum_{s>t} f_t(u \to v),$$

and we have the following corollary.

Corollary 12.

$$\mathbf{Pr}(\boldsymbol{A}_t(v)) = \frac{(1 + O(T\pi_v))}{(1 + (1 + O(T\pi_v))\pi_v/R_v)^t} + o(e^{-t/KT}).$$

4 Oblivious particles

4.1 Cover time for k particles walking independently

The proof of the following lemma follows directly from the one given in [5] with the simplification made in later papers (e.g. [7]), that we only consider first visits after T. It is obtained from Corollary 12 by substituting $R_T \sim \theta_r$ for tree-like vertices.

Lemma 13. *Let $\mathcal{G}_r$ denote the set of r-regular random graphs. Let T_H be a mixing time given by of Lemma 10. Let*

$$p = \frac{\theta_r}{n}. \tag{14}$$

There exists a subset $\mathcal{H}$ of $\mathcal{G}_r$ of size $(1 - o(1))|\mathcal{G}_r|$ such that the following properties hold for $G \in \mathcal{H}$.

(i) Let $\boldsymbol{A}_t(v)$ be the event that a walk starting at a fixed vertex x does not visit v during steps $T, ..., t$. For all $v \in V$

$$\mathbf{Pr}(\boldsymbol{A}_t(v)) \leq (1-p)^{-(t+o(t))} + O(Te^{-t/(2KT)}),$$

where $K > 0$ is constant. Moreover, if v is tree-like then

$$\mathbf{Pr}(\boldsymbol{A}_t(v)) = (1-p)^{-(t+o(t))} + O(Te^{-t/(2KT)}).$$

(ii) Let $\boldsymbol{A}_t(u,v)$ be the event that a walk starting at a fixed vertex x does not visit u or v during steps $T,...,t$. There exists a set $S \subseteq V$ of size $n^{1-o(1)}$ such that for all $u,v \in S$

$$\mathbf{Pr}(\boldsymbol{A}_t(u,v)) = (1+o(1))\mathbf{Pr}(\boldsymbol{A}_t(u))\mathbf{Pr}(\boldsymbol{A}_t(v)) + O(Te^{-t/(2KT)}).$$

Let $\boldsymbol{A}_{k,t}(v)$ be the event that no agent visits vertex v in steps $T,...,t$. As the particles are independent, and $T = T_H$ is a mixing time for all k particles, we have that,

$$\mathbf{Pr}(\boldsymbol{A}_{k,t}(v)) = (\mathbf{Pr}(\boldsymbol{A}_{k,t}(v)))^k = e^{-kp(t+o(t))}$$

and similarly for $\mathbf{Pr}(\boldsymbol{A}_{k,t}(u,v))$. The proof of Theorem 2 is a straightforward adaptation of the proof of Theorem 1 as given in Lemma 13.

Upper bound on cover time. Let $t_k^* = (\theta_r n/k)\ln n$, then for suitably large t, the event $\boldsymbol{A}_{k,t}(v)$ that vertex v is unvisited in $T,...,t$ is at most $e^{-kp(t+o(t))}$. Choosing $t_0 = (1+\epsilon)t^*$ where $\epsilon \to 0$ sufficiently slowly, and substituting this value into the upper bound proof given in section 5.1 of [5], we find that $C_G(k) \leq t_0 = (1+o(1))t_k^*$.

Lower bound on cover time. Choosing $t_1 = (1-\epsilon)t_k^*$ where $\epsilon \to 0$ sufficiently slowly, and substituting this value into the lower bound proof given in section 5.2 of [5], we find that there is a set of vertices S, given by Lemma 13 above which **whp** are not all covered at time t_1. The conclusion is that $C_G(k) \geq t_1$.

This completes the proof of Theorem 2. □

4.2 Particles with finite life span

Particles with breakdown. The first part of Theorem 3 follows directly from Lemma 13. Let $\mathcal{B}_t(v)$ be the event that the particle P breaks down at $t \geq T_H$, and v is unvisited by P after T_H. Then

$$\mathbf{Pr}(\mathcal{B}_t(v)) \leq \frac{b}{n}\left(1-\frac{b}{n}\right)^{t-1}\left(1-\frac{\theta_r}{n}\right)^{t+o(t)}. \tag{15}$$

Now,

$$\sum_{t=T}^{\infty} \frac{b}{n}\left(1-\frac{b}{n}\right)^{t-1}\left(1-\frac{\theta_r}{n}\right)^{t+o(t)} = \frac{(1+o(1))b}{b+\theta_r}.$$

The k particles are walking independently and part (a) follows immediately.

If v is tree-like then there is equality in (15). Part (b) follows since **whp** $T = o(n)$ and almost all vertices are tree-like. □

5 Probability two or more particles meet at a given step

The main task of this section is to estimate the probability two or more particles meet at a given step.

We note the following result (see e.g. [11]), for a random walk on the line $= \{0, ..., a\}$ with absorbing states $\{0, a\}$, and transition probabilities q, p, s for moves left, right and looping respectively. Starting at vertex z, the probability of absorption at the origin 0 is

$$\rho(z, a) = \frac{(q/p)^z - (q/p)^a}{1 - (q/p)^a} \le \left(\frac{q}{p}\right)^z, \tag{16}$$

provided $q \le p$. Similarly, for a walk starting at z on the line $\{0, 1, ..., \}$, with absorbing states $\{0, \infty\}$, the probability of absorption at the origin is $\rho(z) = (q/p)^z$.

We first consider the case of a meeting between two particles.

Lemma 14. *Let G be a typical r-regular graph, and let v be a vertex of G, tree-like to depth $L_1 = \lfloor \epsilon_1 \log_r n \rfloor$. Suppose that at time zero, two independent random walks $(\mathcal{W}_1, \mathcal{W}_2)$ start from v. Let (x, y) denote the position of the particles at any step. Let $S = \{(u, u) : u \in V\}$. Let f_T be the probability of a first return to S within $T = T_\Gamma$ steps given that the walks leave v by different edges at time zero. Then*

$$f_T = \frac{1}{(r-1)^2} + O(n^{-\Omega(1)}).$$

Proof

We write $f_T = g_T + h_T$ where g_T is the probability of a first return to S up to time L_1. Assume the walks leave v by distinct edges at time 0, let x_t, y_t denote the positions of the particles after t steps and let $Y_t = \text{dist}(x_t, y_t)$.

To estimate g_T we extend $N(v, L_1)$ to an infinite r-regular tree $\mathcal{T}$ rooted at v. Let X_t be the distance between the equivalent pair of particles walking in $\mathcal{T}$. Thus provided $t \le L$, we have that $Y_t = X_t$, and $g_T = \mathbf{Pr}(\exists t \in [1, L_1] : \ X_t = 0)$. The values of X_t are as follows: Initially $X_1 = 2$. If $X_t = 0$, then $\mathbf{Pr}(X_{t+1} = 0) = 1/r$, $\mathbf{Pr}(X_{t+1} = 2) = (r-1)/r$. If $X_t > 0$, then

$$X_t = \begin{cases} X_{t-1} - 2 & \text{with probability } q = \frac{1}{r^2} \\ X_{t-1} & \text{with probability } s = \frac{2(r-1)}{r^2} \\ X_{t-1} + 2 & \text{with probability } p = \left(\frac{r-1}{r}\right)^2. \end{cases} \tag{17}$$

Finally let Z_t be a walk on the even numbers $\{0, \pm 2, \pm 4, ...\}$ of the infinite line, with $Z_1 = 2$, and with transition probabilities p, q, s. By coupling Z_t and X_t, we have inductively that

$X_t \geq Z_t$. Note that

$$\mathbf{E}(Z_t - Z_{t-1}) = 2 - \frac{4}{r}. \tag{18}$$

Now let g_∞ denote the probability that $\{\exists t \geq 1 : X_t = 0\}$, i.e. the particles meet in $\mathcal{T}$. Equation (16) implies that

$$g_\infty = \frac{1}{(r-1)^2}. \tag{19}$$

Furthermore,

$$g_\infty = g_T + g'_T \tag{20}$$

where g'_T is the probability that $\{\exists t > L_1 : X_t = 0\}$, i.e. the particles meet after L_1 steps. But, using (16) once again, we see that

$$\begin{aligned} g'_T &\leq \mathbf{Pr}(X_{L_1} \leq L_1/2) + \left(\frac{1}{(r-1)^2}\right)^{L_1/4} \\ &\leq O(n^{-\Omega(1)}) + \left(\frac{1}{(r-1)^2}\right)^{L_1/4}. \end{aligned} \tag{21}$$

As $\mathbf{Pr}(Z_{L_1} \leq L_1/2) \geq \mathbf{Pr}(X_{L_1} \leq L_1/2)$, the bound $O(n^{-\Omega(1)})$ follows from (18) and the Hoeffding inequality for the sums of bounded random variables Z_{L_1}. We use $\Omega(1)$ throughout this proof, instead of providing explicit constants, but remark that, as the right hand side of (18) is at least 2/3 for any r, we can insert absolute constants independent of r.

It follows from (19), (20) and (21) that we can write

$$g_T = \frac{1}{(r-1)^2} + O(n^{-\Omega(1)}). \tag{22}$$

Recall that h_T is the probability of a first return to S after time L_1. We next prove that $h_T = O(n^{-\Omega(1)})$. Let $h_T = h'_T + h''_T$ where

$$h'_T = \mathbf{Pr}\left(\text{The walks meet during steps } \{L_1, ..., T\} \text{ and } Y_{L_1} > L_1/2\right),$$

and we know from (21) that

$$h''_T \leq \mathbf{Pr}(Y_{L_1} \leq L_1/2) = O(n^{-\Omega(1)}). \tag{23}$$

Let $\sigma \leq T$ be the step at which the particles first meet again, and let s be the *last* step less than σ at which the distance between the particles is $L_1/2$ or more. Let $x = x_s,\ y = y_s$ denote the positions of the particles at time s. Let ρ_1, ρ_2 denote the particles at x, y respectively.

Let $N = N(x, L_1)$, the neighbourhood of depth at most L_1 centered at x. It follows from property P4 that there are most two paths $xPy,\ xP'y$ between x and y in N, both of length at least L_2.

Suppose there is a single path xPy. Consider the particle ρ_1. Either ρ_1 moves at least $L_1/4$ down xPy, at some step $s < t \leq \sigma$, or if not, then ρ_2 must do; as we now show. Suppose first that both particles stay within N until they meet, then ρ_2 must move at least $L_1/4$ along xPy to meet ρ_1. Suppose next that $s < t \leq \sigma$ is the first step at which the boundary of N visited by either particle, and suppose this particle is ρ_1 (or both). As the distance between the particles is at most $L_1/2$, then ρ_2 has moved at least $L_1/4$ along xPy by that step.

Suppose next there are two paths xPy, $xP'y$ not internally disjoint, and $|P'| \geq |P| \geq L_1/2$. Let w be the mid-point of P so that $xPy = xRwSy$ (resp. w' of P' etc). Then (e.g.) each half-path $xRw, xR'w'$ has a section of length at least $L_1/12$ in common with or disjoint from the other. Repeating the argument above, at least one of these sections must be walked by one of the particles.

Thus there is one of at most four fixed sub-paths of length $L_1/12$ in G which one of the particles has to traverse, an event of probability $O((1/(r-1))^{L_1/12})$, (see (16)). As there are at most T ways of choosing s, and T starting times for traversing the sub-path, an upper bound of $O(T^2(r-1)^{-L_1/12})$ follows. □

Using Lemma 14, we can calculate the expected number of returns to the diagonal S of H_k for k particles. Recall the definitions of $\Gamma(S)$ and $\gamma(S)$, the contraction of S.

Lemma 15. *Let k be the number of particles walking on the underlying graph G. Let $\mathcal{W}_\gamma$ be a random walk in Γ starting at γ. Let f^* denote the probability that $\mathcal{W}_\gamma$ makes a first return to γ within T_Γ steps. Then*

$$f^* = \frac{1}{r-1} + O\left(\frac{k^2}{n^{\Omega(1)}}\right).$$

If $k \leq n^\epsilon$ for a small constant ϵ, then $f^* \sim \frac{1}{r-1}$.

Proof Let $S = \{(v_1, ..., v_k) :$ at least two v_i are the same$\}$. The particles are at the components of the vector corresponding to the vertex in question. Every vertex in S has degree r^k in H_k, and the size of of S is at most $\binom{k}{2}n^{k-1}$. On the other hand, there are at least $N_2(k)$ vertices of S with exactly two replicates, where

$$N_2(k) \geq \binom{k}{2}n^{k-1} - \binom{k}{3}n^{k-2} - \binom{k}{2}\binom{k-2}{2}n^{k-2} = \binom{k}{2}n^{k-1}\left(1 - O\left(\frac{k^2}{n}\right)\right).$$

Thus the total degree of $\gamma(S)$ is

$$d(\gamma) = \binom{k}{2}n^{k-1}r^k\left(1 - O\left(\frac{k^2}{n}\right)\right). \tag{24}$$

Similarly the loop degree of $\gamma(S)$ is

$$d_\ell(\gamma) = \binom{k}{2}n^{k-1}r^{k-1}\left(1 + O\left(\frac{k^2}{n}\right)\right),$$

the correction being for where a different pair of particles coincide at the next step.

A single walk in Γ is related to k independent walks in the underlying graph G. The point of difference being the moves into and out of $\gamma(S)$. First returns to $\gamma(S)$ can be of several types. The simplest type is a loop return (type O), for example two particles move to the same neighbour. If this does not occur, we distinguish four cases. In the first case (type A), there were initially exactly two particles coincident at a tree-like vertex of G, which meet up again at some vertex. In the second case (type B), the coincident particles do not meet up again, but instead some other particles which were not initially coincident meet up. In the third case (type C), three or more particles coincide, either initially or finally. In the fourth case (type D), the coincident particles are initially at a non-tree-like vertex of G. Thus we can write

$$f^* = f_O + f_A + O(f_B + f_C + f_D),$$

where $f_O = \frac{d_\ell(\gamma)}{d(\gamma)}$. For type A returns,

$$f_A = \frac{d'(\gamma)}{d(\gamma)} f_{T_\Gamma} \tag{25}$$

Here $d'(\gamma)$ counts the non-loop edges of γ corresponding to tree-like vertices of G, and f_{T_Γ} is from Lemma 14. Thus $d'(\gamma) = (d(\gamma) - d_\ell(\gamma))(1 - O(k^2 n^{\epsilon_0}/n) - O(k^2/n))$. This follows from P3, as G is typical, and the value of $N_2(k)$ above. Thus

$$f_O + f_A = \frac{1}{r-1} + O(n^{-\Omega(1)}).$$

We can estimate f_B as follows: Of the vertices of S, at most $\nu_1 = \binom{k}{2}^2 r^{L_1} n^{k-2}$, have another pair of entries within the same neighbourhood of depth L and at most $\nu_2 = \binom{k}{2} k r^{L_1} n^{k-2}$, have an entry within the neighbourhood of a coincident pair. For particles distance at least L_1 apart, the probability they coincide in T_Γ steps is $O(n^{-\Omega(1)})$, by the analysis of Lemma 14. Thus

$$f_B = O\left(\frac{(\nu_1+\nu_2) r^k}{d(\gamma)} + \frac{k^2}{n^{\Omega(1)}}\right) = O\left(\frac{k^2}{n^{\Omega(1)}}\right).$$

Finally,

$$\begin{aligned} f_C &= O\left(\frac{k^3 n^{k-2} r^k}{d(\gamma)} + \frac{k^3 r^{2L} n^{k-2} r^k}{d(\gamma)} + \frac{k^2}{n^{\Omega(1)}}\right) = O\left(\frac{k^2}{n^{\Omega(1)}}\right) \\ f_D &\leq \frac{n^{2\epsilon_0} r^L k^2 n^{k-2}}{d(\gamma)} = O\left(\frac{k^2}{n^{\Omega(1)}}\right), \qquad \text{see P3.} \end{aligned}$$

The expression for f_C arises as follows: The term $O\left(\frac{k^3 n^{k-2} r^k}{d(\gamma)}\right)$ is the probability 3 or more particles coincident initially. The term $O\left(\frac{k^3 r^{2L} n^{k-2} r^k}{d(\gamma)}\right)$ is the probability that two particles are initially in the L_1-neighbourhood of a third vertex. The term $O\left(\frac{k^2}{n^{\Omega(1)}}\right)$ is the probability that at least two particles, initially at distant greater than L_1 meet within time T. □

Corollary 16. *For typical graphs and k particles, the expected number of returns to γ in T_Γ steps is*

$$R_{\gamma(S)} = \theta_r + O\left(\frac{k^2}{n^{\Omega(1)}}\right). \tag{26}$$

If $k \leq n^\epsilon$ for a small constant ϵ, then $R_{\gamma(S)} \sim \theta_r$. □

By an *occupied vertex*, we mean a vertex visited by at least one particle at that time step. The next lemma concerns what happens during the first mixing time, when the particles start from general position, and also the separation of the occupied vertices when a meeting occurs.

Corollary 17. *For typical graphs G and $k \leq n^\epsilon$ particles,*
(i) Suppose two (or more) particles meet at time $t > T_\Gamma$. Let p_L be the probability that the minimum separation between some pair of occupied vertices is less than L. Then $p_L = O(k^2 r^L/n)$.
(ii) Suppose the particles start walking on G with minimum separation

$$\ell = \alpha(\max\{\ln\ln n, \ln k\}).$$

Then

$$\mathbf{Pr}(\textit{Some pair of particles meet during } T_\Gamma) = o(1).$$

Proof (i) At most $\nu = \binom{k}{2}^2 n^{k-2} r^L r^k$ edges incident with γ have a pair of occupied vertices within distance L. After the mixing time T_Γ, the probability the move to γ which causes the meeting is made using such an edge is $(1 + o(1))\nu/d(\gamma)$.

(ii) By the analysis of Lemma 14 the probability some pair of particles meet during $T_\Gamma = O(k \log n)$ steps is

$$O\left(\frac{k^2 \ell T_\Gamma}{(r-1)^{\ell/4}}\right) = O\left(\frac{k^3 \ell \log n}{(r-1)^{\ell/4}}\right) = o(1),$$

for suitable choice of α and small $\epsilon > 0$, where $k = O(n^\epsilon)$.

□

6 Conditions of the first visit time lemma

We next check that the conditions of Lemma 11 hold with respect to the vertex γ of the graph Γ. Thus in this section, $T = T_\Gamma$, and $v = \gamma$. The conditions are:

(a) $\min_{|z| \leq 1+\lambda} |R_T(z)| \geq \theta$, for some constant $\theta > 0$, and $\lambda = 1/KT$ for suitably large K.

(b) $T^2\pi_v = o(1)$ and $T\pi_v = \Omega(n^{-2})$.

Condition (a) follows from Lemma 18 below. Condition (b) is easily disposed of. Recall from Lemma 10, that $T = O(k \ln n)$. From Lemma 15, $\pi_\gamma = d(\gamma)/(nr)^k$ where $1 \le d(\gamma)/n^{k-1}r^k \le k^2$. Thus $T^2\pi_\gamma = o(1)$ provided $k \le n^{1/5}$.

Lemma 18. *For $|z| \le 1+\lambda$, there exists a constant $\theta > 0$ such that $|R_T(z)| \ge \theta$.*

Proof Let $r_t = r_{t,A} + r_{t,B}$ where $r_{t,A}$ is the probability of a loop return ($t = 1$), or type A return at time $t \le L_1$. Thus $R_T(z) = R_A(z) + R_B(z)$ where $R_A(z) = \sum_{t=0}^{L_1} r_{t,A} z^t$.

The arguments in the proof of Lemma 14 show that

$$\phi = \sum_{t=L_1+1}^{T_\Gamma} r_t z^t = O(n^{-\Omega(1)}),$$

and thus

$$|R_B(z)| \le \phi + (1+\lambda)^T T_\Gamma (f_B + f_C + f_D) = O(n^{-\Omega(1)}) + O\left(\frac{k^3 \ln n}{n^{\Omega(1)}}\right) = o(1).$$

As $|R_T(z)| \ge |R_A(z)| - |R_B(z)|$, we have $|R_T(z)| \ge |R_A(z)| - o(1)$.

As in Lemma 14, let Y_t be the distance between the particles during the first L_1 steps. For $1 \le t \le L_1$ let

$$b_t = \mathbf{Pr}(Y_t = 0, Y_1, ..., Y_{t-1} > 0, \max_{i=1,...,t} Y_i < L_1),$$

be the probability that the walks first meet at step t. Let $B(z) = \sum_{t=1}^{L_1} b_t z^t$, and let $A(z) = \sum_{t=0}^{\infty} a_t z^t = 1/(1-B(z))$. Thus

$$R_A(z) = \sum_{t=0}^{L_1} r_{t,A} z^t = \sum_{t=0}^{L_1} a_t z^t = \sum_{t=0}^{\infty} a_t z^t - \sum_{t=L_1+1}^{\infty} a_t z^t =$$
$$\sum_{t=0}^{\infty} a_t z^t - O(((1+\lambda)\zeta)^t) = \sum_{t=0}^{\infty} a_t z^t - o(1),$$

as we now explain. By coupling Y_t and Z_t as in Lemma 14, and again applying the Hoeffding Lemma to Z_t, there is an absolute constant $0 < \zeta < 1$, such that $a_t = O(\zeta^t)$.

From Lemma 15, $B(1) = f_A + f_O = \frac{1}{r-1} + o(1)$ and so for $|z| \le 1+\lambda$

$$B(|z|) \le B(1+\lambda) \le B(1)(1+\lambda)^T \le \frac{e^{1/K}}{r-1} + o(1) < e^{1/K}.$$

So,

$$|R_A(z)| + o(1) = \left|\frac{1}{1-B(z)}\right| \geq \frac{1}{1+B(|z|)} \geq \frac{1}{1+e^{1/K}}.$$

□

7 Results for interacting particles

From Corollary 12 and Section 6 and Corollary 16 we see

Theorem 19. *Let $\boldsymbol{A}_k(t)$ be the event that a first meeting among the k particles after the mixing time T_Γ, occurs after step t. Let $p_k = \binom{k}{2}\frac{\theta_r}{n}(1+O(n^{-\Omega(1)}))$. Then*

$$\mathbf{Pr}(\boldsymbol{A}_k(t)) = (1+o(1))(1-p_k)^t + O(T_\Gamma e^{-t/2KT_\Gamma}).$$

Let $\boldsymbol{B}_{s,k}(t)$ be the event that a first meeting between a given set of s particles and another set of k particles after the mixing time T_Γ, occurs after step t. Let $q_{sk} = sk\frac{\theta_r}{n}(1+O(n^{-\Omega(1)}))$. Then

$$\mathbf{Pr}(\boldsymbol{B}_{s,k}(t)) = (1+o(1))(1-q_{sk})^t + O(T_\Gamma e^{-t/2KT_\Gamma}).$$

□

Corollary 20. *Let M_k (resp. $M_{s,k}$) be the time at which a first meeting of the particles occurs, then $\mathbf{E}(M_k) = (1+o(1))/p_k$ (resp. $\mathbf{E}(M_{s,k}) = (1+o(1))/q_k$).*

This follows from $\mathbf{E}(M_k) = \sum_{t\geq T}\mathbf{Pr}(\boldsymbol{A}_k(t))$ and $p_kT_\Gamma = o(1)$ etc. □

The proof of Theorem 4 follows from Theorem 19 and Corollary 17.

7.1 Expected broadcast time: Theorem 5

We allow the particles which met, time $T = T_G$ to re-mix after an encounter. This happens $k-1$ times. Recall that $T_\Gamma = O(kT)$. From Corollary 17, the event that some particles meet during one of these mixing times has probability $O(k^3T/n^{\Omega(1)}) = o(1)$ (by assumption).

Assuming this does not happen, the expected time $\mathbf{E}(B_k)$, for a given agent to broadcast to all other agents is,

$$\begin{aligned}\mathbf{E}(B_k) &= O(kT) + \sum_{s=1}^{k-1} \frac{(1+o(1))}{q_{sk}} \\ &\sim n\theta_r \sum_{s=1}^{k-1} \frac{1}{s(k-s)}, \\ &= \frac{2n\theta_r}{k} H_{k-1},\end{aligned}$$

where H_k is the k-th harmonic number. On the assumption that k is large,

$$\mathbf{E}(B_k) \sim \frac{2\theta_r \ln k}{k} n.$$

□

7.2 Expected time to coalescence; sticky particles: Theorem 7

Let S_k be the time for all the particles to coalesce, when there are originally k sticky particles walking in the graph. Then,

$$\begin{aligned}\mathbf{E}(S_k) &= O(kT) + \sum_{s=1}^{k} \frac{(1+o(1))}{p_s} \\ &\sim n\theta_r \sum_{s=1}^{k} \frac{2}{s(s-1)}.\end{aligned}$$

Noting that $\sum_{s=1}^{\infty} 1/(s(s-1)) = 1$ we see that for large k,

$$\mathbf{E}(S_k) \sim 2\theta_r n.$$

□

7.3 Expected time to extinction; explosive particles: Theorem 8

Let D_k be the time to extinction, when there are originally $k = 2\ell$ explosive particles walking in the graph. Then

$$\begin{aligned}\mathbf{E}(D_k) &= O(kT) + \sum_{s=1}^{\ell} \frac{(1+o(1))}{p_{2s}} \\ &\sim n\theta_r \sum_{s=1}^{\ell} \frac{2}{2s(2s-1)}.\end{aligned}$$

Noting that $\sum_{s=1}^{\infty} 1/(2s(2s-1)) = \ln 2$ we see that for large k,

$$\mathbf{E}(D_k) \sim 2\theta_r \ln 2\, n.$$

□

8 Conclusions

We have extended the results of [5] to deal with multiple random walks. In particular we have shown once again the usefulness of Lemma 11 in the context of random walks on expander graphs of high girth.

References

[1] D. Aldous. Some inequalities for reversible Markov chains. J. London Math. Soc. (2) 25, (1982) 564-576.

[2] D. Aldous and J. Fill. Reversible Markov Chains and Random Walks on Graphs, http://stat-www.berkeley.edu/pub/users/aldous/RWG/book.html.

[3] R. Aleliunas, R.M. Karp, R.J. Lipton, L. Lovász and C. Rackoff. Random Walks, Universal Traversal Sequences, and the Complexity of Maze Problems. *Proceedings of the 20th Annual IEEE Symposium on Foundations of Computer Science* (1979) 218-223.

[4] N. Alon, C. Avin, M. Kouchý, G. Kozma, Z. Lotker, M. Tuttle. Many random walks are faster then one, to appear.

[5] C. Cooper and A. M. Frieze. The cover time of random regular graphs, *SIAM Journal on Discrete Mathematics*, 18, 728-740 (2005).

[6] C. Cooper and A. M. Frieze. The cover time of the preferential attachment graph. *Journal of Combinatorial Theory Series B*, 97(2), 269-290 (2007).

[7] C. Cooper and A. M. Frieze. The cover time of the giant component of a random graph. RSA (accepted 2007).

[8] D. Coppersmith, P. Tetali and P. Winkler. Collisions among random walks on a graph. SIAM Journal on Discrete Mathematics, 6, 363-374 (1993). T

[9] U. Feige. A tight upper bound for the cover time of random walks on graphs, *Random Structures and Algorithms*, 6 (1995) 51-54.

[10] U. Feige. A tight lower bound for the cover time of random walks on graphs, *Random Structures and Algorithms*, 6 (1995) 433-438.

[11] W. Feller, *An Introduction to Probability Theory, Volume I,* (Second edition) Wiley (1960).

[12] J. Friedman. A proof of Alon's second eignevalue conjecture, Memoirs of the A.M.S., to appear.

[13] L. Lovász. Random walks on graphs: a survey. Bolyai Society Mathematical Studies, Combinatorics, Paul Erdös is Eighty (Volume 2) Keszthely (Hungary), (1993) 1-46.

[14] A. Sinclair. Improved bounds for mixing rates of Markov chains and multicommodity flow. CPC 1 (1992) 351-370.