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The L^2-cutoff for reversible Markov processes

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Abstract

We consider the problem of proving the existence of an L^2-cutoff for families of ergodic Markov processes started from given initial distributions and associated with reversible (more, generally, normal) Markov semigroups. This includes classical examples such as families of finite reversible Markov chains and Brownian motion on compact Riemannian manifolds. We give conditions that are equivalent to the existence of an L^2-cutoff and describe the L^2-cutoff time in terms of the spectral decomposition. This is illustrated by several examples including the Ehrenfest process and the biased (p,q)-random walk on the non-negative integers, both started from an arbitrary point.

Keywords: L^2-cutoff; Markov semigroups; Normal operators

1. Introduction

Consider a (time-homogeneous) Markov chain on a finite set Ω with one-step transition kernel K. Let $K^l(x,\cdot)$ denote the probability distribution of this chain at time l starting from the state x. Assuming irreducibility and aperiodicity, it is known that

$$\lim_{l\to\infty} K^l(x,\cdot) = \pi$$

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where π is the unique invariant probability of K on Ω. Set $k_x^l = K^l(x,\cdot)/\pi$, the relative density of $K^l(x,\cdot)$ w.r.t. π. For $1 \leqslant p \leqslant \infty$, set

$$D_p(x,l) = \left\| k_x^l - 1 \right\|_{\ell^p(\pi)} = \begin{cases} \max_y\{|k_x^l(y)-1|\} & \text{if } p=\infty, \\ (\sum_y |k_x^l(y)-1|^p \pi(y))^{1/p} & \text{if } 1 \leqslant p < \infty. \end{cases}$$

For $p=1$, this is exactly twice the total variation distance between $K^l(x,\cdot)$ and π and, for $p=2$, it is the so-called chi-square distance. For any $\epsilon > 0$, set

$$T_p(x,\epsilon) = \min\bigl\{l \geqslant 1:\ D_p(x,l) \leqslant \epsilon\bigr\}. \tag{1.1}$$

The concept of cutoff was introduced by Aldous and Diaconis in [1–3] to capture the fact that many ergodic Markov processes appear to converge abruptly to their stationary measure. We refer the reader to [5,6,15,19] for detailed discussions and the description of various examples. In its simplest form, the cutoff phenomenon in L^2 for a family of finite ergodic Markov chains (with given starting points) $(\Omega_n, x_n, K_n, \pi_n)$ is defined as follows. There is an L^2-cutoff with cutoff sequence t_n if

$$\lim_{n\to\infty} D_{n,2}(x_n, at_n) = \begin{cases} 0 & \text{if } a \in (1,\infty), \\ \infty & \text{if } a \in (0,1). \end{cases}$$

Here $D_{n,2}$ denotes the chi-square distance on Ω_n relative to π_n.

In [5], the authors discussed a number of variants of this definition and produced, in the reversible case, a necessary and sufficient condition for the existence of a max-L^2-cutoff, that is, a cutoff for $\max_{x\in\Omega} D_2(x,\cdot)$ (some of the results in [5] holds for L^p, $1 < p < \infty$).

The aim of the present paper is twofold. Our first goal is to establish a criterion for the existence of an L^2-cutoff for families of Markov processes starting from specific initial distributions when the associated semigroup is normal (i.e., commutes with its adjoint on a proper Hilbert space). Our second goal is to derive formulas for the L^2-cutoff time sequence using spectral information. To attain these two goals, we will take advantage of the very specific structure of the chi-square distance and its direct relationship with spectral decomposition. This is in contrast to the techniques and results of [5] which do not involve much spectral theory and treats L^p-distances, $1 < p < \infty$ as well as L^2. The following theorem illustrates the goals outled above.

Theorem 1.1. *Let $\Omega = \{0,1,2,\dots\}$ and K be the Markov kernel of the birth and death chain on Ω with uniform birth rate $p \in (0,1/2)$, uniform death rate $1-p$ and $K(0,0) = 1-p$. Let x_n be a sequence of states in Ω. Then, the discrete time family of birth and death chains with respective starting states $x_1, x_2, \dots$ presents an L^2-cutoff if and only if x_n tends to infinity. Moreover, if there is a cutoff then*

$$t_n = \frac{\log(1-p) - \log p}{-\log(4p(1-p))} x_n$$

is a cutoff time sequence as $n \to \infty$.

We will also obtain variants of this result that involve finite state spaces $\Omega_n = \{0,1,\dots,n\}$ and birth and death rates (p_n, q_n) that are allowed to depend on n. Our second main example is the Ehrenfest chain, treated in Theorem 6.3. The treatment of these examples occupies the bulk of the paper and illustrates very well the delicacies of the cutoff phenomenon in the context of specified starting distributions.

This paper is organized as follows. In Section 2, we recall various notions of cutoffs and quote useful results from [5]. In Section 3, we give criteria for the existence of a cutoff as well as formulas for the cutoff times in the case of families of Laplace transforms. The main result of Section 3 is Theorem 3.5 which is the technical cornerstone of this work. In Section 4, we observe that the chi-square distance between the distribution of a normal Markov process and its invariant probability measure can be expressed as a Laplace transform. This gives criteria and formula for cutoffs of families of ergodic normal Markov processes (normal here means that the generator is a normal operator, i.e., commutes with its adjoint). In Section 5, we spell out the results in the case of families of finite Markov chains (in discrete and continuous time). See Theorems 5.1, 5.3 and Theorems 5.4, 5.5. In Section 6, we apply these results to study the cutoff phenomenon for the Ehrenfest chain started at an arbitrary point. See Theorems 6.3, 6.5. In Section 7, we prove Theorem 1.1 and a number of related results. In Section 8, we study a family of birth and death chains on $\{-n,\dots,n\}$ containing examples whose stationary measure has either a unique maximum or a unique minimum at 0.

2. Cutoff terminology

In this section, we recall various notions of cutoffs and a series of related results from [5]. The notion of cutoff can be developed for any family of non-increasing functions taking values in $[0,\infty]$. The following definition treats functions defined on the natural integers. We refer the reader to [5] for additional details and examples.

Definition 2.1. Let $\mathbb{N}$ be the set of non-negative integers. For $n \geqslant 1$, let $f_n : \mathbb{N} \to [0,\infty]$ be a non-increasing function vanishing at infinity. Assume that

$$M = \limsup_{n\to\infty} f_n(0) > 0.$$

Then the family $\mathcal{F} = \{f_n\colon n = 1,2,\dots\}$ is said to present:

(i) A *precutoff* if there exist a sequence of positive numbers t_n and constants $b > a > 0$ such that

$$\lim_{n\to\infty} f_n(k_n) = 0, \qquad \liminf_{n\to\infty} f_n(l_n) > 0,$$

where $k_n = \min\{j \geqslant 0\colon j > bt_n\}$ and $l_n = \max\{j \geqslant 0\colon j < at_n\}$.

(ii) A *cutoff* if there exists a sequence of positive numbers t_n such that, for all $\epsilon \in (0,1)$,

$$\lim_{n\to\infty} f_n\big(\overline{k_n}(\epsilon)\big) = 0, \qquad \lim_{n\to\infty} f_n\big(\underline{k_n}(-\epsilon)\big) = M,$$

where $\overline{k_n}(\epsilon) = \min\{j \geqslant 0\colon j > (1+\epsilon)t_n\}$ and $\underline{k_n}(\epsilon) = \max\{j \geqslant 0\colon j < (1+\epsilon)t_n\}$.

(iii) A (t_n, b_n)*-cutoff* if $t_n > 0$, $b_n \geqslant 0$, $b_n = o(t_n)$ and

$$\lim_{c\to\infty} \overline{F}(c) = 0, \qquad \lim_{c\to-\infty} \underline{F}(c) = M,$$

where, for $c \in \mathbb{R}$, $\bar{k}(n,c) = \min\{j \in \mathbb{N}\colon\ j > t_n + cb_n\}$, $\underline{k}(n,c) = \max\{j \in \mathbb{N}\colon\ j < t_n + cb_n\}$ and

$$\overline{F}(c) = \limsup_{n\to\infty} f_n\bigl(\bar{k}(n,c)\bigr), \qquad \underline{F}(c) = \liminf_{n\to\infty} f_n\bigl(\underline{k}(n,c)\bigr). \tag{2.1}$$

This definition agrees with the one used in [5]. In (ii) and (iii), $(t_n)_1^\infty$ is referred to as a cutoff sequence and b_n as a window with respect to t_n. Obviously, (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

Remark 2.1. When the f_n's are functions on $[0,\infty)$, cutoffs are defined in a similar way. For a precutoff, we set $l_n = at_n$ and $k_n = bt_n$. For a cutoff, $\overline{k_n}$ and $\underline{k_n}$ are replaced respectively by $(1+\epsilon)t_n$ and $(1-\epsilon)t_n$. These notions of precutoff, cutoff and their cutoff sequences coincide with the notion in [5]. For a (t_n,b_n)-cutoff in continuous time, we require $b_n > 0$ and use $\bar{k}(n,c) = \underline{k}(n,c) = t_n + cb_n$. Assuming $b_n > 0$, this agrees with the (t_n,b_n)-cutoff of [5].

In Definition 2.1(iii), the window of a cutoff captures explicitly how sharp the cutoff is. It is quite sensitive to the choice of the cutoff sequence t_n. Window optimality is addressed in the following definition (when $b_n > 0$ this definition is equivalent to the one in [5]).

Definition 2.2. Let $\mathcal{F}$ and M be as in Definition 2.1. Assume that $\mathcal{F}$ presents a (t_n,b_n)-cutoff. Then, the cutoff is called:

(i) weakly optimal if, for any (t_n,d_n)-cutoff for $\mathcal{F}$, one has $b_n = O(d_n)$,
(ii) optimal if, for any (s_n,d_n)-cutoff for $\mathcal{F}$, we have $b_n = O(d_n)$. In this case, b_n is called an optimal window for the cutoff,
(iii) strongly optimal if

$$0 < \underline{F}(c) \leqslant \overline{F}(-c) < M \quad \forall c > 0.$$

Remark 2.2. An optimal window is a minimal window (in the sense of order) over all cutoff sequences and, hence, an optimal cutoff is also a weakly optimal cutoff, i.e. (ii) $\Rightarrow$(i). If a (t_n,b_n)-cutoff is strongly optimal, it is easy to see that $b_n > 0$ for all n and that $\inf_n b_n > 0$ if the domain of the functions in $\mathcal{F}$ is $\mathbb{N}$ (b_n may tend to 0 when the domain is $[0,\infty)$). Hence, a strongly optimal (t_n,b_n)-cutoff implies that for any $-\infty < c_1 < c_2 < \infty$ we have

$$0 < f_n(t_n + c_2 b_n) \leqslant f_n(t_n + c_1 b_n) < M$$

for n large enough. This implies that if (s_n,d_n) is another cutoff sequence for $\mathcal{F}$, then the window d_n has order at least b_n, and thus (iii) $\Rightarrow$ (ii).

Remark 2.3. Let $\mathcal{F}$ be a family of functions defined on $\mathbb{N}$. If $\mathcal{F}$ has a (t_n,b_n)-cutoff with $b_n \to 0$, instead of looking for the optimal window, it is better to determine the limsup and liminf of the sequences $f_n([t_n])$, where $[t]$ is any integer in $[t-1,t+1]$.

Remark 2.4. For any family $\{f_n : T \to [0,\infty],\ n = 1,2,\dots\}$ with $T = [0,\infty)$, a necessary condition for a strongly optimal (t_n,b_n)-cutoff is that

$$0 < \liminf_{n\to\infty} f_n(t_n) \leqslant \limsup_{n\to\infty} f_n(t_n) < M.$$

When $T=\mathbb{N}$, we need instead

$$0<\liminf f_n\big(\lceil t_n\rceil\big)\leqslant \limsup f_n\big(\lfloor t_n\rfloor\big)<M.$$

The cutoff phenomenon and its optimality are closely related to the way the functions in $\mathcal{F}$ converge to 0. This is captured by the following simple concept.

Definition 2.3. Let f be an extended real-valued non-negative function defined on T, which is either $\mathbb{N}$ or $[0,\infty)$. For $\epsilon>0$, set

$$T(f,\epsilon)=\inf\big\{t\in T\colon\ f(t)\leqslant\epsilon\big\}$$

if the right-hand side above is non-empty and let $T(f,\epsilon)=\infty$ otherwise.

In the context of ergodic Markov chains, $T(f,\epsilon)$ is interpreted as the ϵ-mixing time. The simplest relationship with the notions of cutoff discussed above is as follows. Let $\mathcal{F}=\{f_n: T\to[0,\infty]\colon n=1,2,\ldots\}$ be a family of non-increasing functions vanishing at infinity. Assume that $M=\limsup_{n\to\infty} f_n(0)>0$. Then $\mathcal{F}$ has a cutoff if and only if

$$\forall\epsilon,\eta\in(0,M),\qquad \lim_{n\to\infty} T(f_n,\epsilon)/T(f_n,\eta)=1.$$

See [5, Propositions 2.3–2.4] for further relationships and details.

We end this section with a technical result concerning cutoffs and optimality which is useful in either proving or disproving a cutoff and its optimality when the cutoff or window sequences contain both bounded and unbounded subsequences. The proof is straightforward and is omitted.

Proposition 2.1. *Let T be $[0,\infty)$ or $\mathbb{N}$. Consider $\mathcal{F}=\{f_n:T\to[0,\infty],\ n=1,2,\ldots\}$ as a family of non-increasing functions vanishing at infinity. For any subsequence $\xi=(\xi_i)$ of positive integers, denote by $\mathcal{F}_\xi$ the subfamily $\{f_{\xi_i},\ i=1,2,\ldots\}$. Assume that $M=\limsup_{n\to\infty} f_n(0)>0$. For $T=[0,\infty)$, the following are equivalent.*

(i-1) *$\mathcal{F}$ has a cutoff (resp. (t_n,b_n)-cutoff).*
(i-2) *For any subsequence $\xi=(\xi_i)$, the family $\mathcal{F}_\xi$ has a cutoff (resp. (t_{ξ_n},b_{ξ_n})-cutoff).*
(i-3) *For any subsequence $\xi=(\xi_i)$, we may choose a further subsequence $\xi'=(\xi'_i)$ such that the family $\mathcal{F}_{\xi'}$ has a cutoff (resp. $(t_{\xi'_n},b_{\xi'_n})$-cutoff).*

Moreover, assume that $\mathcal{F}$ has a (t_n,b_n)-cutoff. Then the following are equivalent.

(ii-1) *$\mathcal{F}$ has an optimal (resp. weakly, strongly optimal) (t_n,b_n)-cutoff.*
(ii-2) *For any subsequence $\xi=(\xi_i)$, the family $\mathcal{F}_\xi$ has an optimal (resp. weakly, strongly optimal) (t_{ξ_n},b_{ξ_n})-cutoff.*
(ii-3) *For any subsequence $\xi=(\xi_i)$, we may choose a further subsequence $\xi'=(\xi'_i)$ such that the family $\mathcal{F}_{\xi'}$ has an optimal (resp. weakly, strongly optimal) $(t_{\xi'_n},b_{\xi'_n})$-cutoff.*

For $T=\mathbb{N}$, all equivalences remain true if $t_n\to\infty$, $\liminf_{n\to\infty} b_n>0$ and, for some $\delta\in(0,M)$, $T(f_n,\delta)\to\infty$.

3. Cutoffs for Laplace transforms

In this section, we deal with cutoffs for family of functions which are Lebesgue–Stieltjes integral of exponential functions, that is, generalized Laplace transforms. Such functions appear naturally in the context of chi-square distance for reversible Markov process. More precisely, if the infinitesimal generator is self-adjoint and the initial distribution has an L^2 Radon–Nikodym derivative w.r.t. the invariant probability measure, then the square of the chi-square distance to stationarity is such an integral. This will be discussed in details in Section 4.

3.1. Laplace transforms

For $n \geqslant 1$, let $V_n : (0,\infty) \to (0,\infty)$ be a non-decreasing and right-continuous function. Consider f_n as a Lebesgue–Stieltjes integral defined by

$$f_n(t) = \int\limits_{(0,\infty)} e^{-t\lambda}\, dV_n(\lambda) \quad \forall t \geqslant 0. \tag{3.1}$$

It is easy to see that f_n is non-increasing. Observe that sums of exponential functions are of this special type. For example, let

$$f_n(t) = \sum_{i \geqslant 1} a_{n,i} e^{-t\lambda_{n,i}} \quad \forall t \geqslant 0,$$

where $a_{n,i} \geqslant 0$ and $\lambda_{n,i+1} \geqslant \lambda_{n,i} > 0$ for $n \geqslant 1$, $i \geqslant 1$. Then f_n is of the form in (3.1) with $a_{n,0} = \lambda_{n,0} = 0$, $\lambda_n = \lambda_{n,1}$ and

$$V_n(t) = \sum_{i=0}^{j-1} a_{n,i}, \quad \text{for } \sum_{i=0}^{j-1} \lambda_{n,i} < t \leqslant \sum_{i=0}^{j} \lambda_{n,i} \quad \text{and } j \geqslant 1.$$

Lemma 3.1. *Let $V : (0,\infty) \to (0,\infty)$ be a non-decreasing and right-continuous function and let f be a function on $[0,\infty)$ defined by*

$$f(t) = \int\limits_{(0,\infty)} e^{-t\lambda}\, dV(\lambda).$$

Assume that V is bounded. Then f is analytic on $(0,\infty)$.

Proof. See [21, Theorem 5, p. 57]. □

The following is an application of the above lemma which is helpful when examining cutoffs and their optimality.

Lemma 3.2. *For $n \geqslant 1$, let f_n be a function on $[0,\infty)$ defined by (3.1). Assume that $\sup_n f_n(0) < \infty$. Then, for any sequence of positive numbers $(t_n)_1^\infty$, there exists a subsequence*

$(t_{n_k})_{k\geqslant 1}$ *such that the sequence* $g_k : a \mapsto f_{n_k}(at_{n_k})$ *converges uniformly on any compact subset of* $(0,\infty)$ *to an analytic function on* $(0,\infty)$*. Moreover, if* c_{n_k} *is such that* $|c_{n_k}| = o(t_{n_k})$*, then*

$$\forall a > 0, \quad \lim_{k\to\infty} f_{n_k}(at_{n_k} + c_{n_k}) = \lim_{k\to\infty} f_{n_k}(at_{n_k}).$$

Proof. See Appendix A. □

Remark 3.1. Let f_n be the function as in Lemma 3.2 and let t_n be any sequence of positive numbers. If t_n tends to infinity, we may choose a subsequence $(n_k)_1^\infty$ such that $a \mapsto f_{n_k}([at_{n_k}])$ converges to a function analytic on $(0,\infty)$, where $[t]$ is any integer in $[t-1, t+1]$.

The following two corollaries are simple applications of Lemma 3.2.

Corollary 3.3. *Let* $(f_n)_1^\infty$ *be as in Lemma* 3.2 *such that* $f_n(0)$ *is bounded. For any sequence of positive numbers* t_n*, set*

$$F_1(a) = \limsup_{n\to\infty} f_n(at_n), \qquad F_2(a) = \liminf_{n\to\infty} f_n(at_n), \quad \forall a > 0.$$

Then F_1 *and* F_2 *are continuous on* $(0,\infty)$*. Furthermore, either* $F_1 > 0$ (*resp.* $F_2 > 0$) *or* $F_1 \equiv 0$ (*resp.* $F_2 \equiv 0$).

Proof. See Appendix A. □

Remark 3.2. By Remark 3.1, if $t_n \to \infty$, Corollary 3.3 also holds with the following replacements

$$F_1(a) = \limsup_{n\to\infty} f_n\big([at_n]\big), \qquad F_2(a) = \liminf_{n\to\infty} f_n\big([at_n]\big),$$

where $[t]$ is any integer in $[t-1, t+1]$.

Corollary 3.4. *Let* $\mathcal{F} = \{f_n : [0,\infty) \to [0,\infty]\colon\ n = 1, 2, \ldots\}$ *be a family of functions defined by* (3.1)*. Assume that* $\mathcal{F}$ *has a* (t_n, b_n)*-cutoff with* $b_n > 0$ *and let* $\overline{F}$, $\underline{F}$ *be functions in* (2.1).

(i) *If* $\overline{F}(0) < \infty$*, then, on the set* $(0,\infty)$*, either* $\overline{F} > 0$ (*resp.* $\underline{F} > 0$) *or* $\overline{F} \equiv 0$ (*resp.* $\underline{F} \equiv 0$).
(ii) *Assume that* $\underline{F}(0) > 0$*. If* $\underline{F}(c) = 0$ (*resp.* $\overline{F}(c) = 0$) *for some* $c > 0$*, then* $\overline{F}(c) = \infty$ (*resp.* $\underline{F}(c) = \infty$) *for all* $c < 0$.
(iii) *If* $t_n = T(f_n, \delta)$*, then the conclusions in* (i) *and* (ii) *hold true without the assumptions on* $\overline{F}(0)$ *and* $\underline{F}(0)$*. That is, for* (i)*, either* $\overline{F} > 0$ (*resp.* $\underline{F} > 0$) *or* $\overline{F} \equiv 0$ (*resp.* $\underline{F} \equiv 0$)*. For* (ii)*, if* $\underline{F}(c) = 0$ (*resp.* $\overline{F}(c) = 0$) *for some* $c > 0$*, then* $\overline{F}(c) = \infty$ (*resp.* $\underline{F}(c) = \infty$) *for all* $c < 0$.

Proof. See Appendix A. □

Remark 3.3. By Remarks 3.1–3.2, Corollary 3.4 also holds for families of functions defined on $\mathbb{N}$ if one assumes that $b_n \to \infty$.

3.2. Cutoffs for Laplace transforms

The following theorem is one of the main technical results of this paper. It provides a simple criterion to inspect cutoffs. If V is a non-decreasing and right-continuous function on $(0,\infty)$, we also denote by V the measure on $(0,\infty)$ such that $V((a,b]) = V(b) - V(a)$.

Theorem 3.5. *For $n \geqslant 1$, let $f_n : [0,\infty) \to [0,\infty]$ be a function defined by* (3.1) *and set $M = \liminf_{n\to\infty} f_n(0)$. Assume that $f_n(t)$ vanishes at infinity for $n \geqslant 1$.*

(i) *If $M < \infty$, then the family has no precutoff.*
(ii) *If $M = \infty$, let $t_n = T(f_n, \delta)$ and*

$$\lambda_n = \lambda_n(C) = \inf\bigl\{\lambda\colon\ V_n\bigl((0,\lambda]\bigr) > C\bigr\}. \tag{3.2}$$

Then the family has a cutoff if and only if there exist constants $\delta, \epsilon, C \in (0,\infty)$ such that
(a) $t_n\lambda_n \to \infty$.
(b) $\int_{(0,\lambda_n)} e^{-\epsilon\lambda t_n}\, dV_n(\lambda) \to 0$.

Furthermore, if (a) *and* (b) *hold, then the family has a (t_n, λ_n^{-1})-cutoff.*

Proof. See Appendix A. □

Remark 3.4. If there is a cutoff for $(f_n)_1^\infty$, then (a) and (b) hold for any positive triple (δ, ϵ, C).

Remark 3.5. It follows from the proof of this result that if f_n is an extended real-valued function defined on $\mathbb{N}$, then Theorem 3.5(i) can fail.

The next theorem is a discrete time version of Theorem 3.5.

Theorem 3.6. *For $n \geqslant 1$, let $f_n : \mathbb{N} \to [0,\infty]$ be a function defined by* (3.1) *and set $M = \liminf_{n\to\infty} f_n(0)$ and $t_n = T(f_n, \delta)$. Assume that $f_n(t)$ vanishes at infinity for all $n \geqslant 1$ and $t_n \to \infty$ for some $\delta > 0$.*

(i) *If $M < \infty$, then the family has no precutoff.*
(ii) *If $M = \infty$, then the family has a cutoff if and only if there exists $C > 0$ and $\epsilon > 0$ such that Theorem* 3.5 (a)–(b) *hold true.*

Furthermore, if Theorem 3.5 (a)–(b) *are satisfied, then the family has a (t_n, γ_n^{-1})-cutoff, where $\gamma_n = \min\{\lambda_n, 1\}$.*

Proof. See Appendix A. □

Remark 3.6. As in Remark 3.4, if, in Theorem 3.6, there is a cutoff for the family $(f_n)_1^\infty$ with cutoff time tending to infinity, then (a) and (b) are true for any positive constants C, δ, ϵ.

Remark 3.7. It has been implicitly proved in the appendix that, for the family of functions f_n defined on $[0,\infty)$, Theorem 3.6 applied to the family of restricted functions $\{f_n|_{\mathbb{N}}\colon n = 1, 2, \ldots\}$

still holds true if $t_n = T(f_n|_{\mathbb{N}}, \delta)$ is replaced by $T(f_n, \delta)$ defined in Theorem 3.5. Furthermore, Remark 3.6 is also true under this replacement.

The next proposition is concerned with the optimality of cutoffs for Laplace transforms.

Proposition 3.7. *Let $\mathcal{F} = \{f_n : T \to [0,\infty] \mid n = 1,2,\ldots\}$ be a family of functions of the form* (3.1). *Assume that $\mathcal{F}$ has a (t_n, b_n)-cutoff where $t_n = T(f_n, \delta)$ for all $n \geqslant 1$ with $\delta \in (0,\infty)$ and $b_n > 0$. Let $\overline{F}$ and $\underline{F}$ be functions in* (2.1). *For $T = [0,\infty)$, the (t_n, b_n)-cutoff is*

(i) *weakly optimal iff it is optimal iff $\underline{F}(c) > 0$ for some $c > 0$,*
(ii) *strongly optimal iff $\overline{F}(c) < \infty$ for all $c < 0$.*

For $T = \mathbb{N}$, the above remains true if $b_n \to \infty$.

Proof. See Appendix A. □

3.3. The cutoff time of Laplace transforms

Theorems 3.5 and 3.6 can be used to examine the existence of a cutoff by checking whether the product of $T(f_n, \delta)$ and λ_n tends to infinity or not. By Remarks 3.4 and 3.6, the constant C appearing in the definition of λ_n can be taken to be any positive number and, hence, the only unknown term that needs to be studied is the δ-mixing time $T(f_n, \delta)$. Understanding this quantity with any precision is a difficult task. In this section, we describe potential cutoff time sequences in different terms.

Theorem 3.8. *Let $\mathcal{F} = \{f_n : [0,\infty) \to [0,\infty]\colon n = 1,2,\ldots\}$ be a family of functions defined by* (3.1) *which vanish at infinity. For $n \geqslant 1$ and $C > 0$, let $\lambda_n = \lambda_n(C)$ be the constant defined in* (3.2) *and set*

$$\tau_n = \tau_n(C) = \sup_{\lambda \geqslant \lambda_n} \left\{ \frac{\log(1 + V_n((0,\lambda]))}{\lambda} \right\}. \tag{3.3}$$

Then $\mathcal{F}$ has a cutoff if and only if, for some $C > 0$ and $\epsilon > 0$,

(a) $\tau_n \lambda_n \to \infty$,
(b) $\int_{(0,\lambda_n)} e^{-\epsilon\lambda\tau_n}\, dV_n(\lambda) \to 0$.

Moreover, if (a) *and* (b) *hold true, then $\mathcal{F}$ has a (t_n, b_n)-cutoff with*

$$t_n = \tau_n, \qquad b_n = \lambda_n^{-1} w(\tau_n \lambda_n),$$

where $w : (0,\infty) \to (0,\infty)$ is any function satisfying

$$\lim_{t\to\infty} \frac{w(t)}{t} = 0, \qquad \liminf_{t\to\infty} e^{w(t)}\left(1 - e^{-w^2(t)/t}\right) > 0. \tag{3.4}$$

In particular, $w(t) = \log t$ is a function qualified for (3.4) *and $b_n = \lambda_n^{-1}\log(\tau_n\lambda_n)$.*

Proof. See Appendix A. □

Remark 3.8. It is shown in the proof of Theorem 3.8 that $\tau_n(C) \leqslant T(f_n, \frac{C}{C+1})$, $C > 0$.

Remark 3.9. In contrast with Theorem 3.5, Theorem 3.8 does not require explicitly that $f_n(0) \to \infty$ (i.e., $M = \infty$) for a cutoff. However, this is in fact contained implicitly in Theorem 3.8(a).

Remark 3.10. What is proved in the appendix is that, if a family has a cutoff then the conditions (a)–(b) hold for any $C > 0$, $\epsilon > 0$. This means that either (a)–(b) hold for all positive constants C, ϵ or one of (a) or (b) must fail for all C, ϵ. Hence, when using Theorem 3.8 to inspect the existence of a cutoff, one needs to check (a) and (b) only for one arbitrary pair (C, ϵ). In practice, this is a very important remark.

Remark 3.11. The second condition in (3.4) implies $\lim_{t\to\infty} w(t) = \infty$. It follows that the window given by Theorem 3.8 has order strictly larger than that the one in Theorem 3.5 and, hence, cannot be optimal.

The following is a discrete time version of Theorem 3.8.

Theorem 3.9. *For $n \geqslant 1$, let $f_n : \mathbb{N} \to [0, \infty]$ be the function defined by* (3.1). *For $C > 0$, let $\lambda_n = \lambda_n(C)$ and $\tau_n = \tau_n(C)$ be the quantities defined by* (3.2) *and* (3.3). *Assume that either, for some $C > 0$, $\tau_n \to \infty$ or, for some $\delta > 0$, $T(f_n, \delta) \to \infty$. Then $\mathcal{F}$ has a cutoff if and only if Theorem* 3.8 (a) *and* (b) *hold for some $C > 0$ and $\epsilon > 0$. Moreover, if* (a) *and* (b) *hold, then $\mathcal{F}$ has a (t_n, b_n)-cutoff with*

$$t_n = \tau_n, \qquad b_n = \max\{\lambda_n^{-1} w(\tau_n \lambda_n), 1\},$$

where w is a function satisfying (3.4).

Proof. See Appendix A. □

Remark 3.12. Remark 3.8 holds true in discrete time cases.

Remark 3.13. Concerning functions defined on $[0, \infty)$ and their restriction to the natural integers, the proof of Theorem 3.9 shows that under the assumption of $\tau_n \to \infty$, the existence of cutoff in Theorems 3.8 and 3.9 are equivalent and both families share the same cutoff type if the window is at least 1. Thus, by Remark 3.10, if the family in Theorem 3.9 has a cutoff, then Theorem 3.8(a), (b) (in both discrete time and continuous time setting) hold true for all $C > 0$ and $\epsilon > 0$.

4. The main results

Spectral theory is a standard tool to study the L^2-convergence of Markov processes to their stationarity. In particular, in the general context of reversible Markov processes, the square of the chi-square distance can be expressed in terms of the spectral decomposition of the infinitesimal generator and written in the form of (3.1). In the following, we start by recalling the definition of ergodic Markov processes discussed in [5].

4.1. Markov processes and transition functions

In what follows, let the time to be either $\mathbb{N} = \{0, 1, 2, \ldots\}$ or $[0, \infty)$. A Markov transition function on a space Ω equipped with a σ-algebra $\mathcal{B}$, is a family of probability measures $p(t, x, \cdot)$ indexed by $t \in T$ ($T = [0, \infty)$ or $\mathbb{N}$) and $x \in \Omega$ such that $p(0, x, \Omega \setminus \{x\}) = 0$ and, for each $t \in T$ and $A \in \mathcal{B}$, $p(t, x, A)$ is $\mathcal{B}$-measurable and satisfies

$$p(t+s, x, A) = \int_{\Omega} p(s, y, A) p(t, x, dy).$$

We say that a Markov process $X = (X_t, t \in T)$ with filtration $\mathcal{F}_t = \sigma(X_s\colon s \leqslant t) \subset \mathcal{B}$ admits $p(t, x, \cdot)$, $t \in T$, $x \in \Omega$, as transition function if

$$E(f \circ X_s \mid \mathcal{F}_t) = \int_{\Omega} f(y) p(s-t, X_t, dy)$$

for all $0 < t < s < \infty$ and all bounded measurable f. The measure $\mu_0(A) = P(X_0 \in A)$ is called the initial distribution of the process X. All finite dimensional marginals of X can be expressed in terms of μ_0 and the transition function. In particular,

$$\mu_t(A) = P(X_t \in A) = \int p(t, x, A)\, \mu_0(dx).$$

Given a Markov transition function $p(t, x, \cdot)$, $t \in T$, $x \in \Omega$, for any bounded measurable function f, set

$$P_t f(x) = \int f(y) p(t, x, dy). \tag{4.1}$$

For any measure ν on $(\Omega, \mathcal{B})$ with finite total mass, set

$$\nu P_t(A) = \int p(t, x, A)\, \nu(dx).$$

We say that a probability measure π is invariant if $\pi P_t = \pi$ for all $t \in T$. In the general setting, invariant measures are not necessarily unique.

4.2. L^2-distances, mixing time and cutoffs

The Markov processes of interest in this paper are ergodic in the sense that, for some initial measure μ of interest, the sequence μ_t converges (in some sense) to a probability measure as t tends to infinity. A simple argument shows that this limit must be an invariant probability measure.

We now introduce the chi-square distance measuring the convergence to stationarity. Let p_t be a Markov transition function on Ω with invariant probability measure π. Let μ be another probability measure on $(\Omega, \mathcal{B})$. For $t \geqslant 0$, if μP_t is absolutely continuous with respect to π with density $h(t, \mu, \cdot)$, we set

$$D_2(\mu, t) = \left(\int_{\Omega} \left| h(t, \mu, x) - 1 \right|^2 \pi(dx) \right)^{1/2}. \tag{4.2}$$

In the case that such a density does not exist, $D_2(\mu, t)$ is set to be infinity. It is an exercise to show that the absolute continuity of μP_t w.r.t π implies that of μP_{t+s} for all $s > 0$. Moreover, the map $t \mapsto D_2(\mu, t)$ is non-increasing (see., e.g., [5]).

Definition 4.1 *(Mixing time).* For any $\epsilon > 0$, set

$$T_2(\mu, \epsilon) = T\big(D_2(\mu, t), \epsilon\big) = \inf\big\{t \in T\colon\ D_2(\mu, t) \leqslant \epsilon\big\}.$$

If the infimum is taken over an empty set, $T_2(\mu, \epsilon) = \infty$.

This quantity, the so-called L^2-mixing time of a Markov transition function with initial distribution μ, plays an important role in the quantitative analysis of ergodic Markov processes.

For any Markov transition function $p(t, x, \cdot)$, $t \in T$ and $x \in \Omega$, let P_t be the operator defined in (4.1). Extend P_t as a bounded operator on the Hilbert space $L^2(\Omega, \pi)$.

Definition 4.2. The spectral gap λ of $p(t, x, \cdot)$, $t \in T$, is the supremum of all constants c such that

$$\forall f \in L^2(\Omega, \pi),\ \forall t \in T, \quad \big\|(P_t - \pi) f\big\|_2 \leqslant e^{-ct} \|f\|_2.$$

Remark 4.1. If $T = [0, \infty)$ and P_t is a strongly continuous semigroup of contractions on $L^2(\Omega, \pi)$, then λ can be characterized using the infinitesimal generator A of $P_t = e^{tA}$. That is,

$$\lambda = \inf\big\{\langle -Af, f\rangle\colon\ f \in \mathrm{Dom}(A),\ \text{real-valued},\ \pi(f) = 0,\ \pi\big(f^2\big) = 1\big\}.$$

In general, λ is not in the spectrum of A but, assuming that $\mathrm{Dom}(A) = \mathrm{Dom}(A^*) = D$ then λ is in the spectrum of any self-adjoint extension of the symmetric operator $(\frac{1}{2}(A + A^*), D)$.

Remark 4.2. If $T = \mathbb{N}$, then $e^{-\lambda}$ is the second largest singular value of the operator P_1 on $L^2(\Omega, \pi)$, namely,

$$\lambda = -\log\big(\|P_1 - \pi\|_{L^2(\Omega,\pi) \to L^2(\Omega,\pi)}\big).$$

Consider a family of measurable spaces $(\Omega_n, \mathcal{B}_n)$ indexed by $n = 1, 2, \ldots$. For each, n, let $p_n(t, x, \cdot)$ with $t \in T$, $x \in \Omega_n$, be a Markov transition function with invariant measure π_n and spectral gap λ_n. Fix a sequence of probability measures μ_n on Ω_n and let $f_n(t) = D_{n,2}(\mu_n, t)$ be the L^2-distance defined in (4.2). Then, the family $\{p_n(t, \mu_n, \cdot)\colon\ n \geqslant 1\}$ is said to have an L^2-cutoff (resp. L^2-precutoff and (t_n, b_n)–L^2-cutoff) if $\{f_n\colon\ n \geqslant 1\}$ has a cutoff (resp. precutoff and (t_n, b_n)-cutoff) in the sense of Definition 2.1. The following proposition gives a sufficient condition for the L^2-cutoff.

Proposition 4.1. *(See [5, Theorem 3.3].) Referring to the setting and notation introduced above, assume that $D_2(\mu_n, t)$ vanishes as t tends to infinity and set $t_n = T_{n,2}(\mu_n, \epsilon) = \inf\{t \in T\colon D_{n,2}(\mu_n, t) \leqslant \epsilon\}$.*

- (i) *Assume that $T = [0, \infty)$ and $t_n\lambda_n \to \infty$. Then the family $\{p_n(t, \mu_n, \cdot)\colon n \geqslant 1\}$ has a (t_n, λ_n^{-1})-L^2-cutoff.*
- (ii) *Assume that $T = \mathbb{N}$ and $t_n\gamma_n \to \infty$ where $\gamma_n = \max\{\lambda_n, 1\}$. Then the family $\{p_n(t, \mu_n, \cdot)\colon n \geqslant 1\}$ has a (t_n, γ_n^{-1})-L^2-cutoff.*

We refer the reader to [4,5] for further results in this direction. The goal of the present work is to provide a necessary and sufficient condition for an L^2-cutoff and to describe the cutoff time using spectral theory.

4.3. The L^2-distance for normal Markov kernels

Let $T = [0, \infty)$ or $\mathbb{N}$. A Markov transition function $p(t, \cdot,\cdot)$, $t \in T$, with invariant probability measure π is called normal if, for $t \in T \cap [0, 1]$, the operator $P_t : L^2(\Omega, \pi) \to L^2(\Omega, \pi)$ defined by (4.1) is normal, that is, $P_t P_t^* = P_t^* P_t$ on $L^2(\Omega, \pi)$. In the case that P_t is a strongly continuous semigroup with infinitesimal generator A, the normality of $p(t, \cdot,\cdot)$ is equivalent to that of A. When P_t is normal,

$$\|P_t - \pi\|_{L^2(\Omega,\pi)\to L^2(\Omega,\pi)} = e^{-\lambda t}, \quad \forall t > 0.$$

Our next goal is to obtain a spectral formula for the chi-square distance. See Theorems 4.4–4.5 below.

Lemma 4.2. *Let $\{P_t\colon t > 0\}$ be a strongly continuous semigroup of contractions associated to a transition function $p(t, x, \cdot)$, $x \in \Omega$, $t \geqslant 0$, by (4.1). Let A be its infinitesimal generator. Assume that A is normal and let $\{E_B\colon B \in \mathcal{B}(\mathbb{C})\}$ be a resolution of the identity corresponding to $-A$, where $\mathcal{B}(\mathbb{C})$ is the Borel algebra over $\mathbb{C}$. Set*

$$C_0 = \{bi\colon b \in \mathbb{R}\}, \qquad C_1 = \{a + bi\colon a > 0,\ b \in \mathbb{R}\}.$$

Then, for $g \in L^2(\Omega, \pi)$,

$$\lim_{t\to\infty} \|P_t g\|_2 = \|E_{C_0} g\|_2.$$

In particular, if $\|P_t g - \pi(g)\|_2 \to 0$ as t tends to infinity, then $E_{C_0} g = \pi(g)$ and

$$\big\|P_t g - \pi(g)\big\|_2^2 = \int_{C_1} e^{-2\mathrm{Re}(\gamma)t}\, d\langle E_\gamma g, g\rangle_\pi.$$

Proof. Let $C = C_0 \cup C_1$. Since $(P_t)_{t>0}$ are contractions, the spectrum of $-A$ is contained in C. By the spectral theorem, for all $g \in L^2(\Omega, \pi)$,

$$\|P_t g\|_2^2 = \int_C e^{-2\,\mathrm{Re}(\gamma)t}\, d\langle E_\gamma g, g\rangle_\pi = \|E_{C_0} g\|_2^2 + \int_{C_1} e^{-2\,\mathrm{Re}(\gamma)t}\, d\langle E_\gamma g, g\rangle_\pi.$$

For a reference on the resolution of the identity for normal operators, see [18]. □

Lemma 4.3. *Let $\{P_t\colon t>0\}$ be as in Lemma* 4.2 *with infinitesimal generator A and spectral gap λ. Let $\sigma(-A)$ be the spectrum of $-A$ and $\widetilde{\lambda}=\inf\{\mathrm{Re}(c)\colon \mathrm{Re}(c)>0,\ c\in\sigma(-A)\}$, where* $\mathrm{Re}(c)$ *denotes the real part of c. Then,*

(i) $\lambda \leqslant \widetilde{\lambda}$.
(ii) *Assume that V is a dense subspace of $L^2(\Omega,\pi)$ and the following limit holds*

$$\lim_{t\to\infty}\left\|P_t g-\pi(g)\right\|_2=0,\quad \forall g\in V.$$

Then $\lambda=\widetilde{\lambda}$. In particular, if $\lambda>0$, then $\lambda=\widetilde{\lambda}$.

Remark 4.3. The converse of Lemma 4.3(i) is not always true and a typical example is to consider reducible finite Markov chains.

Proof of Lemma 4.3. For the spectrum of $-A$, since P_t is a contraction for $t>0$, $\sigma(-A)$ is a subset of the half plane $\{a+bi\colon a\geqslant 0,\ b\in\mathbb{R}\}$. In the case that P_t is normal, we may choose, by the spectral decomposition, a resolution of the identity $\{E_B\colon B\in\mathcal{B}(\mathbb{C})\}$ corresponding to $-A$ such that

$$\langle -Ag,g\rangle=\int_{\sigma(-A)}\gamma\,d\langle E_\gamma g,g\rangle_\pi,\quad \forall g\in\mathcal{D}(A), \tag{4.3}$$

where $\mathcal{B}(\mathbb{C})$ is the Borel algebra over $\mathbb{C}$ and $\mathcal{D}(A)$ is the domain of A. By Remark 4.1, λ can be obtained by the formula

$$\inf\bigl\{\langle -Ag,g\rangle_\pi\colon g\in\mathcal{D}(A),\ \pi(g)=0,\ \pi\bigl(g^2\bigr)=1\bigr\}.$$

For (i), note that if $\lambda=0$, then obviously $\lambda\leqslant\widetilde{\lambda}$. We now assume that $\lambda>0$. Fix $\delta\in(0,\lambda)$ and let $B_\delta=\{c\in\mathbb{C}\colon 0<\mathrm{Re}(c)<\delta\}$ and $T=E_{B_\delta}$. Using (4.3), one may easily compute that, for $g\in\mathcal{D}(A)$ with $\pi(g)=0$,

$$\bigl\langle T(-A-\delta)g,g\bigr\rangle_\pi=\bigl\langle(-A-\delta)(Tg),Tg\bigr\rangle_\pi\geqslant(\lambda-\delta)\|Tg\|_2^2\geqslant 0$$

and

$$\bigl\langle T(-A-\delta)g,g\bigr\rangle_\pi=\int_{B_\delta}(\gamma-\delta)\,d\langle E_\gamma g,g\rangle_\pi.$$

Since $\langle T(-A-\delta)g,g\rangle_\pi$ is real, the above identity can be rewritten as

$$\bigl\langle T(-A-\delta)g,g\bigr\rangle_\pi=\int_{B_\delta}\bigl[\mathrm{Re}(\gamma)-\delta\bigr]\,d\langle E_\gamma g,g\rangle_\pi\leqslant 0.$$

Combining the above three inequalities, we obtain that $E_{B_\delta}g=0$ for $g\in\mathcal{D}(A)$ satisfying $\pi(g)=0$. It is also clear that $E_{B_\delta}\mathbf{1}=0$ (since $0\notin B_\delta$ and $\mathbf{1}$ is contained in the range of $E_{\{0\}}$). Thus, using the fact $\overline{\mathcal{D}(A)}=L^2(\Omega,\pi)$, we have $E_{B_\delta}=0$ on $L^2(\Omega,\pi)$. Finally, because $\sigma(-A)$

is exactly the essential range of the function $\Psi(t) = t$ w.r.t. $\{E_B\colon B \in \mathcal{B}(\mathbb{C})\}$, B_δ and $\sigma(-A)$ are mutually disjoint, which implies $\delta \leqslant \widetilde{\lambda}$ for $\delta \in (0, \lambda)$. This proves the first part.

For (ii), it remains to show that $\widetilde{\lambda} \leqslant \lambda$. Obviously, this inequality holds for $\widetilde{\lambda} = 0$. For the case $\widetilde{\lambda} > 0$, we set $C_1 = \{a + bi\colon a \geqslant \widetilde{\lambda},\ b \in \mathbb{R}\}$. By Lemma 4.2,

$$\forall g \in V, \quad \left\| P_t g - \pi(g) \right\|_2^2 = \int_{C_1} e^{-2\operatorname{Re}(\gamma)t}\, d\langle E_\gamma g, g\rangle_\pi \leqslant e^{-2\widetilde{\lambda}t} \|g\|_2^2.$$

Since V is dense in $L^2(\Omega, \pi)$, the above holds true on $L^2(\Omega, \pi)$. Thus, $\widetilde{\lambda} \leqslant \lambda$. $\square$

We are now ready to compute the L^2-distance, $D(\mu, t)$, using the spectral information of the infinitesimal generator A of P_t.

Theorem 4.4. *Let $\{P_t\colon t > 0\}$ be as in Lemma* 4.2 *with infinitesimal generator A and spectral gap $\lambda > 0$. Assume that A is normal and $\{E_B\colon B \in \mathcal{B}(\mathbb{C})\}$ is a resolution of identity for $-A$. If μ is a probability measure with an L^2-density f w.r.t. π, then*

$$D_2(\mu, t)^2 = \int_{C(\lambda)} e^{-2\operatorname{Re}(\gamma)t}\, d\langle E_\gamma f, f\rangle_\pi,$$

where $C(\lambda) = \{c \in \mathbb{C}\colon \operatorname{Re}(c) \geqslant \lambda\}$.

Proof. Let $d(\mu P_t) = f_t\, d\pi$. Then for $g \in L^2(\Omega, \pi)$,

$$\langle g, f_t\rangle_\pi = (\mu P_t)(g) = \mu(P_t g) = \langle P_t g, f\rangle_\pi = \left\langle g, P_t^* f\right\rangle_\pi,$$

where P_t^* denotes the adjoint operator of P_t. This implies that $f_t = P_t^* f$. Since P_t is normal, it is obvious that $\|P_t g\|_2 = \|P_t^* g\|_2$ for all $g \in L^2(\Omega, \pi)$ and, hence,

$$D_2(\mu, t)^2 = \left\| P_t^*(f - \mathbf{1}) \right\|_2^2 = \left\| P_t f - \pi(f) \right\|_2^2 = \int_{C(\lambda)} e^{-2\operatorname{Re}(\gamma)t}\, d\langle E_\gamma f, f\rangle_\pi$$

where the last equality uses Lemmas 4.2 and 4.3. $\square$

The discrete time version for Theorem 4.4 is as follows. The proof is similar.

Theorem 4.5. *Let $\{P_t\colon t \in \mathbb{N}\}$ be the family of contractions on $L^2(\Omega, \pi)$ defined in* (4.1) *with spectral gap $\lambda > 0$. Assume that P_1 is a normal operator whose corresponding resolution of the identity is $\{E_B\colon B \in \mathcal{B}(\mathbb{C})\}$. If μ is a probability measure with an L^2-density f w.r.t. π, then for $t \in \mathbb{N}$,*

$$D_2(\mu, t)^2 = \int_{\widetilde{C}(\lambda)} |\gamma|^{2t}\, d\langle E_\gamma f, f\rangle_\pi,$$

where $\widetilde{C}(\lambda) = \{c \in \mathbb{C}\colon |c| \leqslant \lambda\}$.

4.4. The L^2-cutoff time for families of normal Markov kernels

Now, we shall assume that the initial probability has an L^2-density (with respect to the invariant probability). In this case, Theorems 4.4 and 4.5 imply that the L^2-distance between a normal Markov transition function and its stationary measure is a Laplace transform. Thus, the results in Section 3 are applicable. In detail, let $(\Omega_n, \mathcal{B}_n)$ be a measurable space and $p_n(t, x, \cdot)$, $t \in T$ ($T = [0, \infty)$ or $\mathbb{N}$) and $x \in \Omega_n$, be a Markov transition function on Ω_n with invariant probability measure π_n. Let $P_{n,t}$ be the operator defined by

$$P_{n,t} g(x) = \int_{\Omega_n} g(y) p_n(t, x, dy), \quad \forall g \in L^2(\Omega_n, \pi_n),\ t \in T, \tag{4.4}$$

and μ_n be a probability on $(\Omega_n, \mathcal{B}_n)$ with L^2-density f_n w.r.t. π_n.

- For $T = [0, \infty)$, assume that $P_{n,t}$ is normal, strongly continuous, with positive spectral gap λ_n, and that the infinitesimal generator of $P_{n,t}$ has resolution of the identity $\{E_{n,B}\colon B \in \mathcal{B}(\mathbb{C})\}$, where $\mathcal{B}(\mathbb{C})$ is the Borel algebra over $\mathbb{C}$. For $\lambda > 0$, let S_λ be the strip $\{c \in \mathbb{C}\colon \mathrm{Re}(c) \in (0, \lambda]\}$ and set

$$V_n(\lambda) = \langle E_{n,S_\lambda} f_n, f_n \rangle_{\pi_n}. \tag{4.5}$$

- For $T = \mathbb{N}$, assume that $P_{n,1}$ is normal with positive spectral gap λ_n and resolution of the identity $\{E_{n,B}\colon B \in \mathcal{B}(\mathbb{C})\}$. For $\lambda > 0$, let A_λ be the annulus $\{c \in \mathbb{C}\colon |c| \in [e^{-\lambda}, 1)\}$ and set

$$V_n(\lambda) = \langle E_{n,A_\lambda} f_n, f_n \rangle_{\pi_n}. \tag{4.6}$$

As a consequence of Theorems 4.4 and 4.5, the L^2-distance is given by

$$D_{n,2}(\mu_n, t)^2 = \int_{[\lambda_n, \infty)} e^{-2\lambda t}\, dV_n(\lambda).$$

To state the main results of this paper, for $\delta > 0$ and $C > 0$, set

$$\begin{cases} t_n(\delta) = T_{n,2}(\mu_n, \delta) = \inf\{t \in T\colon\ D_{n,2}(\mu_n, t) \leqslant \delta\}, \\ \lambda_n(C) = \inf\{\lambda\colon\ V_n([\lambda_n, \lambda]) > C\}, \\ \tau_n(C) = \sup\left\{\dfrac{\log(1 + V_n([\lambda_n, \lambda]))}{2\lambda}\colon\ \lambda \geqslant \lambda_n(C)\right\}. \end{cases} \tag{4.7}$$

Further, set

$$\begin{cases} \gamma_n = \lambda_n(C)^{-1}, \quad b_n = \lambda_n(C)^{-1} \log(\lambda_n(C)\tau_n(C)) \quad \text{if } T = [0, \infty), \\ \gamma_n = \max\{1, \lambda_n(C)^{-1}\}, \quad b_n = \max\{1, \lambda_n(C)^{-1} \log(\lambda_n(C)\tau_n(C))\} \quad \text{if } T = \mathbb{N}. \end{cases}$$

If $T = \mathbb{N}$, we assume in addition that either $\tau_n(C)$ or $t_n(\delta)$ tends to infinity, for some C or some δ.

Theorem 4.6. *Referring to the setup and notation described in* (4.4)–(4.7)*,*

(i) *If* $\liminf_{n\to\infty} \pi_n(f_n^2) < \infty$*, then* $\{p_n(t,\mu_n,\cdot)\colon n \geqslant 1\}$ *has no* L^2*-precutoff.*
(ii) *If* $\pi_n(f_n^2) \to \infty$*, then the following are equivalent.*
  (a) $\{p_n(t,\mu_n,\cdot)\colon t \in [0,\infty)\}$ *has an* L^2*-cutoff.*
  (b) *For some positive constants* C, δ, ϵ*,*

$$\lim_{n\to\infty} t_n(\delta)\lambda_n(C) = \infty, \qquad \lim_{n\to\infty} \int\limits_{[\lambda_n,\lambda_n(C))} e^{-\epsilon\lambda t_n(\delta)}\, dV_n(\lambda) = 0.$$

  (c) *For some positive constants* C, ϵ*,*

$$\lim_{n\to\infty} \tau_n(C)\lambda_n(C) = \infty, \qquad \lim_{n\to\infty} \int\limits_{[\lambda_n,\lambda_n(C))} e^{-\epsilon\lambda\tau_n(C)}\, dV_n(\lambda) = 0.$$

Further,
– *If* (b) *holds, then* $\{p_n(t,\mu_n,\cdot)\colon n \geqslant 1\}$ *has a* $(t_n(\delta),\gamma_n)$*-*L^2*-cutoff.*
– *If* (c) *holds, then* $\{p_n(t,\mu_n,\cdot)\colon n \geqslant 1\}$ *has a* $(\tau_n(C),b_n)$*–*L^2*-cutoff.*

Proof. Immediate from Theorems 3.5 and 3.8 □

Remark 4.4. By Remark 3.4, if the family $\{p_n(t,\mu_n,\cdot)\colon t \in [0,\infty)\}$ has an L^2-cutoff, then Theorem 4.6 (b) and (c) hold for any positive C, δ, ϵ. Similarly, by Remark 3.6, if the family $\{p_n(t,\mu_n,\cdot)\colon t \in \mathbb{N}\}$ has an L^2-cutoff with the L^2-mixing time $T_2(\mu_n,\delta)$ tending to infinity, then Theorem 4.6 (b) and (c) are true for any positive C, δ, ϵ.

5. Applications to finite Markov chains

In this section, we spell out how our main results apply to normal Markov chains on finite state spaces. Let Ω be a finite set and K be an irreducible Markov kernel on Ω with invariant probability measure π. Denote by $p^d(t,\cdot,\cdot)$ the associated discrete time Markov transition function, that is, $p^d(t,x,y) = K^t(x,y)$, $t \in \mathbb{N}$. Let $p^c(t,\cdot,\cdot)$ be the associated continuous time Markov transition function defined by

$$p^c(t,x,y) = e^{-t(I-K)}(x,y) = e^{-t}\sum_{n=0}^{\infty} \frac{t^n}{n!} K^n(x,y), \quad t \geqslant 0. \tag{5.1}$$

To facilitate applications, we discuss continuous and discrete time separately.

For $n \geqslant 1$, let Ω_n be a finite set and K_n be an irreducible Markov kernel on Ω_n with invariant probability π_n. Let μ_n be some given initial distribution with density f_n with respect to π_n. We assume that K_n is normal. Its eigenvalues and eigenfunctions will be ordered in different ways in the discrete and continuous time cases. We let $p_n^c(t,x,y)$, $p_n^d(t,x,y)$ be the corresponding continuous and discrete time transition functions.

5.1. Continuous time

Let $\beta_{n,0} = 1, \beta_{n,1}, \dots, \beta_{n,|\Omega_n|-1}$ be the eigenvalues of K_n with orthonormal eigenvectors $\psi_{n,0} \equiv 1, \psi_{n,1}, \dots, \psi_{n,|\Omega_n|-1}$ on $L^2(\Omega_n, \pi_n)$, ordered in such a way that

$$\operatorname{Re}\beta_{n,i} \geqslant \operatorname{Re}\beta_{n,i+1}, \quad \forall 1 \leqslant i \leqslant |\Omega_n| - 2.$$

Let $\lambda_{n,i} = 1 - \operatorname{Re}\beta_{n,i}$ and set, for $C > 0$,

$$j_n = j_n(C) = \min\left\{ j \geqslant 1\colon \sum_{i=1}^{j} \left|\mu_n(\psi_{n,i})\right|^2 > C \right\} \tag{5.2}$$

and

$$\tau_n = \tau_n(C) = \max_{j \geqslant j_n} \left\{ \frac{\log\left(\sum_{i=0}^{j} |\mu_n(\psi_{n,i})|^2\right)}{2\lambda_{n,j}} \right\}. \tag{5.3}$$

Note that

$$D^c_{n,2}(\mu_n, t) = \left(\sum_{i \geqslant 1} \left|\mu(\psi_{n,i})\right|^2 e^{-2\lambda_{n,i} t} \right)^{1/2} \tag{5.4}$$

and set

$$t_n = t_n(\delta) = T^c_{n,2}(\mu_n, \delta) = \inf\left\{ t \geqslant 0\colon D^c_{n,2}(\mu_n, t) \leqslant \delta \right\}. \tag{5.5}$$

Theorem 4.6 yields the following result.

Theorem 5.1. *Referring to the above setting and notation,*

(i) *If* $\liminf_{n\to\infty} \pi_n(f_n^2) < \infty$, *then* $\{p_n^c(t, \mu_n, \cdot)\colon n \geqslant 1\}$ *has no* L^2*-precutoff.*
(ii) *If* $\pi_n(f_n^2) \to \infty$, *then the following are equivalent.*
  (a) $\{p_n^c(t, \mu_n, \cdot)\colon n \geqslant 1\}$ *has an* L^2*-cutoff.*
  (b) *For some positive constants* C, ϵ, δ,

$$\lim_{n\to\infty} t_n \lambda_{n,j_n} = \infty, \qquad \lim_{n\to\infty} \sum_{i=1}^{j_n - 1} \left|\mu_n(\psi_{n,i})\right|^2 e^{-\epsilon t_n \lambda_{n,i}} = 0.$$

  (c) *For some positive constants* C, ϵ,

$$\lim_{n\to\infty} \tau_n \lambda_{n,j_n} = \infty, \qquad \lim_{n\to\infty} \sum_{i=1}^{j_n - 1} \left|\mu_n(\psi_{n,i})\right|^2 e^{-\epsilon \tau_n \lambda_{n,i}} = 0.$$

Furthermore, in case (ii), *if* (b)/(c) *holds, then* $\{p_n^c(t, \mu_n, \cdot)\colon n \geqslant 1\}$ *has a* $(t_n, \lambda_{n,j_n}^{-1})$*–*$L^2$*-cutoff and a* (τ_n, b_n)*–*L^2*-cutoff with* $b_n = \lambda_{n,j_n}^{-1} \log(\tau_n \lambda_{n,j_n})$.

Remark 5.1. By Remark 3.8, $\tau_n(C) \leqslant T_2^c(\mu_n, \frac{C}{C+1})$ for $C > 0$.

Remark 5.2. Theorem 5.1 is useful in proving an L^2-cutoff if there is indeed one but, as stated, in order to disprove the existence of an L^2-cutoff, one has to show that Theorem 5.1 (b) and (c) fail for all C, δ, ϵ. In fact, as stated in Remark 4.4, a stronger version of Theorem 5.1 says that if p_n^c has an L^2-cutoff, then (b) holds for any triple (C, ϵ, δ) and (c) holds for any pair (C, ϵ). Hence, to disprove the existence of an L^2-cutoff, we only need to check that (b) or (c) fails for some constants C, ϵ, δ.

The following is a simple application of Theorem 5.1 which deals with the L^2-cutoff for a specific class of chains, those whose spectral gap is bounded away from 0 as in the case of expander graphs. The notation is as above.

Corollary 5.2. *Assume that K_n is normal and λ_n^{-1} is bounded. Then the family $\{p_n^c(t, \mu_n, \cdot)\colon n \geqslant 1\}$ has an L^2-cutoff if and only if $\pi_n(f_n^2) \to \infty$.*

Furthermore, if $\pi_n(f_n^2) \to \infty$, then the family $\{p_n^c(t, \mu_n, \cdot)\colon n \geqslant 1\}$ presents a strongly optimal $(t_n, 1)$-L^2-cutoff, where t_n is the constant in (5.5) and δ is any positive constant.

Proof. The first part of this corollary is obvious from Theorem 5.1 whereas the second part follows from

$$\delta e^{-2c} \leqslant D_{n,2}^c(\mu_n, t_n + c) \leqslant \delta e^{-c\lambda_n}, \quad \forall -t_n < c < 0$$

and

$$\delta e^{-c\lambda_n} \leqslant D_{n,2}^c\big(\mu_n, t_n + c\lambda_n^{-1}\big) \leqslant \delta e^{-2c}, \quad \forall c > 0. \qquad \square$$

5.2. Discrete time

To treat the discrete time case, order the eigenvalues $\beta_{n,0} = 1, \beta_{n,1}, \ldots, \beta_{n,|\Omega_n|-1}$ and orthonormal eigenvectors $\psi_{n,0} \equiv 1, \psi_{n,1}, \ldots, \psi_{n,|\Omega_n|-1}$ in such a way that

$$|\beta_{n,i}| \geqslant |\beta_{n,i+1}|, \quad \forall 1 \leqslant i \leqslant |\Omega_n| - 2.$$

Set $\lambda_{n,i} = -\log|\beta_{n,i}|$ and define $j_n = j_n(C)$ and $\tau_n = \tau_n(C)$ by (5.2). The L^2-distance takes the form

$$D_{n,2}^d(\mu_n, t) = \bigg(\sum_{i \geqslant 1} \big|\mu(\psi_{n,i})\big|^2 |\beta_{n,i}|^{2t}\bigg)^{1/2}. \tag{5.6}$$

For $\delta > 0$, set

$$t_n = t_n(\delta) = T_{n,2}^d(\mu_n, \delta) = \inf\big\{t \geqslant 0\colon D_{n,2}^d(\mu_n, t) \leqslant \delta\big\}. \tag{5.7}$$

Theorem 5.3. *Referring to the above setting and notation and assuming that either $t_n(\delta) \to \infty$ for some $\delta > 0$ or $\tau_n(C) \to \infty$ for some $C > 0$, we have:*

(i) *If* $\liminf_{n\to\infty}\pi_n(f_n^2)<\infty$, *then* $\{p_n^d(t,\mu_n,\cdot)\colon n\geqslant 1\}$ *has no* L^2*-precutoff.*
(ii) *If* $\pi_n(f_n^2)\to\infty$, *then the following are equivalent.*
 (a) $\{p_n^d(t,\mu_n,\cdot)\colon n\geqslant 1\}$ *has an* L^2*-cutoff.*
 (b) *For some positive constants* C,ϵ,δ,

$$\lim_{n\to\infty} t_n\lambda_{n,j_n}=\infty,\qquad \lim_{n\to\infty}\sum_{i=1}^{j_n-1}\left|\mu_n(\psi_{n,i})\right|^2|\beta_{n,i}|^{\epsilon t_n}=0.$$

 (c) *For some positive constants* C,ϵ,

$$\lim_{n\to\infty} \tau_n\lambda_{n,j_n}=\infty,\qquad \lim_{n\to\infty}\sum_{i=1}^{j_n-1}\left|\mu_n(\psi_{n,i})\right|^2|\beta_{n,i}|^{\epsilon \tau_n}=0.$$

In case (ii)*, if* (b)/(c) *holds, then* $\{p_n^d(t,\mu_n,\cdot)\colon n\geqslant 1\}$ *has a* (t_n,γ_n^{-1})*–*L^2*-cutoff with* $\gamma_n=\min\{\lambda_{n,j_n},1\}$ *and a* (τ_n,b_n)*-*L^2*-cutoff with* $b_n=\max\{\lambda_{n,j_n}^{-1}\log(\tau_n\lambda_{n,j_n}),1\}$.

Remark 5.3. Remarks 5.1 and 5.2 remain true in discrete time cases. One also easily obtains a discrete version of Corollary 5.2 under the assumption that the eigenvalues $\beta_{n,i}$, $0\leqslant i\leqslant|\Omega_n|-1$, $\beta_{n,0}=1$ of the normal operator K_n satisfy $\inf\{1-|\beta_{n,i}|,|\beta_{n,i}|\colon 1\leqslant i\leqslant|\Omega_n|-1,\ n\geqslant 1\}>0$.

5.3. Invariant kernels

Next, we specialize Theorems 5.1–5.3 to the case when the kernels K_n are invariant under some transitive group action, i.e., for each n, there is a group G_n acting transitively on Ω_n and such that

$$K_n(gx,gy)=K_n(x,y)\quad \forall g\in G_n,\ x,y\in\Omega_n. \tag{5.8}$$

If $|\Omega_n|\nrightarrow\infty$, then the families $\{p_n^c(t,x_n,\cdot)\colon n\geqslant 1\}$ and $p_n^d(t,x_n,\cdot)$ have no L^2-precutoff so we assume that $|\Omega_n|\to\infty$. The notable contribution of these results is in the explicit spectral description of the cutoff time τ_n.

Theorem 5.4 *(Continuous time). Assume that* $|\Omega_n|\to\infty$ *and that* K_n *satisfies* (5.8) *and is irreducible normal with eigenvalues* $\beta_{n,0}=1,\beta_{n,1},\dots,\beta_{n,|\Omega_n|-1}$, $\operatorname{Re}\beta_{n,i}\geqslant\operatorname{Re}\beta_{n,i+1}$, $\forall 1\leqslant i\leqslant |\Omega_n|-2$. *Let* $\lambda_{n,i}=1-\operatorname{Re}\beta_{n,i}$, $\lambda_n=\lambda_{n,1}$ *and set*

$$\tau_n=\sup_{j\geqslant 1}\left\{\frac{\log(j+1)}{2\lambda_{n,j}}\right\},\qquad t_n=T_{n,2}^c(x_n,\delta). \tag{5.9}$$

The following properties are equivalent.

(a) $\{p_n^c(t,x_n,\cdot)\colon n\geqslant 1\}$ *has a* L^2*-cutoff.*
(b) $t_n\lambda_n\to\infty$ *for some* $\delta>0$.
(c) $\tau_n\lambda_n\to\infty$.

Furthermore, if (b)/(c) *holds, then* $\{p_n^c(t, x_n, \cdot)\colon n \geqslant 1\}$ *has a* (t_n, λ_n^{-1})*-*L^2*-cutoff and a* (τ_n, b_n)*-*L^2*-cutoff with* $b_n = \lambda_n^{-1}\log(\tau_n\lambda_n)$.

Theorem 5.5 *(Discrete time). Assume that* $|\Omega_n| \to \infty$ *and that* K_n *satisfies* (5.8) *and is irreducible normal with eigenvalues* $|\beta_{n,i}| \geqslant |\beta_{n,i+1}|$, $0 \leqslant i \leqslant |\Omega_n| - 2$. *Set* $\lambda_{n,i} = -\log|\beta_{n,i}|$, $\lambda_n = \lambda_{n,1}$, *and let* τ_n *be defined in terms of these* $\lambda_{n,i}$ *as in* (5.9). *Set also* $t_n = T_{n,2}^d(x_n, \delta)$. *Assume that either* $t_n \to \infty$ *for some* $\delta > 0$ *or* $\tau_n \to \infty$. *Then the following are equivalent.*

(a) $\{p_n^d(t, x_n, \cdot)\colon n \geqslant 1\}$ *has an* L^2*-cutoff.*
(b) $t_n\lambda_n \to \infty$ *for some* $\delta > 0$.
(c) $\tau_n\lambda_n \to \infty$.

Furthermore, if (b)/(c) *holds, then* $\{p_n^d(t, x_n, \cdot)\colon n \geqslant 1\}$ *has a* (t_n, γ_n^{-1})*-*L^2*-cutoff with* $\gamma_n = \min\{\lambda_n, 1\}$ *and a* (τ_n, b_n)*-*L^2*-cutoff with* $b_n = \max\{\lambda_n^{-1}\log(\tau_n\lambda_n), 1\}$.

6. The Ehrenfest chain

The Ehrenfest chain is one of the most celebrated example of finite Markov chain. Its state space is $\Omega_n = \{0, \dots, n\}$ and its kernel is given by

$$K_n(i, i+1) = 1 - \frac{i}{n}, \qquad K_n(i+1, i) = \frac{i+1}{n}, \quad \forall 0 \leqslant i < n. \tag{6.1}$$

It is clear that K_n is irreducible with stationary distribution $\pi_n(i) = \binom{n}{i}2^{-n}$, $i \in \{0, 1, \dots, n\}$. Note that K_n is periodic. The L^2-distance of the Ehrenfest chain to its stationary measure has been studied by many authors. By lifting the chain to a walk on the hypercube, the representation theory of $(\mathbb{Z}_2)^n$ can be used to identify the eigenvalues and eigenvectors of the Ehrenfest chain and to compute the L^2-distance. The following well-known result gives a description on the eigenvalues and eigenvectors of K_n.

Theorem 6.1. *The matrix* K_n *defined in* (6.1) *has eigenvalues*

$$\beta_{n,i} = 1 - \frac{2i}{n}, \quad 0 \leqslant i \leqslant n,$$

with $L^2(\pi_n)$*-normalized right eigenvectors*

$$\psi_{n,i}(x) = \binom{n}{i}^{-1/2} \sum_{k=0}^{i} (-1)^k \binom{x}{k}\binom{n-x}{i-k}, \quad 0 \leqslant i, x \leqslant n. \tag{6.2}$$

Proof. See, e.g., [7]. The vectors $\psi_{n,i}$ are in fact the Krawtchouk polynomials (up to a constant multiple) and the desired properties are the orthogonality and recurrence relation of Krawtchouk polynomials. See [14,16]. □

To apply our main result using the above spectral information, we need to study $\psi_{n,i}$. Recall the classical notation

$$ {}_rF_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_r)_k}{(b_1)_k\cdots(b_s)_k}\frac{z^k}{k!} $$

where, for $a \in \mathbb{R}$ and $n \geqslant 0$, $(a)_n$ is the Pochhammer symbol defined by

$$ (a)_0 = 1, \qquad (a)_n = a(a+1)\cdots(a+n-1), \quad \forall n \geqslant 1. $$

Using this notation, Krawtchouk polynomials are defined by

$$ P_i(x,p,n) = {}_2F_1\left(\begin{matrix} -i,-x \\ -n \end{matrix}\,\middle|\, \frac{1}{p}\right) $$

for $i = 0,1,\dots,n$. Then, the eigenvector $\psi_{n,i}$ of K_n can be rewritten as

$$ \psi_{n,i}(x) = \binom{n}{i}^{1/2} P_i(x,1/2,n). \tag{6.3} $$

The recurrence relation for $P_i(j,1/2,n)$ is

$$ (n-2x)P_i(x,1/2,n) = (n-i)P_{i+1}(x,1/2,n) + iP_{i-1}(x,1/2,n). \tag{6.4} $$

Note that this is exactly saying that $\beta_{n,i}$ and $\psi_{n,i}$ are eigenvalues and eigenvectors for K_n. Using the above identity, we are able to apply the results from Section 5 to the Ehrenfest chain.

6.1. *The continuous time Ehrenfest process*

The transition function of the continuous time Ehrenfest process is given by $p_n^c(t,\cdot,\cdot) = e^{-t(I-K_n)}$.

Theorem 6.2. *Given starting states x_n, the family $\mathcal{F}_c$ of the continuous time Ehrenfest chains $\{p_n^c(t,x_n,\cdot),\ n=1,2,\dots\}$ has an L^2-cutoff if and only if*

$$ \lim_{n\to\infty} \frac{|n-2x_n|}{\sqrt{n}} = \infty. \tag{6.5} $$

Our second result concerns the L^2-cutoff time and the optimality of window sequences.

Theorem 6.3. *Referring to the Ehrenfest family $\mathcal{F}_c$, Assume that* (6.5) *holds and let*

$$ t_n = \frac{n}{2}\log\frac{|n-2x_n|}{\sqrt{n}}. $$

Then, there exist universal positive constants A, N such that for all $n \geqslant N$

$$ e^{-2c} \leqslant D_{n,2}(x_n, t_n + cn) \leqslant Ae^{-2c}, $$

where the first inequality holds true for all real c with $t_n + cn \geqslant 0$ and the second inequality is true for $c > 0$.

Remark 6.1. By Proposition 3.7, this result says that there is an optimal (t_n, n)–L^2-cutoff. In fact, the (t_n, n)–L^2-cutoff is strongly optimal.

Using the relation between the first and the second eigenvectors of K_n, we also obtain a result concerning the total variation cutoff or equivalently, the L^1-cutoff. The details of the proof are omitted.

Theorem 6.4. *Referring to the Ehrenfest family $\mathcal{F}_c$, let $D_{n,\mathrm{TV}}(x_n, t)$ be the total variation distance between the distribution of the nth chain at time t starting from x_n and π_n. Then, for all $n \geqslant 1$ and all c such that $t_n + cn \geqslant 0$, we have*

$$D_{n,\mathrm{TV}}(x_n, t_n + cn) \geqslant 1 - 8e^{4c}.$$

Remark 6.2. By Theorems 6.3–6.4, if (6.5) holds, then there is a (t_n, n)-total variation cutoff.

Before proving these results, we make some analysis on the eigenvectors $\psi_{n,i}$. Let $x_n = \frac{1}{2}(n + y_n)$ with $|y_n| \leqslant n$. Using (6.3), the recurrence relation of Krawtchouk polynomials in (6.4) yields the following identity

$$\begin{aligned} a_{n,i+1} &= \frac{-y_n}{\sqrt{(i+1)(n-i)}} a_{n,i} - \sqrt{\frac{i(n-i+1)}{(i+1)(n-i)}} a_{n,i-1} \\ &= -A_{n,i} b_n a_{n,i} - B_{n,i} a_{n,i-1} \end{aligned} \tag{6.6}$$

where $a_{n,i} = \psi_{n,i}(x_n)$, $b_n = y_n/\sqrt{n}$ and

$$A_{n,i} = \sqrt{\frac{n}{(i+1)(n-i)}}, \qquad B_{n,i} = \sqrt{\frac{i(n-i+1)}{(i+1)(n-i)}}. \tag{6.7}$$

To compute $a_{n,i}$ using the above iterative formula, one needs the boundary conditions $a_{n,1}$ and $a_{n,2}$, which can be easily determined using the formula given in Theorem 6.1. They are

$$a_{n,1} = -b_n, \qquad a_{n,2} = \sqrt{\frac{n}{2(n-1)}} \big(b_n^2 - 1\big). \tag{6.8}$$

Concerning the L^2-distance, the symmetry of the chain implies that there is no loss of generality in assuming $x_n \geqslant n/2$, that is, $y_n \geqslant 0$. (Otherwise, one only needs to replace x_n with $n - x_n$ without any change on the L^p-distance.) From now on, we assume that $x_n \geqslant n/2$. Before starting the proofs, let $\lambda_{n,i}$, $j_n(C)$ and $\tau_n(C)$ be as in Theorem 5.1. It can be easily seen from Theorem 6.1 that $\lambda_{n,i} = 2i/n$. Also, note that (6.5) is equivalent to $b_n \to \infty$.

Proof of Theorem 6.2. We first prove that (6.5) implies an L^2-cutoff. Since $|\psi_{n,1}(x_n)| = b_n \to \infty$, we have $j_n(1) = 1$ for n large enough. This implies that

$$\tau_n(1) \geqslant \frac{\log |a_{n,1}|}{\lambda_{n,1}}$$

and, hence, $\lambda_{n,1}\tau_n \geqslant \log |a_{n,1}| \to \infty$ as $n \to \infty$. By Theorem 5.1, $\mathcal{F}_c$ has an L^2-cutoff.

Suppose now that $b_n \nrightarrow \infty$. By Proposition 2.1, we can assume that b_n is bounded from above, say by B. Observe that, by (6.8),

$$a_{n,1}^2 + a_{n,2}^2 = b_n^2 + \frac{n}{2(n-1)}\left(b_n^2 - 1\right)^2 \geqslant 1/2.$$

This implies that $j_n(1/2) \leqslant 2$ for all n. Also, it is obvious from (6.7) that

$$A_{n,i} \leqslant 1, \qquad B_{n,i} \leqslant 1, \quad \forall 0 \leqslant i < n.$$

By setting $\gamma = \sup_n b_n \vee 1$ using these inequalities and (6.6), one derives that

$$|a_{n,i+1}| \leqslant \gamma |a_{n,i}| + |a_{n,i-1}|, \quad \forall 1 \leqslant i < n,\ n \geqslant 1.$$

Then, an inductive argument along with the fact $|a_{n,1}| \leqslant B$ and $|a_{n,2}| \leqslant B^2$ for $n > 1$ implies that

$$|a_{n,i}| \leqslant B^2(\gamma + 1)^i, \quad \forall 1 \leqslant i \leqslant n,$$

which gives

$$\sum_{i=0}^{j} |a_{n,i}|^2 \leqslant B^4 \sum_{i=0}^{j} (\gamma + 1)^{2i} \leqslant B^4 (\gamma + 1)^{2j+1}, \quad \forall j \leqslant n.$$

Hence,

$$\tau_n(1/2)\lambda_{n,j_n(1/2)} \leqslant \log B^4 + \log(\gamma + 1) \sup_{1 \leqslant j \leqslant n} \frac{2j+1}{j} < \infty.$$

By Theorem 5.1, this shows there is no L^2-cutoff. □

Proof of Theorem 6.3. Assume that $b_n \to \infty$ and set $f_n(c) = D_{n,2}(x_n, t_n + cb_n)$ where t_n is the sequence defined in Theorem 6.3, that is, $t_n = \frac{1}{2} n \log b_n$. To prove the desired result, we need to investigate $a_{n,i}$ or instead $A_{n,i}$ and $B_{n,i}$. The following claim is the only fact we need.

Claim: There exists $N > 0$ such that

$$\frac{i+1}{i} A_{n,i} + \frac{i+1}{i-1} B_{n,i} b_n^{-2} \leqslant 1, \quad \forall 2 \leqslant i \leqslant n - 3,\ n \geqslant N.$$

To prove this claim, set $\theta \in (1, 4/3)$. Then, for $2 \leqslant i \leqslant (1-1/\theta)n$,

$$\frac{i+1}{i}A_{n,i} = \frac{\sqrt{\theta(i+1)}}{i} \times \sqrt{\frac{n}{\theta(n-i)}} \leqslant \frac{\sqrt{3\theta}}{2} < 1$$

and, for $(1-1/\theta)n < i \leqslant n/2$,

$$\frac{i+1}{i}A_{n,i} \leqslant \left(1 + \frac{\theta}{(\theta+1)n}\right)\frac{\theta^2}{(\theta+1)n} \leqslant \frac{2}{n}.$$

Summarizing, we get

$$\sup_{2\leqslant i\leqslant n/2,\, n\geqslant 3} A_{n,i} = \alpha < 1.$$

When $i > n/2$, since $(i+1)/i$ is decreasing in i and $A_{n,i} = A_{n,n-i-1}$, the same bound holds. The claim is then proved by choosing N large enough so that

$$\frac{i+1}{i-1}B_{n,i}b_n^{-2} \leqslant 3b_n^{-2} < 1-\alpha, \quad \forall n \geqslant N.$$

Returning to the proof of Theorem 6.3, we apply the triangle inequality in (6.6) to get

$$|a_{n,i+1}| \leqslant A_{n,i}b_n a_{n,i} + B_{n,i}a_{n,i-1}. \tag{6.9}$$

Let N be the integer chosen in the claim above. Then, this iterative inequality and induction yields

$$|a_{n,i}| \leqslant \frac{2}{i}b_n^i, \quad 1 \leqslant i \leqslant n-2,\ n \geqslant N.$$

Using (6.9) for $i = n-2$ and $n-1$ implies that

$$|a_{n,i}| \leqslant \frac{\beta}{i}b_n^i, \quad 1 \leqslant i \leqslant n,\ n \geqslant N,$$

for some β bigger than 2. To finish the proof, recall that

$$D_{n,2}(x,t)^2 = \sum_{i=1}^{n} \left|\psi_{n,i}(x_n)\right|^2 e^{-2t\lambda_{n,i}}.$$

Hence, for $c > 0$ and $n \geqslant N$,

$$D_{n,2}(x_n, t_n + cn)^2 \leqslant \left(\beta^2 \sum_{i=1}^{\infty} \frac{1}{i^2}\right) e^{-4c},$$

and for $c \in \mathbb{R}$ and $n \geqslant 1$,

$$D_{n,2}(x_n, t_n + cn)^2 \geqslant \left|\psi_{n,1}(x_n)\right|^2 e^{-2(t_n+cn)\lambda_{n,1}} = e^{-4c}. \qquad \square$$

Proof of Remark 6.1. (The cutoff is strongly optimal.) The proof is similar to that of Theorem 6.3. Fix $\widetilde{c} < 0$ and choose $N = N(\widetilde{c}) > 0$ such that

$$e^{-2\widetilde{c}}\sqrt{\frac{2}{N+1}} + e^{-4\widetilde{c}} b_n^{-2} \leqslant 1 \quad \forall n \geqslant N.$$

Note that, for $N \leqslant i \leqslant n - N - 1$ and $n \geqslant 2N$,

$$\frac{n}{(i+1)(n-i)} \leqslant \frac{n}{(N+1)(n-N)} \leqslant \frac{2}{N+1}.$$

Combining the above two inequalities then gives

$$e^{-2\widetilde{c}} A_{n,i} + e^{-4\widetilde{c}} B_{n,i} b_n^{-2} \leqslant 1 \quad \forall N \leqslant i < n - N, \ n \geqslant 2N.$$

As before, the iterative inequality (6.9) implies that $|a_{n,i}| \leqslant \beta e^{2\widetilde{c}i} b_n^i$ for all $1 \leqslant i \leqslant n$ and $n \geqslant 2N$, where β is a universal constant only depending on $\widetilde{c}$. Hence, for $c > \widetilde{c}$,

$$D_{n,2}(x_n, t_n + cn)^2 \leqslant \beta^2 \sum_{i=1}^{\infty} e^{4(\widetilde{c}-c)i} < \infty.$$

By Proposition 3.7, the family has a strongly optimal (t_n, n)–L^2-cutoff. □

6.2. *Discrete time Ehrenfest chains*

Let K_n' be the Markov kernel obtained by

$$K_n' = \frac{1}{n+1} I_{n+1} + \frac{n}{n+1} K_n \tag{6.10}$$

where K_n is the Ehrenfest kernel and I_n is the $n \times n$ identity matrix. As a consequence of Theorem 6.1, K_n' has eigenvalues $\beta_{n,i}' = 1 - \frac{2i}{n+1}$ with corresponding eigenvectors $\psi_{n,i}$ given by (6.2).

To apply Theorem 5.3 to this chain, we need to reorder the eigenvalues. For $1 \leqslant i \leqslant n/2$, let

$$\lambda_{n,2i-1}' = \lambda_{n,2i}' = -\log\left(1 - \frac{2i}{n+1}\right), \qquad \psi_{n,2i-1}' = \psi_{n,n-i+1}, \qquad \psi_{n,2i}' = \psi_{n,i}. \tag{6.11}$$

Then, the L^2-distance $D_{n,2}'$ in the discrete time case is given by

$$D_{n,2}'(x,t) = \sum_{i=1}^{n} \left|\psi_{n,i}'(x)\right|^2 e^{-2t\lambda_{n,i}'}. \tag{6.12}$$

Write (6.3) in the form

$$\psi_{n,i+1}(x) = -\frac{n-2x}{\sqrt{n}} A_{n,i} \psi_{n,i}(x) - B_{n,i} \psi_{n,i-1}(x), \quad 1 \leqslant i < n,$$

and

$$\psi_{n,1}(x) = \frac{n-2x}{\sqrt{n}}\psi_{n,0}(x), \qquad \psi_{n,n-1}(x) = \frac{n-2x}{\sqrt{n}}\psi_{n,n}(x),$$

where $\psi_{n,0} \equiv 1$, $\psi_{n,n}(x) = (-1)^x$ with

$$A_{n,i} = \sqrt{\frac{n}{(i+1)(n-i)}}, \qquad B_{n,i} = \sqrt{\frac{i(n-i+1)}{(i+1)(n-i)}}.$$

Note that for $1 \leqslant i \leqslant n/2$,

$$A_{n,n-i} = A_{n,i}/B_{n,i}, \qquad B_{n,n-i} = 1/B_{n,i}.$$

This implies

$$\psi_{n,n-i-1}(x) = -\frac{n-2x}{\sqrt{n}}A_{n,i}\psi_{n,n-i}(x) - B_{n,i}\psi_{n,n-i+1}(x).$$

Since $\psi_{n,i+1}$ and $\psi_{n,n-i-1}$ are derived by the same iterative formulae with respective initial values 1 and $(-1)^x$, they are related as follows.

$$\psi_{n,n-i}(x) = (-1)^x\psi_{n,i}(x) \quad \forall x, i \in \{0,1,\dots,n\}.$$

This identity implies

$$\left|\psi'_{n,1}(x)\right| = 1, \qquad \left|\psi'_{n,2i}(x)\right| = \left|\psi'_{n,2i+1}(x)\right| \quad \forall 1 \leqslant i \leqslant n/2. \tag{6.13}$$

This discussion will be used for the proof of the following theorem.

Theorem 6.5. *Let $\Omega_n = \{0,1,\dots,n\}$ and $\mathcal{F}' = \{(\Omega_n, K'_n, \pi_n)\colon n = 1,2,\dots\}$ be the family of Markov chains given by* (6.10) *with starting states $(x_n)_1^\infty$. Then, the following are equivalent.*

(i) $|n-2x_n|/\sqrt{n} \to \infty$;
(ii) *The family $\mathcal{F}'$ has an L^2-cutoff.*

Furthermore, if (i) *holds true, then $\mathcal{F}'$ has a strongly optimal (t_n, n)–L^2-cutoff and a (t_n, n)-total variation cutoff, where*

$$t_n = \frac{n}{2}\log\frac{|n-2x_n|}{\sqrt{n}}.$$

Proof. The standard way to prove the above result would be to apply Theorem 5.3. Here, instead, we bound $D'_{n,2}$ using the L^2-distance, $D_{n,2}$, of the Ehrenfest process discussed in Theorem 6.3. In detail, by (6.11), (6.12) and (6.13), we have

$$D'_{n,2}(x,t)^2 \leqslant e^{-2\lambda'_{n,1}t} + 2\sum_{i=1}^{[n/2]}\left|\psi'_{n,2i}(x)\right|^2 e^{-2\lambda'_{n,2i}t} \leqslant e^{-2\lambda'_{n,1}t} + D_{n,2}(x,t)^2$$

where the last inequality uses the fact $\log(1-t) \leqslant -t$ for all $t \in (0,1)$, which implies that $\lambda'_{n,2i} \geqslant 2i/n = 1-\beta_{n,i}$ and $\beta_{n,i}$ is the term defined in Theorem 6.1. For the lower bound, we use the second term in the series of $D'_{n,i}(x,t)^2$, that is,

$$D'_{n,2}(x,t)^2 \geqslant \left|\psi'_{n,2}(x)\right|^2 e^{-2\lambda'_{n,2}t} = \frac{(n-2x)^2}{n} e^{-2\lambda'_{n,2}t}.$$

By writing $\lambda'_{n,2} = 2/n(1+c_n)$, it can be easily shown that $c_n = O(1/n)$.

Recall that $x_n = (n+y_n)/2$ with $0 \leqslant y_n \leqslant n$. By Theorem 6.3, if $y_n/\sqrt{n}$ is bounded, then there exists $\epsilon > 0$ such that the ϵ-mixing time $T^d_{n,2}(x_n,\epsilon)$ of $D'_{n,2}$ is of order at most n. (In fact, it is of order n using the lower bound obtained above.) Let $j_n(C)$ be the integer in Theorem 5.3. It is clear that $j_n(1) = 1$ and, hence, $\lambda'_{n,1} = \lambda'_{n,2}$ and

$$T_{n,2}(x_n,\epsilon)\lambda'_{n,1} = O(n) \times \frac{2}{n}(1+c_n) = O(1).$$

By Theorem 5.3, $\mathcal{F}'$ has no L^2-cutoff. In the case $y_n/\sqrt{n} \to \infty$, Theorem 6.3 and Remark 6.1 imply that

$$\limsup_{n\to\infty} D_{n,2}(x_n, t_n + cn) \begin{cases} < \infty & \text{if } c < 0, \\ \leqslant Ae^{-2c} & \text{if } c > 0, \end{cases}$$

where A is a constant and $t_n = (n/2)\log(y_n/\sqrt{n})$. Using the bounds for $D'_{n,2}$ established above, we get

$$\limsup_{n\to\infty} D'_{n,2}(x_n, t_n + cn) \begin{cases} < \infty & \text{if } c < 0, \\ \leqslant Ae^{-2c} & \text{if } c > 0 \end{cases}$$

and

$$\liminf_{n\to\infty} D'_{n,2}(x_n, t_n + cn) \geqslant e^{-2c} \quad \forall c \in \mathbb{R}.$$

This implies that $\mathcal{F}'$ has a strongly optimal (t_n,n)–L^2-cutoff. The proof of the total variation cutoff is as in the continuous time case. □

7. Constant rate birth and death chains

This section applies the main results of this paper to the study of constant rate birth and death chains. Finding the L^2-cutoff of family of Markov chains from arbitrary starting points is a difficult task that requires a great deal of spectral information. The following examples illustrate this very well. First, we treat families of finite constant rate birth and death chains on $\{0,\dots,n\}$ with n tending to infinity and arbitrary constant rates p_n, q_n. Second, we discuss the case when the state space is the countable set $\{0,1,\dots\}$.

7.1. Chains of finite length

Karlin and McGregor [12,13] observed that the spectral analysis of any given birth and death chain can be treated as an orthogonal polynomial problem. This sometimes leads to the exact computation of the spectrum. See, e.g., [10,12,13,20] and also [17] for a somewhat different approach based on continued fractions.

The families of interest here are of the following simple type. For $n \geqslant 1$, let $\Omega_n = \{0, 1, \dots, n\}$ and let K_n be the Markov kernel of a birth and death chain on Ω_n with constant rates

$$p_n(x) = p_n, \qquad q_n(x) = q_n = 1 - p_n, \quad \forall 0 \leqslant x \leqslant n, \tag{7.1}$$

where $p_n(x)$ and $q_n(x)$ denote respectively the birth rate and the death rate with the usual convention that $q_n(0) = r_n(0)$, $p_n(n) = r_n(n)$ are holding probabilities. This has stationary (reversible) distribution π_n given by

$$\pi_n(x) = c_n\left(\frac{p_n}{q_n}\right)^x, \quad \text{with } c_n = \left(1 - \frac{p_n}{q_n}\right)\left[1 - \left(\frac{p_n}{q_n}\right)^{n+1}\right]^{-1}. \tag{7.2}$$

Set

$$\beta_{n,0} = 1, \qquad \beta_{n,j} = 2\sqrt{p_n q_n}\cos\frac{j\pi}{n+1}, \quad \forall 1 \leqslant j \leqslant n, \tag{7.3}$$

and let $\psi_{n,j}$ be a vector on Ω_n defined by $\psi_{n,0} \equiv 1$ and, for $1 \leqslant j \leqslant n$ and $x \in \Omega_n$,

$$\psi_{n,j}(x) = C_{n,j}\left\{\left(\frac{q_n}{p_n}\right)^{(x+1)/2}\sin\frac{j(x+1)\pi}{n+1} - \left(\frac{q_n}{p_n}\right)^{(x+2)/2}\sin\frac{jx\pi}{n+1}\right\}, \tag{7.4}$$

where $C_{n,j}^{-2} = c_n(n+1)q_n\lambda_{n,j}/(2p_n^2)$ and $\lambda_{n,j} = 1 - \beta_{n,j}$. Then, $\beta_{n,j}$ is an eigenvalue of K_n with corresponding normalized eigenvector $\psi_{n,j}$. See [9, Chapter XVI.3].

Let $x_n \in \Omega_n$, $n \geqslant 1$, be a sequence of initial states and set as before $D_{n,2}^{\gamma}(x_n, t)$, $\gamma \in \{c, d\}$, to be the L^2-distance for the nth chain starting from x_n(c denotes the continuous time case and d stands for the discrete time case). Then, by Theorem 5.1(i) and Theorem 5.3(i), a necessary condition for the family $\{D_{n,2}^{\gamma}(x_n, t),\ n = 1, 2, \dots\}$, $\gamma \in \{c, d\}$, to have a cutoff is $\pi_n(x_n) \to 0$ as $n \to \infty$. The following lemma gives an equivalent condition for such a limit using p_n and x_n.

Lemma 7.1. *For $n \geqslant 1$, let $\pi_n(\cdot)$ be the probability defined in* (7.2) *with* $p_n \in (0, 1/2)$. *Then, for $x_n \in \{0, 1, \dots, n\}$, $\pi_n(x_n) \to 0$ if and only if*

$$\lim_{n\to\infty}\left(\frac{1}{1-2p_n} + \frac{x_n}{p_n}\right) = \infty.$$

Proof. Set, for $n \geqslant 1$, $b_n = (p_n/q_n)^{x_n}$. Then, $\pi_n(x_n) = b_n c_n$. Assume that $\pi_n(x_n) \to 0$. Using the fact $\log(1 + t) \leqslant t$, we have

$$\log c_n = -\log\bigl(1 + p_n/q_n + \cdots + (p_n/q_n)^n\bigr) \geqslant -(q_n - p_n)^{-1}$$

and

$$\log b_n = -x_n \log\left(1 + \frac{q_n - p_n}{p_n}\right) \geqslant -\frac{x_n}{p_n}.$$

Thus, $b_n c_n \to 0$ implies $x_n/p_n + 1/(q_n - p_n) \to \infty$ as desired.

For the other direction, assume that $\limsup_{n\to\infty} \pi_n(x_n) > 0$. Since $b_n \leqslant 1$ and $c_n \leqslant 1$, we may choose a subsequence $(n_k)_{k\geqslant 1}$ such that $\inf_k b_{n_k} > 0$ and $\inf_k c_{n_k} > 0$. Consider the following identity.

$$1 - c_n = (p_n/q_n)\big(1 - c_n(p_n/q_n)^n\big).$$

This implies that

$$c_n \to 0 \quad \Leftrightarrow \quad p_n \to 1/2$$

and, hence,

$$\limsup_{k\to\infty} p_{n_k} = p < 1/2.$$

Using the last observation and the fact $\inf_k b_{n_k} > 0$, it is clear that x_{n_k} has to be bounded. Concerning the value of x_{n_k}, let $A = \{n_k\colon x_{n_k} = 0\} = \{n'_k\colon k \geqslant 1\}$ and $B = \{n_k\colon k = 1, 2, \ldots\} \setminus A = \{n''_k\colon k \geqslant 1\}$. Observe that $|A| = \infty$ or $|B| = \infty$ must hold. In the former case, it is easy to see that

$$\liminf_{n\to\infty}\left(\frac{1}{1-2p_n} + \frac{x_n}{p_n}\right) \leqslant \limsup_{k\to\infty} \frac{1}{1-2p_{n'_k}} \leqslant \frac{1}{1-2p} < \infty.$$

In the latter case, since $x_{n''_k} \geqslant 1$ and $\inf_k b_{n''_k} \geqslant \inf_k b_{n_k} > 0$, it must be true that $\inf_k p_{n''_k} > 0$. Hence, we obtain

$$\liminf_{n\to\infty}\left(\frac{1}{1-2p_n} + \frac{x_n}{p_n}\right) \leqslant \limsup_{k\to\infty} \frac{1}{1-2p_{n''_k}} + \limsup_{k\to\infty} \frac{x_{n''_k}}{p_{n''_k}}$$
$$\leqslant \frac{1}{1-2p} + \frac{\sup_k x_{n''_k}}{\inf_k p_{n''_k}} < \infty. \qquad \square$$

The next theorem concerns the L^2-cutoff for these birth and death chains and the associated cutoff time.

Theorem 7.2. *Referring to the setting introduced above, for $n \geqslant 1$ and $\gamma \in \{c, d\}$, let $p_n^\gamma(t,\cdot,\cdot)$ be the (continuous/discrete) associated Markov transition function. Fix a sequence of states $x_n \in \Omega_n$. Assume that $0 < p_n < 1/2$. Then, for $\gamma \in \{c, d\}$, the family $\{p_n^\gamma(t, x_n, \cdot)\colon n \geqslant 1\}$ has an L^2-cutoff if and only if*

$$\lim_{n\to\infty} x_n\left(\frac{q_n}{p_n} - 1\right) = \infty. \tag{7.5}$$

Moreover, if the above condition holds, then, for $\gamma \in \{c,d\}$, the family $p_n^\gamma(t, x_n, \cdot)$ has a (t_n^γ, b_n^γ)–L^2-cutoff where

$$t_n^c = \frac{x_n(\log q_n - \log p_n)}{2(1-2\sqrt{p_n q_n})}, \qquad t_n^d = \left\lfloor \frac{x_n(\log q_n - \log p_n)}{-\log(4p_n q_n)} \right\rfloor,$$

and

$$b_n^c = \frac{\log\log(q_n/p_n)^{x_n}}{1-2\sqrt{p_n q_n}}, \qquad b_n^d = \frac{\log\log(q_n/p_n)^{x_n}}{-\log(4p_n q_n)}.$$

Remark 7.1. Note that the cutoff time (if there is an L^2-cutoff) is independent of the size of the state space.

Remark 7.2. Concerning the case $p_n \equiv p \in (0, 1/2)$, by Theorem 7.2, the existence of the L^2-cutoff for $p_n^\gamma(t, x_n, \cdot)$, $\gamma \in \{c, d\}$, is equivalent to the condition $x_n \to \infty$. As a consequence of Theorem 7.2, if $x_n \to \infty$, the family $p_n^\gamma(t, x_n, \cdot)$, $\tau \in \{c, d\}$, has a (t_n^γ, b_n^γ)–L^2-cutoff with

$$t_n^c = \frac{(\log q - \log p)x_n}{2(1-2\sqrt{pq})}, \qquad t_n^d = \frac{(\log q - \log p)x_n}{-\log(4pq)}, \qquad b_n^c = b_n^d = \log x_n.$$

Diaconis and Saloff-Coste proved in [8] that both families $p_n^c(t, n, \cdot)$ and $p_n^d(t, n, \cdot)$ have a separation cutoff at time $\frac{n}{q-p}$. One can check that

$$\frac{\log q - \log p}{2(1-2\sqrt{pq})} > \frac{\log q - \log p}{-\log(4pq)} > \frac{1}{q-p} \quad \forall p \in (0, 1/2). \tag{7.6}$$

Thus, $t_n^c \geqslant t_n^d$ and the L^2-cutoff occurs later than the separation cutoff (this is not always true). Note that the window given here is not optimal. For example, in continuous time case, it can be proved directly using the expression in (5.4) and the formulas (7.3), (7.4) that $p_n^c(t, n, \cdot)$ has a strongly optimal $(t_n^c, 1)$–L^2-cutoff, where the strong optimality uses Corollary 5.2. Similarly, for any integer m, the L^2-distance between $p_n^d(t_n^d + m, n, \cdot)$ and π_n always converges to 0 as $n \to \infty$.

Remark 7.3. In the case $p_n \to 0$, the equivalent condition for the existence of the L^2-cutoff is $x_n \to \infty$. If this holds true, then the family $p_n^\gamma(t, x_n, \cdot)$ has a (t_n^γ, b_n^γ)–L^2-cutoff with

$$t_n^c = \frac{1}{2} x_n \log(1/p_n), \qquad b_n^c = \log x_n$$

and

$$t_n^d = x_n, \qquad b_n^d = \frac{\log x_n}{\log(1/p_n)}.$$

Note that t_n^c and t_n^d are of different order.

Remark 7.4. In the case $p_n \to 1/2$, write $p_n = \frac{1}{2} - \frac{\delta_n}{n}$, where $\delta_n = o(n)$. By Theorem 7.2, for $\gamma \in \{c, d\}$, $p_n^\gamma(t, x_n, \cdot)$ has an L^2-cutoff if and only if $x_n\delta_n/n \to \infty$. Moreover, if $x_n\delta_n/n \to \infty$, then both families in continuous time and discrete time cases have a (t_n, b_n)–L^2-cutoff with

$$t_n = \frac{nx_n}{\delta_n}, \qquad b_n = x_n + \frac{n^2}{\delta_n^2}\log\frac{x_n\delta_n}{n}.$$

Theorem 7.2 also holds in the case $p_n = q_n = 1/2$ where there is no cutoff. This is well known and we omit the details.

Proof of Theorem 7.2. The proof of Theorem 7.2 involves considering several cases. We shall use the convention that, for any two sequences of positive numbers s_n, t_n,

$$\begin{cases} s_n \sim t_n & \text{if } \lim_{n\to\infty} s_n/t_n = 1; \\ s_n \lesssim t_n & \text{if } \limsup_{n\to\infty}\{s_n/t_n\} < \infty; \\ s_n \asymp t_n & \text{if } s_n \lesssim t_n,\ t_n \lesssim s_n. \end{cases} \tag{7.7}$$

Set $p_n = \frac{1}{2} - \frac{\delta_n}{n}$ and let $x_n \in \{0, 1, \dots, n\}$. Then, for any sequence of pairs (x_n, p_n), there exists a subsequence n_k such that the conjunction of one $A(i)$ and one $B(j)$ holds, where

$$\begin{cases} A(1)\colon\ \delta_{n_k} = o(1); & B(1)\colon\ x_{n_k} \equiv 0; \\ A(2)\colon\ \delta_{n_k} \asymp 1; & B(2)\colon\ x_{n_k} \asymp 1; \\ A(3)\colon\ \delta_{n_k} \to \infty, \quad \delta_{n_k} = o(n_k); & B(3)\colon\ x_{n_k} \to \infty, \quad x_{n_k}\delta_{n_k} = o(n_k); \\ A(4)\colon\ \delta_{n_k}/n_k \to \delta \in (0, 1/2); & B(4)\colon\ x_{n_k} \to \infty, \quad x_{n_k}\delta_{n_k} \asymp n_k; \\ A(5)\colon\ \delta_{n_k}/n_k \to 1/2; & B(5)\colon\ x_{n_k} \to \infty, \quad n_k = o(x_{n_k}\delta_{n_k}). \end{cases}$$

Let $R(i, j)$ denote the case when $A(i)$ and $B(j)$ hold. Clearly, $R(1, 4)$, $R(1, 5)$, $R(2, 5)$, $R(4, 3)$, $R(4, 4)$, $R(5, 3)$ and $R(5, 4)$ can not happen. By Lemma 7.1, it is easy to see that $\pi_n(x_n) \asymp 1$ is equivalent to cases $R(4, 1)$, $R(4, 2)$ and $R(5, 1)$. Thus, by Theorem 5.1, the family of continuous time chains has no L^2-cutoff in those cases. For the family of discrete time chains, we can show that

$$\lim_{n\to\infty} \pi_n\left(\left|\frac{p_n^d(0, 0, \cdot)}{\pi_n(\cdot)} - 1\right|^2\right) = 0 \quad \text{in } R(5, 1)$$

and

$$\forall t > 0 \quad \liminf_{n\to\infty} \pi_n\left(\left|\frac{p_n^d(t, 0, \cdot)}{\pi_n(\cdot)} - 1\right|^2\right) > 0 \quad \text{in } R(4, 1) \text{ and } R(4, 2).$$

This implies that no L^2-cutoff exists, where the case $R(5, 1)$ uses the first equality and cases $R(4, 1)$ and $R(4, 2)$ use the second inequality and Corollary 3.3.

Let $\xi = \{n_k\colon k \geqslant 1\}$ and $\mathcal{F}_\xi$ be the subfamily of $\mathcal{F}$ indexed by ξ. By Proposition 2.1, to prove Theorem 7.2, in cases $R(3, 5)$, $R(4, 5)$, $R(5, 2)$ and $R(5, 5)$, it suffices to show that $\mathcal{F}_\xi$ has an L^2-cutoff. In cases $R(i, j)$ with $i, j \in \{1, 2, 3\}$ and $R(2, 4)$, $R(3, 4)$, it suffices to show that $\mathcal{F}_\xi$ has no L^2-cutoff. To simplify the notations, we write $\mathcal{F}$ for $\mathcal{F}_\xi$. Let $\psi_{n,0}^c = \psi_{n,0}^d \equiv 1$ and set

$$\lambda_{n,i}^c = \lambda_{n,i} = 1 - \beta_{n,i}, \qquad \psi_{n,i}^c = \psi_{n,i}, \quad \forall 1 \leqslant i \leqslant n, \tag{7.8}$$

and, for $1 \leqslant i \leqslant [(n+1)/2]$,

$$\lambda^d_{n,2i-1} = \lambda^d_{n,2i} = -\log \beta_{n,i}, \qquad \psi^d_{n,2i-1} = \psi_{n,i}, \qquad \psi^d_{n,2i} = \psi_{n,n+1-i}. \tag{7.9}$$

As before, c and d represent the continuous time and discrete time cases, and $\mathcal{F}_c$ and $\mathcal{F}_d$ are the corresponding families.

Necessity of (7.5). Consider the cases $R(1,j)$ with $1 \leqslant j \leqslant 3$ and $R(i,j)$ with $i \in \{2,3\}$ and $j \in \{1,2,3,4\}$. Rewrite (7.4) as follows.

$$\psi^2_{n,i}(x_n) = \frac{2\Delta^2_{n,i}}{(n+1)\pi_n(x_n)\lambda_{n,i}}, \quad \forall 1 \leqslant i \leqslant n, \tag{7.10}$$

where $\Delta_{n,i} = \sqrt{p_n}\sin\frac{i(x_n+1)\pi}{n+1} - \sqrt{q_n}\sin\frac{ix_n\pi}{n+1}$. Note that, for $0 \leqslant s \leqslant \pi/2$ and $0 \leqslant t \leqslant \pi$ with $s < t$,

$$\frac{1}{8}(t^2 - s^2) \leqslant \cos s - \cos t \leqslant \frac{1}{2}(t^2 - s^2).$$

This implies that, for $1 \leqslant i \leqslant n$,

$$\lambda_{n,i} = \frac{(q_n - p_n)^2}{(\sqrt{q_n} + \sqrt{p_n})^2} + 2\sqrt{p_n q_n}\left(1 - \cos\frac{i\pi}{n+1}\right) \begin{cases} \geqslant \alpha_{n,i}/5, \\ \leqslant 5\alpha_{n,i} \end{cases} \tag{7.11}$$

where

$$\alpha_{n,i} = \frac{1}{n^2}\left(\delta_n^2 + i^2\sqrt{1 - \frac{4\delta_n^2}{n^2}}\right).$$

Note also that, for $j \in \{1,2,3,4\}$,

$$\pi_n(x_n) \asymp \begin{cases} 1/n & \text{for } R(1,j), R(2,j), \\ \delta_n/n & \text{for } R(3,j). \end{cases} \tag{7.12}$$

Now, we are going to disprove the existence of L^2-cutoff using Theorems 5.1–5.3. We first treat the continuous time cases. For $C > 0$, let $j_n(C)$, $\tau_n(C)$ be as defined in (5.2), (5.3). Step 1 and Step 2 below treat the cases $R(1,j)$ with $j \in \{1,2,3\}$ and $R(2,j)$ with $j \in \{1,2,3,4\}$, whereas Step 3 and Step 4 discuss the cases $R(3,j)$ with $1 \leqslant j \leqslant 4$.

Step 1: There exists $C_0 > 0$ such that $j_n(C_0) \asymp 1$.

Note that $A(1)$ or $A(2)$ implies $p_n \to 1/2$ and $\delta_n = O(1)$. When $A(1)$ holds, by writing

$$\Delta_{n,i} = \sqrt{p_n}\left(\sin\frac{i(x_n+1)\pi}{n+1} - \sin\frac{ix_n\pi}{n+1}\right) + \left(\sqrt{p_n} - \sqrt{q_n}\right)\sin\frac{ix_n\pi}{n+1}, \tag{7.13}$$

we have $|\Delta_{n,1}| \lesssim 1/n$ and $|\Delta_{n,2}| \lesssim 1/n$. In a detailed computation, one can get

$$\begin{cases} |\Delta_{n,1}| \asymp 1/n & \text{if } x_n/n \in [0,3/8] \cup [5/8,1], \\ |\Delta_{n,2}| \asymp 1/n & \text{if } x_n/n \in [3/8,5/8]. \end{cases}$$

Thus, $\Delta_{n,1}^2 + \Delta_{n,2}^2 \asymp n^{-2}$. When $A(2)$ holds, we may choose a constant $\epsilon \in (0, 1/2)$ such that

$$|\Delta_{n,1}| \gtrsim 1/n, \quad \forall x_n \in [0, \epsilon n] \cup [n/2, n]. \tag{7.14}$$

Clearly, for all $k \geqslant 1$, $|\Delta_{n,k}| \lesssim 1/n$ using (7.13). To get a similar bound as in $A(1)$, observe that for fixed $k \geqslant 1$, if $x_n \in [n/(2k), n/k]$, then

$$\begin{aligned} |\Delta_{n,k}| &= \sqrt{p_n}\left(\sin\frac{kx_n\pi}{n+1} - \sin\frac{k(x_n+1)\pi}{n+1}\right) + \left(\sqrt{q_n} - \sqrt{p_n}\right)\sin\frac{kx_n\pi}{n+1} \\ &\geqslant \sqrt{p_n}\left(\sin\frac{kx_n\pi}{n+1} - \sin\frac{(kx_n+1)\pi}{n+1}\right) + \left(\sqrt{q_n} - \sqrt{p_n}\right)\sin\frac{kx_n\pi}{n+1} \gtrsim 1/n, \end{aligned}$$

where the last asymptotic inequality is given by (7.14). Consequently, by setting $K = \lceil 1/(2\epsilon)\rceil$, we have

$$\Delta_{n,1}^2 + \cdots + \Delta_{n,K}^2 \asymp n^{-2}.$$

Hence, in either case of $A(1)$ and $A(2)$, $\sum_{i=1}^K \Delta_{n,i}^2 \asymp n^{-2}$. Plugging this result, (7.11) and (7.12) into (7.10) then gives

$$\sum_{i=1}^K \psi_{n,i}^2(x_n) \geqslant \frac{2(\Delta_{n,1}^2 + \cdots + \Delta_{n,K}^2)}{(n+1)\pi_n(x_n)\lambda_{n,K}} \asymp 1.$$

This proves Step 1.

Step 2: Let C_0 be as in Step 1. Then, $\tau_n(C_0) \asymp n^2$.

In order to prove this fact, we need the following computations.

$$\begin{aligned} |\Delta_{n,i}| &\leqslant \sqrt{p_n}\left|\sin\frac{i(x_n+1)\pi}{n+1} - \sin\frac{ix_n\pi}{n+1}\right| + \left(\sqrt{q_n} - \sqrt{p_n}\right)\left|\sin\frac{ix_n\pi}{n+1}\right| \\ &\leqslant \frac{i\pi}{n}\left(1 + \frac{4x_n\delta_n}{n}\right) \leqslant \frac{(1+4\delta_n)\pi i}{n}. \end{aligned} \tag{7.15}$$

Using the last inequality and (7.10)–(7.12), we obtain

$$\left|\psi_{n,i}^2(x_n)\right| \leqslant \frac{10\pi^2(1+4\delta_n)^2 i^2/n^2}{(n+1)\pi_n(x_n)\alpha_{n,i}} \leqslant \frac{10\pi^2(1+4\delta_n)^2}{(n+1)\pi_n(x_n)\sqrt{1-4\delta_n^2/n^2}} \asymp 1,$$

where the last asymptotic relation is uniform for $1 \leqslant i \leqslant n$. This implies that

$$\sup\left\{\psi_{n,i}^2(x_n)\colon\ 1 \leqslant i \leqslant n,\ n \geqslant 1\right\} = M < \infty$$

and, hence,

$$n^2 \asymp \frac{\log(1+C_0)}{2\lambda_{n,j_n(C_0)}} \leqslant \tau_n(C_0) \leqslant \max_{1\leqslant i\leqslant n} \frac{\log(1+Mi)}{2\lambda_{n,i}} \asymp n^2.$$

This proves Step 2.

It is immediate from Step 1 and Step 2 that $\lambda_{n,j_n(C_0)}\tau_n(C_0) \asymp 1$ and then, by Theorem 5.1, the family $\{p_n^c(t, x_n, \cdot)\colon n = 1, 2, \ldots\}$ has no L^2-cutoff.

In Step 3 and Step 4, we treat the cases $R(3, j)$ with $1 \leqslant j \leqslant 4$.

Step 3: There exists $C_1 > 0$ such that $j_n(C_1) \asymp \delta_n$.

To see the detail, recall those identities introduced in (7.10)–(7.12). It is an immediate result of (7.11) that if $A(3)$ holds, then

$$\lambda_{n,i} \asymp \left(\delta_n^2 + i^2\right)n^{-2}, \quad \text{uniformly for } 1 \leqslant i \leqslant n. \tag{7.16}$$

We first consider $R(3, j)$ with $j \in \{1, 2, 3\}$. In these cases, it is obvious that $x_n\delta_n = o(n)$. Using this fact, one can easily compute

$$\left|\sin\frac{i(x_n+1)\pi}{n+1} - \sin\frac{ix_n\pi}{n+1}\right| \asymp \frac{i}{n}, \qquad \left(\sqrt{q_n} - \sqrt{p_n}\right)\left|\sin\frac{ix_n\pi}{n+1}\right| \asymp \frac{ix_n\delta_n}{n^2} = o\left(\frac{i}{n}\right),$$

where $\asymp$ and $o(\cdot)$ are uniform for $1 \leqslant i \leqslant (n+1)/(4(x_n+1))$. This implies

$$\Delta_{n,i}^2 \asymp i^2/n^2, \quad \text{uniformly for } 1 \leqslant i \leqslant \delta_n. \tag{7.17}$$

By replacing corresponding terms in (7.10) with (7.12), (7.16) and (7.17), we obtain

$$\psi_{n,i}^2(x_n) \asymp \frac{i^2}{\delta_n^3}, \quad \text{uniformly for } 1 \leqslant i \leqslant \delta_n.$$

Thus, for C small enough, $j_n(C) \asymp \delta_n$.

We now consider the case $R(3, 4)$, that is, $\delta_n \to \infty$, $x_n \to \infty$ and $\delta_n x_n \asymp n$. As before, applying (7.12), (7.15) and (7.16) to (7.10) gives

$$\psi_{n,i}^2(x_n) \lesssim \frac{i^2}{\delta_n(\delta_n^2 + i^2)} \quad \text{uniformly for } 1 \leqslant i \leqslant n. \tag{7.18}$$

This implies $j_n(C) \gtrsim \delta_n$ for all $C > 0$. To see the inverse direction, observe that for $\frac{n+1}{2x_n} \leqslant i \leqslant \frac{n+1}{x_n}$,

$$|\Delta_{n,i}| = \sqrt{p_n}\left|\sin\frac{i(x_n+1)\pi}{n+1} - \sin\frac{ix_n\pi}{n+1}\right| + \left(\sqrt{q_n} - \sqrt{p_n}\right)\left|\sin\frac{ix_n\pi}{n+1}\right|. \tag{7.19}$$

This can be easily seen from (7.13). To analyze the right side summation, we compute that

$$\forall \frac{n+1}{2x_n} \leqslant i \leqslant \frac{3(n+1)}{4x_n}, \quad \sin\frac{ix_n\pi}{n+1} \geqslant \frac{1}{2} \geqslant \frac{2x_n i}{3(n+1)},$$

and

$$\forall \frac{3(n+1)}{4x_n} \leqslant i \leqslant \frac{n+1}{x_n}, \quad \left|\sin\frac{i(x_n+1)\pi}{n+1} - \sin\frac{ix_n\pi}{n+1}\right| \geqslant \frac{i\pi}{2(n+1)}.$$

Putting these two inequalities back to (7.19) gives

$$|\Delta_{n,i}| \gtrsim i/n, \quad \text{uniformly for } \frac{n+1}{2x_n} \leqslant i \leqslant \frac{n+1}{x_n}.$$

Hence, by applying this result with (7.10), (7.12) and (7.16), we get

$$\sum_{i=1}^{\lfloor \frac{n+1}{x_n} \rfloor} \psi_{n,i}^2(x_n) \geqslant \sum_{i=\lceil \frac{n+1}{2x_n} \rceil}^{\lfloor \frac{n+1}{x_n} \rfloor} \psi_{n,i}^2(x_n) \gtrsim \sum_{i=\lceil \frac{n+1}{2x_n} \rceil}^{\lfloor \frac{n+1}{x_n} \rfloor} \frac{i^2}{\delta_n(\delta_n^2 + i^2)} \asymp 1.$$

This implies $j_n(C) \lesssim n/x_n \asymp \delta_n$ for C small enough. Consequently, in $R(3,4)$, $j_n(C) \asymp \delta_n$ for C small enough.

Step 4: Let C_1 be as in Step 3. Then, $\tau_n(C_1) \asymp n^2/\delta_n^2$.

Note that, in cases $B(1)$–$B(4)$, $x_n\delta_n/n \lesssim 1$. By the second inequality of (7.15), this implies

$$|\Delta_{n,i}| \lesssim i/n \quad \text{uniformly for } 1 \leqslant i \leqslant n.$$

As before, applying this result with (7.10), (7.12) and (7.16) gives

$$\sup\{\psi_{n,i}^2(x_n)\delta_n\colon\ 1 \leqslant i \leqslant n,\ n \geqslant 1\} = N < \infty.$$

Thus, we have

$$\frac{n^2}{\delta_n^2} \asymp \frac{\log(1 + C_1)}{\lambda_{n,j_n(C_1)}} \leqslant \tau_n(C_1) \leqslant \max_{j_n(C_1) \leqslant i \leqslant n} \frac{\log(1 + iN/\delta_n)}{\lambda_{n,i}} \asymp \frac{n^2}{\delta_n^2}.$$

As a consequence of Step 3 and Step 4, we have $\lambda_{n,j_n(C_1)}\tau_n(C_1) \asymp 1$. By Theorem 5.1, this implies that the family $\{p_n^c(t, x_n, \cdot)\colon\ n = 1, 2, \ldots\}$ has no L^2-cutoff.

The proof for discrete time cases goes in a similar way. Recall in the following the spectral information displayed in (7.9) using the setting given by (7.3) and (7.4). For $1 \leqslant i \leqslant n/2$,

$$\psi_{n,2i-1}^d = \psi_{n,i}, \qquad \psi_{n,2i}^d = \psi_{n,n+1-i}, \tag{7.20}$$

and

$$\lambda_{n,2i-1}^d = \lambda_{n,2i}^d = -\log\beta_{n,i}.$$

Note that

$$\forall 1 \leqslant i \leqslant n, \quad -\log\beta_{n,i} = -\log(1 - \lambda_{n,i}) \geqslant \lambda_{n,i}$$

and, for all $L > 2$,

$$-\log\beta_{n,i} \asymp \lambda_{n,i} \quad \text{uniformly for } 1 \leqslant i \leqslant n/L.$$

Using the above comparison relationship, it is easy to show from the definition of $j_n(C)$ and $\tau_n(C)$ given in (5.2) and (5.3) that Step 1 and Step 2 remain true in cases $R(1, j)$ with $j =$

$1,2,3$ and $R(2,j)$ with $j=1,2,3,4$. As a consequence of Theorem 5.3, the family $\{p_n^d(t,x_n,\cdot)$: $n=1,2,\dots\}$ has no L^2-cutoff.

For cases $R(3,j)$ with $j\in\{1,2,3,4\}$, let C_1 be the constant for families of continuous time chains selected in Step 3. Using (7.20), one can easily show that, for discrete time chains, $j_n(C_1)\lesssim\delta_n$, whereas (7.18) gives $j_n(C)\gtrsim\delta_n$ for all $C>0$. This implies $j_n(C_1)\asymp\delta_n$. A similar proof as that for Step 4 implies $\tau_n(C)\asymp n^2/\delta_n^2$. By Theorem 5.3, the family $\{p_n^d(t,x_n,\cdot)$: $n=1,2,\dots\}$ has no L^2-cutoff.

Sufficiency of (7.5). First of all, recall the notations defined in (7.2)–(7.4) and (7.7)–(7.9), and rewrite (7.10) and $\lambda_{n,i}$ in the following way.

$$\forall 1\leqslant i\leqslant n,\quad \psi_{n,i}^2(x_n)=\frac{2\sin^2((\frac{ix_n}{n+1}+\theta_{n,i})\pi)}{(n+1)\pi_n(x_n)},\tag{7.21}$$

where $\theta_{n,i}\in(1/2,1)$ is such that

$$\sin(\theta_{n,i}\pi)=\frac{\sqrt{p_n}\sin\frac{i\pi}{n+1}}{\sqrt{\lambda_{n,i}}},\qquad \cos(\theta_{n,i}\pi)=\frac{\sqrt{p_n}\cos\frac{i\pi}{n+1}-\sqrt{q_n}}{\sqrt{\lambda_{n,i}}},\tag{7.22}$$

and

$$\begin{aligned}\forall 1\leqslant i\leqslant n,\quad \lambda_{n,i}&=\left(\sqrt{p_n}-\sqrt{q_n}\right)^2+2\sqrt{p_nq_n}\left(1-\cos\frac{i\pi}{n+1}\right)\\&=\frac{4\delta_n^2/n^2}{1+2\sqrt{p_nq_n}}\left(1+O\left(\frac{i^2}{\delta_n^2}\right)\right),\end{aligned}\tag{7.23}$$

where O is uniform for $1\leqslant i\leqslant n$. According to the discussion in the beginning of the proof, only cases $R(3,5)$, $R(4,5)$, $R(5,2)$ and $R(5,5)$ are needed to be considered. Obviously, either of them implies

$$\lim_{n\to\infty}\delta_n=\infty,\qquad \lim_{n\to\infty}\frac{x_n\delta_n}{n}=\infty$$

and further that

$$\pi_n(x)\sim\frac{2\delta_n}{nq_n}\left(\frac{p_n}{q_n}\right)^x.\tag{7.24}$$

We will prove the sufficiency of (7.5) using Theorems 5.1–5.3. For $C>0$, let $j_n^\gamma(C)$ and $\tau_n^\gamma(C)$, $\gamma\in\{c,d\}$, be as defined in (5.2) and (5.3). In what follows, Steps 5, 6 and 7 deal with cases $R(3,5)$, $R(4,5)$ and $R(5,5)$, whereas Step 8 consider $R(5,2)$.

Step 5: For $C>0$, $j_n^d(C)\leqslant 2j_n^c(C)-1$ and $j_n^c(C)\lesssim\lceil\delta_n(p_n/q_n)^{x_n/3}\rceil$.

Clearly, the first inequality follows from the setting $\psi_{n,2i-1}^d=\psi_{n,i}^c$. To see the second one, observe that

$$1-\theta_{n,i}\sim\frac{p_n+\sqrt{p_nq_n}}{2}\times\frac{i}{\delta_n}\quad\text{uniformly for } 1\leqslant i\leqslant n/x_n.$$

This can be proved without difficulty using (7.23). By this fact, one can show that

$$\frac{ix_n}{n+1} + \theta_{n,i} - 1 \sim \frac{ix_n}{n} \quad \text{uniformly for } 1 \leqslant i \leqslant \frac{n}{x_n}.$$

Hence, we have

$$\sin^2\left(\left(\frac{ix_n}{n+1} + \theta_{n,i}\right)\pi\right) \asymp \left(\frac{ix_n}{n}\right)^2 \quad \text{uniformly for } 1 \leqslant i \leqslant n/(2x_n). \tag{7.25}$$

As the above result can hold only for $x_n \leqslant n/2$, we consider two subcases.

Case 1: $x_n \leqslant n/2$. In this case, one may use (7.25) to show that for $1 \leqslant j \leqslant n/(2x_n)$,

$$\log\left(\sum_{i=1}^{j} \left|\psi_{n,i}^c\right|^2(x_n)\right) = \log\left(\left(\frac{q_n}{p_n}\right)^{x_n} \frac{j^3 x_n^2}{\delta_n n^2}\right) + O(1). \tag{7.26}$$

Using the inequality $\log(1+t) \geqslant \frac{1}{2}(t \wedge 1)$ for $t \geqslant 0$, we obtain

$$\frac{q_n}{p_n} \geqslant \frac{\delta_n}{n} \wedge \frac{1}{2} \geqslant \frac{\delta_n}{2n}.$$

This implies

$$\delta_n \left(\frac{p_n}{q_n}\right)^{x_n/3} \leqslant \delta_n \exp\left\{-\frac{x_n \delta_n}{6n}\right\} = \delta_n \times o\left(\frac{n}{x_n \delta_n}\right) = o\left(\frac{n}{x_n}\right). \tag{7.27}$$

Hence, for $C > 0$,

$$j_n(C)^c \asymp \left\lceil \left(\frac{p_n}{q_n}\right)^{x_n/3} \left(\frac{\delta_n n^2}{x_n^2}\right)^{1/3} \right\rceil \lesssim \left\lceil \delta_n \left(\frac{p_n}{q_n}\right)^{x_n/3} \right\rceil.$$

Case 2: $x_n > n/2$. In this case, we go back to (7.10). Note that for $x_n > n/2$,

$$|\Delta_{n,1}| = \sqrt{p_n}\left(\sin\frac{x_n \pi}{n+1} - \sin\frac{(x_n+1)\pi}{n+1}\right) + \left(\sqrt{q_n} - \sqrt{p_n}\right)\sin\frac{x_n \pi}{n+1},$$

where the right side is a sum of positive terms. In a few computations, one can show that for $p_n < 1/4$ or $x_n < 3n/4$,

$$\left(\sqrt{q_n} - \sqrt{p_n}\right)\sin\frac{x_n \pi}{n+1} \gtrsim \frac{1}{n},$$

and for $p_n \geqslant 1/4$ and $x_n \geqslant 3n/4$,

$$\sqrt{p_n}\left(\sin\frac{x_n \pi}{n+1} - \sin\frac{(x_n+1)\pi}{n+1}\right) \asymp \frac{1}{n}.$$

Thus, $|\Delta_{n,1}| \gtrsim n^{-1}$. Applying this result with (7.10), (7.23) and (7.24) gives

$$\psi_{n,1}^2(x_n) \gtrsim \frac{1}{\delta_n^3}\left(\frac{q_n}{p_n}\right)^{x_n}.$$

Moreover, using the fact $\log(1+t) \geqslant \frac{1}{2}(t \wedge 1)$, we have

$$\log \psi_{n,1}^2(x_n) \geqslant x_n \log \frac{q_n}{p_n} + O(\log \delta_n) \gtrsim n\big[(\delta_n/n) \wedge 1\big] + O(\log \delta_n), \tag{7.28}$$

where the most right summation tends to infinity. This implies that for any $C > 0$, $j_n^c(C) = 1$ as n large enough. Then, Step 5 is an immediate result of (7.27).

Step 6: For $C > 0$ and $\gamma \in \{c, d\}$,

$$\tau_n^\gamma(C) \geqslant \frac{x_n \log(q_n/p_n) + O(\log\log(q_n/p_n)^{x_n})}{2\lambda_{n,1}^\gamma}.$$

To prove this inequality, we set, for $n \geqslant 1$,

$$\ell_n = \frac{n}{x_n}\left(x_n \log \frac{q_n}{p_n}\right)^{-1/2}.$$

Using the first inequality of (7.27), one can show that

$$\forall C > 0, \quad \delta_n\left(\frac{p_n}{q_n}\right)^{x_n/3} = o(\ell_n), \qquad \ell_n = o\left(\frac{n}{x_n}\right). \tag{7.29}$$

As the proof for Step 5, we consider the following two cases.

Case 1: $x_n \leqslant n/2$. An immediate result of (7.29) is that for any $C > 0$,

$$j_n^c(C) \leqslant \lceil \ell_n \rceil \leqslant \frac{n}{2x_n}, \quad \text{for } n \text{ large enough.} \tag{7.30}$$

Putting this fact with (7.23), (7.26) and (7.27) together gives

$$\begin{aligned}\tau_n^c(C) &\geqslant \frac{\log \sum_{i=0}^{\lceil \ell_n \rceil} |\psi_{n,i}^c(x_n)|^2}{2\lambda_{n,\lceil \ell_n \rceil}^c} \geqslant \frac{x_n \log(q_n/p_n) + O(\log\log(q_n/p_n)^{x_n})}{2\lambda_{n,1}^c(1 + O((\lceil \ell_n \rceil - 1)^2/\delta_n^2))} \\ &= \frac{x_n \log(q_n/p_n) + O(\log\log(q_n/p_n)^{x_n})}{2\lambda_{n,1}^c} \sim \frac{x_n \log(q_n/p_n)}{2\lambda_{n,1}^c}.\end{aligned} \tag{7.31}$$

For discrete time chains, one can compute without difficulty that

$$\lambda_{n,2i-1}^d = \lambda_{n,2i}^d = -\log\left(2\sqrt{p_n q_n} \cos \frac{i\pi}{n+1}\right) = \lambda_{n,1}^d\left(1 + O\left(\frac{(i-1)^2}{\delta_n^2}\right)\right) \tag{7.32}$$

where O is uniformly for $1 \leqslant i \leqslant n/x_n$. Applying this fact with (7.20) and (7.31), we have

$$\tau_n^d(C) \geqslant \frac{\sum_{i=0}^{2\lceil \ell_n \rceil} |\psi_{n,i}^d(x_n)|^2}{2\lambda_{n,2\lceil \ell_n \rceil}^d} \geqslant \frac{\sum_{i=0}^{\lceil \ell_n \rceil} |\psi_{n,i}^c(x_n)|^2}{2\lambda_{n,1}^d(1+O((\lceil \ell_n \rceil - 1)^2/\delta_n^2))}$$
$$= \frac{x_n \log(q_n/p_n) + O(\log\log(q_n/p_n)^{x_n})}{2\lambda_{n,1}^d} \sim \frac{x_n \log(q_n/p_n)}{2\lambda_{n,1}^d}. \tag{7.33}$$

Case 2: $x_n > n/2$. It has been shown in Case 2 of Step 5 that for any $C > 0$, $j_n^\gamma(C) = 1$ for n large enough. Then, by (7.28), we have

$$\tau_n^\gamma(C) \geqslant \frac{\log(\psi_{n,1}^2(x_n))}{2\lambda_{n,1}^\gamma} = \frac{x_n \log(q_n/p_n) + O(\log(x_n\delta_n/n))}{2\lambda_{n,1}^\gamma}.$$

This proves Step 6 using the first inequality of (7.27).

To determine the existence of the L^2-cutoff using Theorems 5.1 and 5.3, we have to compute $\lambda_{n,j_n^\gamma(C)}^\gamma$. Using (7.23) and (7.32), one can show that

$$\lambda_{n,i}^\gamma \sim \lambda_{n,1}^\gamma \quad \text{uniform for } 1 \leqslant i \leqslant Cn/x_n,\ \gamma \in \{c,d\}$$

where C is any positive constant. By Step 5 and (7.29), it is easy to see that $j_n^\gamma(C) \lesssim n/x_n$. Putting these two results together gives

$$\lambda_{n,j_n^\gamma(C)}^\gamma \sim \lambda_{n,1}^\gamma, \quad \forall C > 0.$$

Then, by Step 6, we obtain that for $\gamma \in \{c,d\}$,

$$\tau_n^\gamma(C)\lambda_{n,j_n^\gamma(C)}^\gamma \sim \tau_n^\gamma(C)\lambda_{n,1}^\gamma \gtrsim x_n \log\left(\frac{q_n}{p_n}\right) \to \infty \quad \text{as } n \to \infty$$

and

$$\sum_{i=1}^{j_n^\gamma(C)-1} \left|\psi_{n,i}^\gamma(x_n)\right|^2 e^{-2\lambda_{n,i}^\gamma \tau_n^\gamma(C)} \leqslant Ce^{-2\lambda_{n,1}^\gamma \tau_n^\gamma(C)} \to 0 \quad \text{as } n \to \infty.$$

As a consequently of Theorems 5.1 and 5.3, the family $\{p_n^\gamma(t,x_n,\cdot)\colon n \geqslant 1\}$ has a $(\tau_n^\gamma(C), l_n^\gamma)$–$L^2$-cutoff with

$$l_n^c = \left(\lambda_{n,1}^c\right)^{-1} \log\left(\tau_n^c(C)\lambda_{n,1}^c\right) \tag{7.34}$$

and

$$l_n^d = \max\left\{1, \left(\lambda_{n,1}^d\right)^{-1} \log\left(\tau_n^d(C)\lambda_{n,1}^d\right)\right\}. \tag{7.35}$$

This proves the sufficiency of the L^2-cutoff for cases $R(3,5)$, $R(4,5)$ and $R(5,5)$. In the next step, we make a detailed computation on the L^2-cutoff times and cutoff windows yielded above.

Step 7: For $C > 0$ and $\gamma \in \{c, d\}$,

$$\tau_n^\gamma(C) = \frac{x_n \log(q_n/p_n) + O(\log\log(q_n/p_n)^{x_n})}{2\lambda_{n,1}^\gamma}.$$

By Step 6, it remains to give an adequate upper bound for $\tau_n^\gamma(C)$. Obviously, by (7.2) and (7.21), we have

$$\psi_{n,i}^2(x_n) \leqslant \frac{2}{(n+1)\pi_n(x_n)} \leqslant \frac{1}{\delta_n}\left(\frac{q_n}{p_n}\right)^{x_n} \quad \forall 1 \leqslant i \leqslant n.$$

This implies for $\gamma \in \{c, d\}$,

$$\tau_n^\gamma(C) \leqslant \max_{j_n^\gamma(C) \leqslant i \leqslant n} \left\{\frac{x_n \log(q_n/p_n) + \log((i+1)/\delta_n)}{2\lambda_{n,i}^\gamma}\right\}. \tag{7.36}$$

We consider two subcases concerning the value of p_n. In the case $p_n < 1/4$, or equivalently $\delta_n \geqslant n/4$, it is obvious from (7.36) that

$$\forall C > 0,\ \gamma \in \{c, d\}, \quad \tau_n^\gamma(C) \leqslant \frac{x_n \log(q_n/p_n) + \log 4}{2\lambda_{n,1}^\gamma}.$$

In the case $p_n \geqslant 1/4$, one can show that there is a constant $N > 0$ such that, for n large enough,

$$\lambda_{n,i}^\gamma \geqslant \lambda_{n,1}^\gamma\left(1 + \frac{i^2 - 1}{N\delta_n^2}\right) \quad \forall 1 \leqslant i \leqslant n,\ \gamma \in \{c, d\}.$$

Using this fact, we may prove that for $\frac{x_n\delta_n^2}{n} < i \leqslant n$,

$$\begin{aligned}\frac{x_n \log(q_n/p_n) + \log((i+1)/\delta_n)}{2\lambda_{n,i}^\gamma} &\leqslant \frac{x_n \log(q_n/p_n)}{2\lambda_{n,1}^\gamma} \times \max_{x_n\delta_n^2/n < i \leqslant n}\left\{\frac{N\log((i+1)/\delta_n)}{(i^2-1)/\delta_n^2}\right\} \\ &= o\left(\frac{x_n \log(q_n/p_n)}{\lambda_{n,1}^\gamma}\right).\end{aligned}$$

Moreover, for $1 \leqslant i \leqslant \frac{x_n\delta_n^2}{n}$,

$$\frac{x_n \log(q_n/p_n) + \log((i+1)/\delta_n)}{2\lambda_{n,i}^\gamma} \leqslant \frac{x_n \log(q_n/p_n) + \log(x_n\delta_n/n))}{2\lambda_{n,1}^\gamma}.$$

Consequently, we get

$$\tau_n^\gamma(C) \leqslant \frac{x_n \log(q_n/p_n) + O(\log(x_n\delta_n/n))}{2\lambda_{n,1}^\gamma}.$$

This proves Step 7 since $\delta_n x_n/n = O((q_n/p_n)^{x_n})$.

Next, we use Step 7 and the conclusion in the end of Step 6 to determine the desired cutoff times and cutoff windows. Set, for $n \geqslant 1$,

$$v_n^\gamma = \frac{x_n \log(q_n/p_n)}{2\lambda_{n,1}^\gamma}, \quad \gamma \in \{c, d\},$$

and

$$w_n^c = \frac{\log\log(q_n/p_n)^{x_n}}{\lambda_{n,1}^c}, \qquad w_n^d = \max\left\{1, \frac{\log\log(q_n/p_n)^{x_n}}{\lambda_{n,1}^d}\right\}.$$

In Step 6, the windows for the L^2-mixing time in (7.34) and (7.35) satisfy

$$l_n^\gamma \asymp w_n^\gamma, \quad \gamma \in \{c, d\}.$$

By Step 7, this derives $\tau_n^\gamma(C) = v_n^\gamma + O(w_n^\gamma)$ and then, by [5, Corollary 2.5(v)], the family $p_n^\gamma(t, x_n, \cdot)$ has a (v_n^γ, w_n^γ)–L^2-cutoff. Consider the following identities.

$$\begin{cases} \lambda_{n,1}^c = \big(1 - 2\sqrt{p_n q_n}\big)\big(1 + O\big(\sqrt{p_n}\delta_n^{-2}\big)\big), \\ \lambda_{n,1}^d = \big(-\log(4p_n q_n)\big)\left(\dfrac{1}{2} + O\big(\delta_n^{-2}\big)\right). \end{cases} \tag{7.37}$$

Let $t_n^c, t_n^d, b_n^c, b_n^d$ be as in Theorem 7.2. Then, (7.37) implies

$$v_n^c = t_n^c + O\left(\frac{\sqrt{p_n}x_n \log(q_n/p_n)}{\delta_n^2(1 - 2\sqrt{p_n q_n})}\right), \qquad v_n^d = t_n^d + O\left(\max\left\{1, \frac{x_n \log(q_n/p_n)}{-\delta_n^2 \log(4p_n q_n)}\right\}\right).$$

Observe that if $p_n \asymp 1$, then

$$\log\frac{q_n}{p_n} = \log\left(1 + \frac{2\delta_n}{p_n n}\right) \asymp \frac{\delta_n}{n},$$

and if $p_n = o(1)$, then $\delta_n \sim n/2$ and

$$\log\frac{q_n}{p_n} \asymp -\log 4p_n q_n = o\big(1/\sqrt{p_n}\big).$$

This implies

$$\frac{\sqrt{p_n}x_n \log(q_n/p_n)}{\delta_n^2(1 - 2\sqrt{p_n q_n})} \lesssim \frac{1}{\delta_n(1 - 2\sqrt{p_n q_n})} = o\big(w_n^c\big)$$

and

$$\frac{x_n \log(q_n/p_n)}{-\delta_n^2 \log(4p_n q_n)} \lesssim \frac{1}{\delta_n}\max\left\{1, \frac{1}{-\log(4p_n q_n)}\right\} = o\big(w_n^d\big).$$

Thus, $v_n^\gamma = t_n^\gamma + O(w_n^\gamma)$ for $\gamma \in \{c, d\}$. Again, by [5, Corollary 2.5(v)], the family $p_n^\gamma(t, x_n, \cdot)$ presents a (t_n^γ, w_n^γ)–L^2-cutoff.

To see b_n^γ is the desired cutoff window, note that in case $R(3,5)$, $R(4,5)$ and $R(5,5)$, it is assumed $\delta_n \to \infty$. By (7.37), one has $w_n^c \asymp b_n^c$ and

$$w_n^d \asymp b_n^d \quad \text{if } \log x_n \gtrsim \log\left(\frac{1}{p_n}\right).$$

As a consequence of [5, Corollary 2.5(v)], $p_n^c(t, x_n, \cdot)$ presents a (t_n^c, b_n^c)–L^2-cutoff and, in case $\log x_n \gtrsim \log(1/p_n)$, the family $p_n^d(t, x_n, \cdot)$ presents a (t_n^d, b_n^d)–L^2-cutoff. It remains to consider the discrete time case with the condition $\log x_n = o(\log(1/p_n))$. In cases $R(3,5)$, $R(4,5)$ and $R(5,5)$, this can happen only if $p_n \to 0$ and $x_n = o(1/p_n)$. Recall the L^2-distance in (5.6) as follows.

$$\left[D_{n,2}^d(x_n, t)\right]^2 = \sum_{j=1}^{n} \frac{2\sin^2\left(\frac{jx_n}{n+1} + \theta_{n,j}\right)\pi}{(n+1)\pi_n(x_n)} \left(4p_n q_n \cos^2 \frac{j\pi}{n+1}\right)^t,$$

where $\theta_{n,j} \in (0, 1/2)$ is the term satisfying (7.22). In the assumption $p_n \to 0$, it is clear that

$$\pi_n(x_n) \sim \left(\frac{p_n}{q_n}\right)^{x_n}, \qquad \theta_{n,j} \sim 1 \quad \text{uniformly for } 1 \leqslant j \leqslant n. \tag{7.38}$$

By setting $s_n = \frac{x_n \log(q_n/p_n)}{-\log(4p_n q_n)}$, the former identity of (7.38) implies $\pi_n(x_n) \sim (4p_n q_n)^{s_n}$ and then for any $\epsilon \in (0, 1/2)$,

$$\sum_{j=[\epsilon n]}^{[(1-\epsilon)n]} \frac{2\sin^2\left(\frac{jx_n}{n+1} + \theta_{n,j}\right)\pi}{(n+1)\pi_n(x_n)} \left(4p_n q_n \cos^2 \frac{j\pi}{n+1}\right)^{s_n} = o(1).$$

Thus, we have

$$\lim_{n\to\infty} \left[D_{n,2}^d(x_n, s_n)\right]^2 \leqslant 2\epsilon + o(1) \quad \forall \epsilon \in (0, 1/2)$$

which yields $D_{n,2}^d(x_n, s_n) \to 0$ as $n \to \infty$. To see $D_{n,2}^d(x_n, s_n - 1)$, it loses no generality to assume that $\lim_{n\to\infty} x_n/n = c \in [0,1]$. For $c \in (0, 1/2]$, we have

$$\begin{aligned}\left[D_{n,2}^d(x_n, s_n - 1)\right]^2 &\gtrsim \frac{1}{p_n} \sum_{j=[1/(4c)]}^{[3/(4c)]} \frac{2\sin^2\left(\frac{jx_n}{n+1} + \theta_{n,j}\right)\pi}{n} \left(\cos \frac{j\pi}{n+1}\right)^{2(s_n-1)} \\ &\asymp \frac{1}{np_n} \asymp \frac{1}{x_n p_n} \to \infty\end{aligned}$$

where the second asymptote uses the second identity in (7.37). In a similar reasoning, if $c = 0$, one has

$$\left[D^d_{n,2}(x_n, s_n - 1)\right]^2 \gtrsim \frac{1}{p_n} \sum_{j=1}^{[n/(2x_n)]} \frac{2\sin^2\left(\frac{jx_n}{n+1} + \theta_{n,j}\right)\pi}{n} \left(\cos \frac{j\pi}{n+1}\right)^{2(s_n-1)}$$
$$\asymp \frac{1}{x_n p_n} \to \infty.$$

The proof for $c \in (1/2, 1]$ is almost the same using the symmetry of sine and cosine functions and, consequently, we achieve $D^d_{n,2}(x_n, s_n - 1) \to \infty$. This proves the desired cutoff.

Step 8: In case $R(5, 2)$, that is, $p_n \to 0$ and $x_n \asymp 1$, we prove the existence of the L^2-cutoff and determine a cutoff time and a cutoff window by computing the L^2-distance in detail instead of using Theorems 5.1 and 5.3. First, let $D^\gamma_{n,2}(x_n, t)$, $\gamma \in \{c, d\}$, be the L^2-distance of the nth chain at time t starting from x_n. Using (5.4), (5.6) and (7.21), one can derive

$$\left(D^c_{n,2}(x_n, t)\right)^2 = \sum_{j=1}^{n} \frac{2\sin^2\left(\frac{jx_n}{n+1} + \theta_{n,j}\right)\pi}{(n+1)\pi_n(x_n)} \exp\left\{-2t\left(1 - 2\sqrt{p_n q_n} \cos \frac{j\pi}{n+1}\right)\right\}$$

and

$$\left(D^d_{n,2}(x_n, t)\right)^2 = \sum_{j=1}^{n} \frac{2\sin^2\left(\frac{jx_n}{n+1} + \theta_{n,j}\right)\pi}{(n+1)\pi_n(x_n)} \left(2\sqrt{p_n q_n} \cos \frac{j\pi}{n+1}\right)^{2t}.$$

Using the second part of (7.38) and the fact $x_n \asymp 1$, we have, for any $M \geqslant 0$,

$$\sum_{j=1}^{n} \frac{\sin^2\left(\frac{jx_n}{n+1} + \theta_{n,j}\right)\pi}{n+1} \left(\cos \frac{j\pi}{n+1}\right)^{Mx_n} \asymp \sum_{j=1}^{n} \frac{\sin^2\left(\frac{jx_n}{n+1} + \theta_{n,j}\right)\pi}{n+1} \asymp 1.$$

Putting all above together, we obtain

$$D^c_{n,2}\left(x_n, \frac{x_n \log(q_n/p_n) + c}{2(1 - 2\sqrt{p_n q_n})}\right) \asymp e^{-c} \quad \forall c \in \mathbb{R}$$

and

$$D^d_{n,2}(x_n, x_n + c) \asymp p_n^{-c/2} \quad \forall c \in \mathbb{Z},\ x_n + c \geqslant 0.$$

Consequently, the continuous time family has a strongly optimal $(\frac{x_n \log(q_n/p_n)}{2(1-2\sqrt{p_n q_n})}, 1)$–$L^2$-cutoff and the discrete time family has a (x_n, c_n)–L^2-cutoff where $(c_n)_1^\infty$ is any sequence of positive numbers tending to 0. The desired cutoff for discrete time cases is obtained due to the facts

$$0 < t^d_n - x_n = \frac{x_n \log(4q_n^2)}{-\log(4p_n q_n)} = o(1), \qquad b^d_n \asymp \frac{1}{\log(1/p_n)} = o(1). \qquad \square$$

7.2. Countable chains

In this section, we consider birth and death chains on $\Omega = \{0, 1, 2, \dots\}$ with transition functions $p^\gamma(t, \cdot, \cdot)$, $\gamma \in \{c, d\}$, associated with the kernel

$$K(x, y) = \begin{cases} p & \text{if } y = x + 1, \\ q & \text{if } y = x - 1 \text{ or } x = y = 0. \end{cases} \tag{7.39}$$

Let π be a probability measurable on Ω given by

$$\forall x \in \Omega, \quad \pi(x) = \frac{q - p}{q} \left(\frac{p}{q}\right)^x.$$

Here, we assume that $p < 1/2$ so that there exists a unique invariant probability measure, which is equal to π, associated with $p^\gamma(t, \cdot, \cdot)$. In order to investigate the L^2-cutoff for families of birth and death chains on Ω using Theorems 5.1 and 5.3, one has to compute the spectral information of this infinite chain and this is given in [11]. Here, we consider another approach without the uses of spectral information but establishing a relationship on the L^2-distances between the distributions of finite and infinite chains and their stationarity. This is the main thought in this section and is realized in the following lemma.

Lemma 7.3. *Let $p^\gamma(t, \cdot, \cdot)$, $\gamma \in \{c, d\}$, be the Markov transition functions given by* (7.39) *with $p < 1/2$. For $m \geqslant 1$, let $p^\gamma_m(t, \cdot, \cdot)$ be a birth and death chain on $\Omega_m = \{0, 1, \dots, m\}$ with constant birth rate p and constant death rate $q = 1 - p$. For $x \in \Omega$ and $y \in \Omega_m$, let $D^\gamma(x, t)$ and $D^\gamma_m(y, t)$ be the L^2-distances associated with $p^\gamma(t, x, \cdot)$ and $p^\gamma_m(t, y, \cdot)$. Then, for $t > 0$ and $x \geqslant 0$ such that $m \geqslant x + t$,*

$$\left(D^d_m(x, t)\right)^2 + 1 = \left[\left(D^d(x, t)\right)^2 + 1\right] \times \left[1 - (p/q)^{m+1}\right]^2,$$
$$\left(D^d(x, t)\right)^2 - \left(D^d_m(x, t)\right)^2 = \left[\left(D^d(x, t)\right)^2 + 1\right](p/q)^{m+1}\left[2 - (p/q)^{m+1}\right].$$

Moreover, for $m \geqslant x$,

$$\left|\left(D^c(x, t)\right)^2 - \left(D^c_m(x, t)\right)^2\right| \leqslant 6\left[\left(D^d(x, 0)\right)^2 + 1\right]\left((p/q)^{m+1} + e^{-t} \sum_{j > m - x} \frac{t^j}{j!}\right).$$

Proof. Let π, π_m be the invariant probabilities associated with $p(t, \cdot, \cdot)$, $p_m(t, \cdot, \cdot)$. Then, the first and second identities follow immediately from the fact $\pi(y) = \pi_m(y)[1 - (p/q)^{m+1}]$ for $0 \leqslant y \leqslant m$ and

$$\left(D^d(x, t)\right)^2 = \sum_{y=0}^{x+t} \left(p^d(t, x, y)\right)^2 / \pi(y) - 1,$$

$$\left(D^d_m(x, t)\right)^2 = \sum_{y=0}^{x+t} \left(p^d_m(t, x, y)\right)^2 / \pi_m(y) - 1.$$

To see the last inequality, set $A=\{0,1,\dots,m-x\}$ and $A^c=\Omega\setminus A$. A simple computation shows that

$$\big(p^c(t,x,y)\big)^2=\bigg(\sum_{i\in A}e^{-t}\frac{t^i}{i!}p^d(i,x,y)\bigg)^2+\bigg(\sum_{j\in A^c}e^{-t}\frac{t^j}{j!}p^d(j,x,y)\bigg)^2 + 2e^{-2t}\sum_{i\in A,\,j\in A^c}\frac{t^{i+j}}{i!j!}p^d(i,x,y)p^d(j,x,y).$$

For the second and third terms on the right side, we may prove using Jensen's and Cauchy inequality that

$$\bigg(\sum_{i\in A^c}e^{-t}\frac{t^i}{i!}p^d(i,x,y)\bigg)^2\leqslant\bigg(\sum_{i\in A^c}e^{-t}\frac{t^i}{i!}\bigg)\sum_{j\in A^c}e^{-t}\frac{t^j}{j!}\big(p^d(j,x,y)\big)^2$$

and

$$\bigg(\sum_{y\in\Omega}\frac{p^d(i,x,y)p^d(j,x,y)}{\pi(y)}\bigg)^2\leqslant\bigg(\sum_{y\in\Omega}\frac{(p^d(i,x,y))^2}{\pi(y)}\bigg)\bigg(\sum_{y\in\Omega}\frac{(p^d(j,x,y))^2}{\pi(y)}\bigg).$$

This implies

$$\bigg|\big(D^c(x,t)\big)^2+1-\sum_{y\in\Omega}\bigg(\sum_{i\in A}e^{-t}\frac{t^i}{i!}p^d(i,x,y)\bigg)^2\frac{1}{\pi(y)}\bigg| \leqslant\big[\big(D^d(x,0)\big)^2+1\big]\sum_{i\in A^c}e^{-t}\frac{t^i}{i!}.$$

Similarly, for the transition functions $p^c_m(t,\cdot,\cdot)$ and $p^d_m(t,\cdot,\cdot)$, we have

$$\bigg|\big(D^c_m(x,t)\big)^2+1-\sum_{y\in\Omega_m}\bigg(\sum_{i\in A}e^{-t}\frac{t^i}{i!}p^d_m(i,x,y)\bigg)^2\frac{1}{\pi_m(y)}\bigg| \leqslant 3\big[\big(D^d_m(x,0)\big)^2+1\big]\sum_{i\in A^c}e^{-t}\frac{t^i}{i!}.$$

Note that

$$\text{for } m\geqslant x,\quad D^d(x,0)=D^d_m(x,0)=\frac{1}{\sqrt{\pi_m(x)}}\leqslant\frac{1}{\sqrt{\pi(x)}}=D^d(x,0)$$

and

$$p^d(i,x,y)=\begin{cases}p^d_m(i,x,y) & \text{for } i\in A,\ y\leqslant m,\\ 0 & \text{for } i\in A,\ y>m.\end{cases}$$

Putting all above together and applying the triangle inequality gives

$$\begin{aligned}\left|\left(D^c(x,t)\right)^2-\left(D^c_m(x,t)\right)^2\right| &\leqslant (p/q)^{m+1}\sum_{y\in\Omega}\left(\sum_{i\in A}e^{-t}\frac{t^i}{i!}p^d(i,x,y)\right)^2\frac{1}{\pi(y)}\\ &\quad+6\left[\left(D^d(x,0)\right)^2+1\right]\sum_{j\in A^c}e^{-t}\frac{t^j}{j!}\\ &\leqslant 6\left[\left(D^d(x,0)\right)^2+1\right]\left((p/q)^{m+1}+\sum_{j\in A^c}e^{-t}\frac{t^j}{j!}\right),\end{aligned}$$

where the last inequality uses Jensen's inequality on the summation w.r.t. i. □

The next theorem concerns birth and death chains on non-negative integers and contains Theorem 7.2.

Theorem 7.4. *Let Ω be the set of non-negative integers. For $n \geqslant 1$, let $p_n \in (0, 1/2)$, $q_n = 1 - p_n$ and let $p^\gamma_n(t,\cdot,\cdot)$, $\gamma \in \{c, d\}$, be the continuous or discrete time Markov transition function on Ω associated with* (7.39) *for $p = p_n$. Then, the family $p^\gamma_n(t, x_n, \cdot)$ with $x_n \in \Omega$ has an L^2-cutoff if and only if* (7.5) *holds. Moreover, if* (7.5) *holds, then for $\gamma \in \{c, d\}$, $p^\gamma_n(t, x_n, \cdot)$ has a (t^γ_n, b^γ_n)–L^2-cutoff, where t^γ_n and b^γ_n are as defined in Theorem* 7.2*.*

Proof of Theorem 7.4. For $n \geqslant 1$ and $\gamma \in \{c, d\}$, let $D^\gamma_n(t)$ be the L^2-distance between $p^\gamma_n(t, x_n, \cdot)$ and its stationary distribution. Let t^γ_n be as in Theorem 7.2 and set

$$s_n = \inf\left\{t > 0\colon\ D^\gamma_n(t) \leqslant 1,\ \gamma \in \{c, d\}\right\} + t^c_n.$$

Note that, for $n \geqslant 1$, we may choose $m_n \geqslant \max\{x_n, m_{n-1} + 1\}$ with $m_0 = 0$ such that

$$\lim_{n\to\infty}\left[\left(D^d_n(x_n,0)\right)^2+1\right]\left((p_n/q_n)^{m_n}+e^{-2s_n}\sum_{i=m_n-x_n}^{\infty}\frac{(2s_n)^i}{i!}\right)=0.$$

Let $\widetilde{p}^d_n(t,\cdot,\cdot)$ be the transition function on Ω_{m_n} satisfying

$$\forall x, y \in \Omega_{m_n},\quad \widetilde{p}^d_n(1,x,y)=\begin{cases} p^d_n(1,x,y) & \text{if } (x,y)\neq(m_n,m_n),\\ p_n & \text{if } (x,y)=(m_n,m_n),\end{cases}$$

and let $\widetilde{p}^c_n(t,\cdot,\cdot)$ be the associated continuous time chain. Let $\widetilde{D}^\gamma_n(t)$ be the L^2-distance between $\widetilde{p}^\gamma_n(t, x_n, \cdot)$ and its stationary distribution. In the above setting, one may prove using Lemma 7.3 that for $\tau \in \{c, d\}$,

$$\sup_{0\leqslant t\leqslant 2s_n}\left|D^\gamma_n(t)-\widetilde{D}^\gamma_n(t)\right| = o(1). \tag{7.40}$$

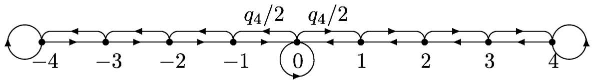

Fig. 1. This is the graph associated with K_4 in (8.2). For undefined arrows, those away from 0 and the loops at 4, −4 have weight q_4, whereas those into 0 and the loop at 0 have weight p_4.

If (7.5) holds true, then by Theorem 7.2, the family $\widetilde{P}_n^\gamma(t, x_n, \cdot)$ has a (t_n^γ, b_n^γ)–L^2-cutoff. Since $t_n^d \leqslant t_n^c \leqslant s_n$, (7.40) implies that $p_n^\gamma(t, x_n, \cdot)$ also has a (t_n^γ, b_n^γ)–L^2-cutoff. Conversely, assume that the family $p_n^\gamma(t, x_n, \cdot)$ has an L^2-cutoff with cutoff time $\bar{s}_n^\gamma$. In this case, it is clear that

$$\limsup_{n\to\infty} \frac{\bar{s}_n^\gamma}{s_n} \leqslant 1, \quad \text{for } \gamma \in \{c, d\}.$$

By (7.5), this implies that $\widetilde{p}_n^\gamma(t, x_n, \cdot)$ also has an L^2-cutoff. As a consequence of Theorem 7.2, we obtain (7.5). This proves Theorem 7.4. □

8. A peak/valley example

Recall that, by Proposition 4.1, a family of ergodic Markov processes has an L^2-cutoff if

$$\lim_{n\to\infty} T_{n,2}(\mu_n, \epsilon)\lambda_n = \infty. \tag{8.1}$$

In the normal case, this sufficient condition for an L^2-cutoff can be regarded as a special case of Theorems 5.1 and 5.3 with $j_n(C) = 1$. However, it is possible that an L^2-cutoff exists but (8.1) fails. That is, (8.1) is not a necessary condition. This is illustrated by the examples in this section.

Consider the following birth and death chain. Let n be a positive integer and K_n be the Markov kernel on $\Omega_n = \{-n, \dots, -1, 0, 1, \dots, n\}$ defined by

$$\begin{aligned} &K_n(-i, -j) = K_n(i, j), \quad \forall i \geqslant 0, j \geqslant 0, \\ &K_n(i, i+1) = K_n(n, n) = q_n, \quad \forall 0 < i < n,\ K_n(0, 1) = q_n/2, \\ &K_n(i+1, i) = K_n(0, 0) = p_n, \quad \forall 0 \leqslant i < n, \end{aligned} \tag{8.2}$$

where $p_n + q_n = 1$. See Fig. 1 for an example of $n = 4$.

Obviously, K_n has invariant probability

$$\pi_n(0) = c_n, \qquad \pi_n(x) = \frac{c_n}{2}\left(\frac{q_n}{p_n}\right)^{|x|}, \quad \forall x \neq 0 \tag{8.3}$$

where

$$c_n = \begin{cases} (1 - q_n/p_n)[1 - (q_n/p_n)^{n+1}]^{-1} & \text{if } p_n \neq q_n, \\ 1/(n+1) & \text{if } p_n = q_n. \end{cases}$$

Using the method in [9, Chapter XVI], K_n has eigenvalues $\beta_{n,0} = 1$ and

$$\beta_{n,1} = \begin{cases} \sqrt{p_n q_n}(a_n + a_n^{-1}) & \text{if } p_n/q_n \leqslant n^2/(n+1)^2, \\ 2\sqrt{p_n q_n}\cos\theta_{n,1} & \text{if } p_n/q_n > n^2/(n+1)^2 \end{cases} \tag{8.4}$$

and

$$\beta_{n,l} = \begin{cases} 2\sqrt{p_n q_n}\cos\theta_{n,j} & \text{if } l = 2j-1 \text{ and } 2 \leqslant j \leqslant n, \\ 2\sqrt{p_n q_n}\cos\frac{j\pi}{n+1} & \text{if } l = 2j \text{ and } 1 \leqslant j \leqslant n \end{cases} \tag{8.5}$$

where, for $1 \leqslant j \leqslant n$, $\theta_{n,j}$ is a solution to

$$\frac{\sin n\theta}{\sin(n+1)\theta} = \sqrt{\frac{p_n}{q_n}}, \quad \theta \in \left(\frac{(j-1)\pi}{n}, \frac{j\pi}{n+1}\right)$$

and a_n solves $f_n(t) = \sqrt{p_n/q_n}$ with

$$f_n(t) = \begin{cases} \frac{t^n - t^{-n}}{t^{n+1} - t^{-n-1}} & \text{if } t \notin \{0, \pm 1\}, \\ 0 & \text{if } t = 0, \\ \frac{n}{n+1} & \text{if } t = 1, \\ \frac{-n}{n+1} & \text{if } t = -1. \end{cases} \tag{8.6}$$

Let $\psi_{n,i}$ be a normalized (in $L^2(\pi_n)$) eigenvector for K_n associated with $\beta_{n,i}$. Then, $\psi_{n,0} = \mathbf{1}$ is the constant function with value 1 and

$$\psi_{n,1}(x) = C_{n,1}\left(\frac{p_n}{q_n}\right)^{|x|/2} \begin{cases} a_n^x - a_n^{-x} & \text{if } p_n/q_n < n^2/(n+1)^2, \\ x & \text{if } p_n/q_n = n^2/(n+1)^2, \\ \sin x\theta_{n,1} & \text{if } p_n/q_n > n^2/(n+1)^2 \end{cases}$$

and

$$\psi_{n,2j-1}(x) = C_{n,2j-1}\left(\frac{p_n}{q_n}\right)^{|x|/2} \sin x\theta_{n,j}, \quad 2 \leqslant j \leqslant n,$$

and

$$\psi_{n,2j}(x) = C_{n,2j}\left(\frac{p_n}{q_n}\right)^{|x|/2} \left\{\sqrt{q_n}\sin\frac{j(|x|+1)\pi}{n+1} - \sqrt{p_n}\sin\frac{j|x|\pi}{n+1}\right\}$$

where

$$C_{n,1}^{-2} = c_n \begin{cases} [\frac{a_n^{2n+1} - a_n^{-2n-1}}{a_n - a_n^{-1}} - (2n+1)] & \text{if } p_n/q_n < n^2/(n+1)^2, \\ n(n+1)(2n+1)/6 & \text{if } p_n/q_n = n^2/(n+1)^2, \\ \frac{1}{2}[n - \frac{\sin n\theta_{n,1}\cos(n+1)\theta_{n,1}}{\sin\theta_{n,1}}] & \text{if } p_n/q_n > n^2/(n+1)^2 \end{cases}$$

and

$$C_{n,2j-1}^{-2} = \frac{c_n}{2}\left[n - \frac{\sin n\theta_{n,j}\cos(n+1)\theta_{n,j}}{\sin\theta_{n,j}}\right], \qquad C_{n,2j}^{-2} = \frac{c_n(n+1)(1-\beta_{n,2j})}{2}.$$

Clearly, $\beta_{n,i} \geqslant \beta_{n,i+1}$ for all $1 \leqslant i < 2n$ and

$$\max\{|\beta_{n,1}|, |\beta_{n,2n}|\} = |\beta_{n,1}| \quad \forall n.$$

Remark 8.1. Note that $1, \mathbf{1}$ and $\beta_{n,2j-1}(x), \psi_{n,2j-1}(x)$ with $1 \leqslant j \leqslant n$ and $x \in \{0, 1, \ldots, n\}$ are the eigenvalues and eigenfunctions of the transition matrix

$$\forall 1 \leqslant x < n, \quad K(x, x+1) = K(n, n) = q_n, \qquad K(x, x-1) = p_n, \qquad K(0, 0) = 1,$$

whereas $1, \mathbf{1}$ and $\beta_{n,2j}, \psi_{n,2j}(n-x)$ with $1 \leqslant j \leqslant n$ and $x \in \{0, 1, \ldots, n\}$ are the eigenvalues and eigenvectors for the transition matrix in (7.1).

With work, the above spectral information leads to the following result.

Theorem 8.1. *Let $\{(\Omega_n, K_n, \pi_n)\colon n = 1, 2, \ldots\}$ be the family introduced in* (8.2) *and let x_n be the initial state of the nth chain. Then, in the continuous and the discrete time cases, if $p_n \geqslant q_n$, the family has an L^2-cutoff if and only if*

$$|x_n|\left(\frac{p_n}{q_n} - 1\right) \to \infty. \tag{8.7}$$

If $p_n < q_n$, then the family has an L^2-cutoff if and only if

$$n(q_n - p_n) \to \infty \quad \text{and} \quad |x_n|(q_n - p_n) \to 0. \tag{8.8}$$

Moreover, if there is an L^2-cutoff, then the cutoff time is $t_n^\gamma(|x_n|)$ if $p_n > q_n$ and $t_n^\gamma(n - |x_n|)$ if $p_n < q_n$, where c and d represent for continuous time and discrete time cases and

$$t_n^c(x) = \frac{x|\log p_n - \log q_n|}{2(1 - \sqrt{p_n q_n})}, \qquad t_n^d(x) = \left\lfloor \frac{x|\log p_n - \log q_n|}{-\log(4 p_n q_n)} \right\rfloor.$$

Remark 8.2. A (non-optimal) window size can be obtained by arguments similar to those in Theorem 7.2. It is not included because it involves additional long computations.

Remark 8.3. As illustrated in (8.5) and (8.15), except perhaps for the second largest one, the eigenvalues of K_n are distributed in way that is very similar to those of the chains treated in Theorem 7.2. In the case $p_n > q_n$, this is true even for the second largest eigenvalue. When $p_n < q_n$, however, the spectral gap $1 - \beta_{n,1}$ is of much smaller order than for the chain in Theorem 7.2 and $\beta_{n,1}$ is separated from the rest of the eigenvalues. In the latter case, it is easy to see from Theorem 8.1 and (8.15) that if there is an L^2-cutoff, then the cutoff time is of order smaller than the inverse of the spectral gap. This means the optimal window size is not directly related to the spectral gap but depends on the rest of the eigenvalues.

Remark 8.4. Theorem 8.1 covers two very different cases depending on whether $p_n >> q_n$ or $p_n << q_n$.

When $p_n > q_n$, the stationary distribution has a sharp peak at 0 and this case is not much different from the one treated in Theorem 7.2. The spectral gap is relatively large in this case (bounded away from 0 when $p_n/q_n > 1$ stays bounded away from 1). To reach stationarity, the walk must have a chance to visit the peak. To present a cutoff, the walk must start far enough from the peak, just as in Theorem 7.2.

When $p_n < q_n$, the stationary measure has a unique valley bottom at 0. In this case, to reach stationarity, the walk must have a chance to cross the bottom. The bottom creates a bottle neck which implies that the spectral gap $1-\beta_{n,1}$ is very close to 0 if $p_n/q_n < 1$ stays bounded away from 1. However, the rest of the spectrum (in the continuous time case, say, i.e., $1-\beta_{n,j}$, $j>1$) is bounded away from 0. In this case, there is no cutoff, except if one starts very close to 0 where the eigenvector associated with the spectral gap takes very small values. This illustrates one of the main feature of the central results of this paper: in order to understand the cutoff and the cutoff time from specified starting points, one may have to drop those eigenvalues (including possibly the spectral gap) whose eigenvectors take very small values at the specified starting points.

Before proving Theorem 8.1, we make some analysis on $1-\beta_{n,1}$, where $\beta_{n,1}$ is defined in (8.4). Set $p_n = 1/2 - \delta_n/n$ and assume first that $|\delta_n| = o(n)$. In the case $p_n/q_n > n^2/(n+1)^2$, the fact $\theta_{n,1} \in (0, \pi/(n+1))$ yields

$$1-\beta_{n,1} = 1 - 2\sqrt{p_nq_n} + 2\sqrt{p_nq_n}(1-\cos\theta_{n,1}) \asymp \begin{cases} \delta_n^2/n^2 + \theta_{n,1}^2 & \text{if } |\delta_n| = O(1), \\ \delta_n^2/n^2 & \text{if } |\delta_n| \to \infty. \end{cases} \tag{8.9}$$

In the subcase $|\delta_n| = O(1)$, one may use the following identity

$$\frac{\sin n\theta_{n,1}}{\sin(n+1)\theta_{n,1}} = \sqrt{\frac{p_n}{q_n}},$$

to derive

$$\sin n\theta_{n,1}\left(\sqrt{\frac{q_n}{p_n}} - 1\right) = \sin(n+1)\theta_{n,1} - \sin n\theta_{n,1} = \int_{n\theta_{n,1}}^{(n+1)\theta_{n,1}} \cos t\, dt. \tag{8.10}$$

This implies

$$\theta_{n,1} \in \begin{cases} (0, \pi/(2n+1) & \text{if } \delta_n > 0, \\ (\pi/(2n+1), \pi/(n+1)) & \text{if } \delta_n < 0. \end{cases} \tag{8.11}$$

Thus, by (8.9), if $|\delta_n| = O(1)$ and $\delta_n < 0$, then $1-\beta_{n,1} \asymp 1/n^2$. For the further subcase $|\delta_n| = O(1)$ and $\delta_n > 0$, consider the following computations.

$$\delta_n \asymp \frac{\sin n\theta_{n,1}}{\theta_{n,1}}\left(\sqrt{\frac{q_n}{p_n}} - 1\right) = \frac{1}{\theta_{n,1}} \int_{n\theta_{n,1}}^{(n+1)\theta_{n,1}} \cos t\, dt = \cos\widetilde{\theta}_n,$$

where $\widetilde{\theta}_n \in (n\theta_{n,1}, (n+1)\theta_{n,1})$. Note that the first asymptote uses the fact $\theta_{n,1} \in (0, \pi/(2n+1))$ whereas the first equality applies (8.10). Hence, if $\delta_n \to 0$ and $\delta_n > 0$, then $\widetilde{\theta}_n \to \frac{\pi}{2}$ or equivalently $n\theta_{n,1} \to \pi/2$. As a consequence of (8.9), if $|\delta_n| = O(1)$ and $\delta_n > 0$, then

$$1 - \beta_{n,1} \asymp \frac{\delta_n^2 + n^2\theta_{n,1}^2}{n^2} \asymp \frac{1}{n^2}.$$

In the case $p_n/q_n = n^2/(n+1)^2$, it is obvious that $\delta_n \sim 1/2$ and $1 - \beta_{n,1} = 1 - 2\sqrt{p_n q_n} \sim 2\delta_n^2/n^2 \sim 1/(2n^2)$. In the case $p_n/q_n < n^2/(n+1)^2$, let $a_n \in (0,1)$ be such that $f_n(a_n) = \sqrt{p_n/q_n}$, where $f_n(t)$ is the function in (8.6). That is,

$$\sqrt{\frac{p_n}{q_n}} = \frac{a_n^n - a_n^{-n}}{a_n^{n+1} - a_n^{-n-1}} = a_n - \frac{(1-a_n^2)a_n^{2n+1}}{1 - a_n^{2n+2}}.$$

Then, we have

$$0 < 1 - a_n < 1 - \sqrt{\frac{p_n}{q_n}} \sim \frac{2\delta_n}{n} \quad \text{as } n \to \infty. \tag{8.12}$$

Write $a_n = 1 - \frac{\zeta_n}{n}$ with $\zeta_n \geqslant 0$. If $|\delta_n| = O(1)$, then the last asymptote implies that $\zeta_n = O(1)$ and

$$a_n^n = \exp\{\zeta_n(1 + o(1))\} \asymp 1, \qquad 1 - a_n^{2n} = 1 - \exp\{2\zeta_n(1 + o(1))\} \asymp \zeta_n,$$

and

$$1 - a_n^{2n+2} = 1 - a_n^2 + a_n^2(1 - a_n^{2n}) \asymp \zeta_n.$$

Thus, we have

$$1 - \beta_{n,1} = \frac{\sqrt{p_n q_n}(1-a_n^2)^2 a_n^{2n-1}}{(1-a^{2n})(1-a^{2n+2})} \asymp \frac{1}{n^2} \quad \text{as } n \to \infty.$$

If $|\delta_n| \to \infty$ or equivalently $\delta_n \to \infty$, one can compute

$$1 - a_n \sim \frac{2\delta_n}{n}, \qquad \left(\frac{1}{a_n}\sqrt{\frac{p_n}{q_n}}\right)^n = \left(1 - \frac{(1-a_n^2)a_n^{2n}}{1 - a_n^{2n+2}}\right)^n \sim 1 \quad \text{as } n \to \infty, \tag{8.13}$$

which yields

$$1 - \beta_{n,1} = \frac{\sqrt{p_n q_n}(1-a_n^2)^2 a_n^{2n-1}}{(1-a^{2n})(1-a^{2n+2})} \sim \frac{8\delta_n^2}{n^2}\left(\frac{p_n}{q_n}\right)^n. \tag{8.14}$$

For the case $|\delta_n| \asymp n$, it is clear that if $\delta_n < 0$, then $1 - \beta_{n,1} \asymp 1$ and if $\delta_n > 0$, then

$$\limsup_{n\to\infty} p_n < 1/2.$$

Note that the function f_n in (8.6) converges uniformly to the identity map on [0, 1]. Thus, in the case $\delta_n \asymp n$, we have

$$\limsup_{n\to\infty} a_n < 1.$$

This implies that the second part of (8.13) holds true and (8.14) becomes

$$1-\beta_{n,1} \asymp \left(\frac{p_n}{q_n}\right)^{n+1/2}.$$

Summarizing from the above discussions, we achieve

$$1-\beta_{n,1} \asymp \begin{cases} 1 & \text{if } \delta_n \to -\infty \text{ and } |\delta_n| \asymp n, \\ \delta_n^2/n^2 & \text{if } \delta_n \to -\infty \text{ and } |\delta_n| = o(n), \\ 1/n^2 & \text{if } |\delta_n| = O(1), \\ (\delta_n^2/n^2)(p_n/q_n)^n & \text{if } \delta_n \to \infty \text{ and } |\delta_n| = o(n), \\ (p_n/q_n)^{n+1/2} & \text{if } \delta_n \asymp n. \end{cases} \tag{8.15}$$

Proof of Theorem 8.1. Recall those notations introduced in (7.7). Write $p_n = 1/2 - \delta_n/n$. In this setting, (8.7) is equivalent to

$$\frac{|x_n\delta_n|}{nq_n} \to \infty \tag{8.16}$$

and (8.8) becomes

$$\delta_n \to \infty \quad \text{and} \quad \frac{|x_n|\delta_n}{n} \to 0. \tag{8.17}$$

Due to the symmetry of the transition probabilities about 0, we can assume that $x_n \geqslant 0$. In the case $x_n = 0$, by binding states i and $-i$ together, the origin chain in (8.2) collapses to the chain in (7.1) with the exchange of p_n and q_n. Then, the results in Theorem 7.2 and Remark 7.4 yield the equivalent conditions in (8.16) and (8.17) and the desired cutoff time. We assume in the following that $x_n \geqslant 1$ and prove this theorem by considering all possible cases of δ_n and x_n.

Throughout this proof, we let $j_n^\gamma(C)$ and $\tau_n^\gamma(C)$ be those defined in (5.2) and (5.3), where c and d denote respectively continuous time cases and discrete time cases. Let $\lambda_{n,j}^c$ and $\lambda_{n,j}^d$ be the rearrangements of $1-\beta_{n,j}$ and $-\log|\beta_{n,j}|$ in the way that

$$\lambda_{n,j}^\gamma \leqslant \lambda_{n,j+1}^\gamma, \quad \forall 1 \leqslant j < 2n,\ \gamma \in \{c,d\}.$$

Similarly, let $\psi_{n,i}^\gamma$ be the rearrangement of $\psi_{n,i}$ according to $\lambda_{n,i}^\gamma$. In this setting, one can see that $\psi_{n,1}^c = \psi_{n,1}^d = \psi_{n,1}$ and

$$\lambda_{n,j}^c = 1-\beta_{n,j} \quad \forall 1 \leqslant j \leqslant 2n, \tag{8.18}$$

and

$$\lambda_{n,2j-1}^d = -\log|\beta_{n,j}|, \qquad \lambda_{n,2j}^d = -\log|\beta_{n,2n-j+1}| \quad \forall 1 \leqslant j \leqslant n. \tag{8.19}$$

Case 1: $|\delta_n| = O(1)$. In this case, it is easy to see that none of (8.16) and (8.17) are satisfied and we shall prove that there is no L^2-cutoff. To achieve this conclusion, one needs to compute $\tau_n^\gamma(C)$ and $j_n^\gamma(C)$ and, first of all, the order of $\psi_{n,i}^2(x)$ should be determined. In the assumption $|\delta_n| = O(1)$, it is clear that

$$(p_n/q_n)^x \asymp 1 \quad \text{uniformly for } |x| \leqslant n,$$

and then the normalizing constant for the stationary distribution π_n satisfies

$$c_n = \frac{1 - q_n/p_n}{1 - (q_n/p_n)^{n+1}} = \frac{1}{1 + q_n/p_n + \cdots + (q_n/p_n)^n} \asymp \frac{1}{n}.$$

In the case $p_n/q_n < n^2/(n+1)^2$, one may apply (8.12) to get

$$t^x \asymp 1 \quad \text{uniformly for } |x| \leqslant n+1,\ t \in \left[a_n, a_n^{-1}\right]$$

and

$$a_n^{-x} - a_n^x = \int_{a_n}^{a_n^{-1}} x t^{x-1}\, dt \asymp x\left(a_n^{-1} - a_n\right) \quad \text{uniformly for } 1 \leqslant |x| \leqslant n.$$

The last asymptote leads to the following estimations,

$$C_{n,1}^{-2} = c_n \sum_{x=1}^{n} \left(a_n^{-x} - a_n^x\right)^2 \asymp n^2\left(a_n^{-1} - a_n\right)^2$$

and

$$\psi_{n,1}^2(x) \asymp \frac{x^2}{n^2} \quad \text{uniformly for } 1 \leqslant |x| \leqslant n.$$

Such a conclusion is obviously true for the case $p_n/q_n = n^2/(n+1)^2$. When $p_n/q_n > n^2/(n+1)^2$, observe that

$$\frac{1}{2}\left(n - \frac{\sin n\theta \cos(n+1)\theta}{\sin\theta}\right) = \sum_{x=1}^{n} \sin^2 x\theta, \qquad \sin^2 x\theta \asymp \frac{1}{\theta} \int_{(x-1)\theta}^{x\theta} \sin^2 t\, dt$$

where the second asymptote holds true uniformly for $\theta \in (0, \pi/(n+1))$, $x \in \{1, 2, \ldots, n\}$ and $n \geqslant 1$. Using these observations, we have for $\theta \in (0, \pi/(n+1))$,

$$\sum_{x=1}^{n} \sin^2 x\theta \asymp \frac{1}{\theta} \int_0^{n\theta} \sin^2 t\, dt \asymp n^3\theta^2.$$

Hence, $C_{n,1}^{-2} \asymp n^2\theta_{n,1}^2$ and

$$\psi_{n,1}^2(x) \lesssim \frac{x^2}{n^2} \quad \text{uniformly for } 1 \leqslant |x| \leqslant n.$$

For $\psi_{n,2j-1}$, the fact $\theta_{n,j} \in [(j-1)\pi/n, j\pi/(n+1)]$ implies

$$\left|\frac{\sin n\theta_{n,j}\cos(n+1)\theta_{n,j}}{\sin\theta_{n,j}}\right| \leqslant \frac{1}{\sin\theta_{n,j}} \leqslant \frac{1}{\sin\frac{\pi}{n+1}} \leqslant \frac{n+1}{2} \tag{8.20}$$

and, hence, $C_{n,2j-1}^{-2} \asymp 1$ uniformly for $1 < j \leqslant n$ and

$$\psi_{n,2j-1}^2(x) \asymp \sin^2 x\theta_{n,j} \leqslant 1 \quad \text{uniformly for } 1 \leqslant |x| \leqslant n,\ 1 < j \leqslant n.$$

To estimate $\psi_{n,2j}$, note that

$$C_{n,2j}^{-2} = \frac{c_n(n+1)(1-\beta_{n,2j})}{2} \asymp 1-\beta_{n,2j} \quad \text{uniformly for } 1 \leqslant j \leqslant n.$$

By setting

$$\sin\widetilde{\theta}_{n,j} = \frac{\sqrt{q_n}\sin\frac{j\pi}{n+1}}{\sqrt{1-\beta_{n,2j}}}, \qquad \cos\widetilde{\theta}_{n,j} = \frac{\sqrt{q_n}\cos\frac{j\pi}{n+1} - \sqrt{p_n}}{\sqrt{1-\beta_{n,2j}}}, \tag{8.21}$$

the last asymptote yields

$$\psi_{n,2j}^2(x) \asymp \sin^2\left(\frac{j|x|\pi}{n+1} + \widetilde{\theta}_{n,j}\right) \leqslant 1 \quad \text{uniformly for } 1 \leqslant |x| \leqslant n,\ 1 \leqslant j \leqslant n.$$

As a consequence of the above discussions, we have

$$\psi_{n,j}^2(x) \lesssim 1 \quad \text{uniformly for } 1 \leqslant |x| \leqslant n,\ 1 \leqslant j \leqslant n. \tag{8.22}$$

Now, it is ready to estimate $j_n^\gamma(C)$ and $\tau_n^\gamma(C)$. By (8.5), (8.15), (8.18) and (8.19), one can compute

$$\lambda_{n,j}^\gamma \gtrsim \frac{j^2}{n^2} \quad \text{uniformly for } 1 \leqslant j \leqslant 2n,\ \gamma \in \{c,d\}$$

and

$$\lambda_{n,j}^\gamma \asymp \frac{j^2}{n^2} \quad \text{uniformly for } 1 \leqslant j \leqslant n,\ \gamma \in \{c,d\}.$$

These two facts and (8.22) then lead to

$$\tau_n^\gamma(C) \lesssim \sup_{j \geqslant j_n^\gamma(C)} \left\{\frac{\log(j+1)}{j^2/n^2}\right\} \leqslant n^2 \quad \forall C > 0,\ \gamma \in \{c,d\}$$

regardless of the initial states $(x_n)_{n=1}^{\infty}$. For $j_n^{\gamma}(C)$, we may choose, by Remark 7.4, Remark 8.1 and Step 1 in the proof of Theorem 7.2, two constants C_0 and N such that

$$\sum_{j=1}^{N}\left|\psi_{n,2j}^{\gamma}(x)\right|^2 \geqslant C_0 \quad \forall 0 \leqslant x \leqslant n,\ n \geqslant 1,\ \gamma \in \{c, d\}.$$

This implies for $\gamma \in \{c, d\}$, $j_n^{\gamma}(C_0) \lesssim 1$ and

$$\tau_n^{\gamma}(C_0) \gtrsim \frac{1}{\lambda_{n, j_n^{\gamma}(C_0)}^{\gamma}} \asymp n^2, \qquad \lambda_{n, j_n^{\gamma}(C_0)}^{\gamma}\tau_n^{\gamma}(C_0) \asymp 1.$$

Hence, by Theorems 5.1 and 5.3, both families in discrete time and continuous time cases have no L^2-cutoffs.

Case 2: $\delta_n \to -\infty$ and $|\delta_n| = o(n)$. In this case, we will prove that the L^2-cutoff exists if and only if $|\delta_n|x_n/n \to \infty$. By (8.5) and the conclusion in (8.15), it is easy to see that

$$\lambda_{n,j}^{\gamma} \gtrsim \frac{\delta_n^2 + j^2}{n^2} \quad \text{uniformly for } 1 \leqslant j \leqslant 2n,\ \gamma \in \{c, d\}$$

and

$$\lambda_{n,j}^{\gamma} \asymp \frac{\delta_n^2 + j^2}{n^2} \quad \text{uniformly for } 1 \leqslant j \leqslant n,\ \gamma \in \{c, d\}.$$

To estimate the order of $|\psi_{n,j}^{\gamma}(x_n)|^2$, we have to determine the constants $C_{n,2j-1}$. First, the normalizing constant c_n in (8.3) satisfies

$$c_n = \frac{1 - q_n/p_n}{1 - (q_n/p_n)^{n+1}} \sim \frac{2|\delta_n|}{n}.$$

For $C_{n,1}$, note that the fact $\delta_n < 0$ implies $\theta_{n,1} \in [\pi/(2n+1), \pi/(n+1)]$ and

$$\sin^2 x\theta_{n,1} \asymp \frac{x^2}{n^2} \quad \text{uniformly for } 1 \leqslant x \leqslant n/2.$$

This yields

$$n \geqslant \sum_{x=0}^{n} \sin^2 x\theta_{n,1} \gtrsim n, \qquad C_{n,1}^{-2} = \frac{c_n}{2}\sum_{x=1}^{n} \sin^2 x\theta_{n,1} \asymp nc_n \asymp |\delta_n|.$$

For $C_{n,2j-1}$, observe that the conclusion developed in (8.20) is also valid here. Thus, we have

$$C_{n,2j-1}^{-2} \asymp nc_n \asymp |\delta_n| \quad \text{uniformly for } 1 < j \leqslant n.$$

Consequently, the above discussion gives

$$\left|\psi_{n,2j-1}(x)\right|^2 \asymp \left(\frac{p_n}{q_n}\right)^{|x|} \frac{\sin^2 x\theta_{n,j}}{|\delta_n|} \quad \text{uniformly for } 1 \leqslant j,\ |x| \leqslant n. \tag{8.23}$$

Consider two subcases, $|\delta_n|x_n = O(n)$ and $|\delta_n|x_n/n \to \infty$. In the former situation, it is easy to check that

$$\left|\psi_{n,2j-1}(x_n)\right|^2 \lesssim \frac{1}{|\delta_n|} \quad \text{uniformly for } 1 \leqslant j \leqslant n.$$

This implies

$$\sup_{1\leqslant j\leqslant n}\left\{\frac{\log\left(1+\sum_{i=1}^{j}|\psi^c_{n,2i-1}(x_n)|^2\right)}{\lambda^c_{n,2j-1}}\right\} \lesssim \frac{n^2}{\delta_n^2}\sup_{1\leqslant j\leqslant n}\left\{\frac{\log(1+j/|\delta_n|)}{1+j^2/\delta_n^2}\right\} \leqslant \frac{n^2}{\delta_n^2} \tag{8.24}$$

and similarly,

$$\sup_{1\leqslant j\leqslant n}\left\{\frac{\log\left(1+\sum_{i=1}^{j}(|\psi^d_{n,4i-3}(x_n)|^2+|\psi^d_{n,4i}(x_n)|^2)\right)}{\lambda^d_{n,4j-3}}\right\} \lesssim \frac{n^2}{\delta_n^2}. \tag{8.25}$$

Recall the conclusions of Step 3 and Step 4 in the proof of Theorem 7.2: There exist M and C_1 such that

$$\sum_{i=1}^{M|\delta_n|}\left|\psi_{n,2i}(x_n)\right|^2 \geqslant C_1$$

and

$$\sup_{M|\delta_n|\leqslant j\leqslant n}\left\{\frac{\log\left(1+\sum_{i=1}^{j}|\psi^c_{n,2i}(x_n)|^2\right)}{\lambda^c_{n,2j}}\right\} \lesssim \frac{n^2}{\delta_n^2} \tag{8.26}$$

and

$$\sup_{M|\delta_n|\leqslant j\leqslant n}\left\{\frac{\log\left(1+\sum_{i=1}^{j}|\psi^d_{n,4i-2}(x_n)|^2+|\psi^d_{2,4i-1}(x_n)|^2\right)}{\lambda^d_{n,4j-2}}\right\} \lesssim \frac{n^2}{\delta_n^2}. \tag{8.27}$$

The first inequality implies $j_n^\gamma(C_1) \lesssim |\delta_n|$ and

$$\forall\gamma\in\{c,d\},\quad \tau_n^\gamma(C_1) \gtrsim \frac{1}{\lambda^\gamma_{n,j_n(C_1)}} \asymp \frac{n^2}{\delta_n^2}.$$

Using the fact $\log(1+a+b) < \log(1+a)+\log(1+b)$ for $a,b>0$, one may conclude from (8.24)–(8.27) that $\tau_n^\gamma(C_1) \lesssim n^2/\delta_n^2$ for $\gamma\in\{c,d\}$, which yields $\tau_n^\gamma(C_1) \asymp n^2/\delta_n^2$ for $\gamma\in\{c,d\}$. Consequently, $\tau_n^\gamma(C_1)\lambda^\gamma_{n,j_n^\gamma(C_1)} \asymp 1$ and, by Theorems 5.1 and 5.3, both families in discrete time and continuous time cases have no L^2-cutoff.

For the subcase $|\delta_n|x_n/n \to \infty$, let $D^\gamma_{n,2}(x_n,t)$ be the L^2-distance for the nth Markov chain. Then, for $\gamma\in\{c,d\}$,

$$\left(D^\gamma_{n,2}(x_n,t)\right)^2 = L^\gamma_{n,1}(t) + L^\gamma_{n,2}(t) \tag{8.28}$$

where

$$L^c_{n,1}(t)=\sum_{j=1}^n\bigl|\psi_{n,2j-1}(x_n)\bigr|^2e^{-2t(1-\beta_{n,2j-1})},\qquad L^d_{n,1}(t)=\sum_{j=1}^n\bigl|\psi_{n,2j-1}(x_n)\bigr|^2|\beta_{n,2j-1}|^{2t}$$

and

$$L^c_{n,2}(t)=\sum_{j=1}^n\bigl|\psi_{n,2j}(x_n)\bigr|^2e^{-2t(1-\beta_{n,2j})},\qquad L^d_{n,2}(t)=\sum_{j=1}^n\bigl|\psi_{n,2j}(x_n)\bigr|^2|\beta_{n,2j}|^{2t}.$$

Note that $L^\gamma_{n,2}(t)$ is exactly the square of the L^2-distance for the chain in (7.1) starting from x_n. In the assumption of $|\delta_n|x_n/n\to\infty$, Theorem 7.2 implies that, for $\gamma\in\{c,d\}$, the family $\{L^\gamma_{n,2}(t)\colon n=1,2,\ldots\}$ presents an L^2-cutoff with cutoff time $nx_n/|\delta_n|$. Using this fact, it remains to show that $\{L^\gamma_{n,1}(t)\colon n=1,2,\ldots\}$ also possesses an L^2-cutoff with the same cutoff time. In detail, write

$$\widetilde{\lambda}^c_{n,j}=1-\beta_{n,2j-1}\quad\forall 1\leqslant j\leqslant n$$

and

$$\widetilde{\lambda}^d_{n,2j-1}=-\log|\beta_{n,2j-1}|,\qquad\widetilde{\lambda}^d_{n,2j}=-\log|\beta_{n,2n-2j+1}|\quad\forall 1\leqslant j\leqslant n/2,$$

and let $\widetilde{\psi}^\gamma_{n,j}$ be the rearrangement of $\psi_{n,2j-1}$ associated with $\widetilde{\lambda}^\gamma_{n,j}$. In this setting, it is clear that $\widetilde{\lambda}^\gamma_{n,j}\leqslant\widetilde{\lambda}^\gamma_{n,j+1}$ for $1\leqslant j<n$ and $\gamma\in\{c,d\}$ and

$$L^\gamma_{n,1}(t)=\sum_{j=1}^n\bigl|\widetilde{\psi}^\gamma_{n,j}\bigr|^2\exp\bigl\{-2t\widetilde{\lambda}^\gamma_{n,j}\bigr\}.$$

Let $\widetilde{j}^\gamma_n(C)$ and $\widetilde{\tau}^\gamma_n(C)$ be those in (5.2) and (5.3) associated with $\widetilde{\lambda}^\gamma_{n,j}$ and $\widetilde{\psi}^\gamma_{n,j}$. Then, by (8.11) and (8.23), we have

$$\sum_{i=1}^j\bigl|\widetilde{\psi}^\gamma_{n,i}(x_n)\bigr|^2\asymp\left(\frac{p_n}{q_n}\right)^{x_n}\frac{j^3x_n^2}{|\delta_n|n^2}\quad\text{uniformly for }1\leqslant j\leqslant\left\lceil\frac{n}{2x_n}\right\rceil.$$

Using this, one can compute

$$\log\left(1+\sum_{i=1}^{\lceil n/2x_n\rceil}\bigl|\widetilde{\psi}^\gamma_{n,i}(x_n)\bigr|^2\right)\geqslant\frac{4x_n|\delta_n|}{n}\bigl(1+o(1)\bigr)\to\infty\quad\text{as }n\to\infty,$$

which gives $\widetilde{j}^\gamma_n(C)\leqslant n/x_n$ for n large enough and

$$\widetilde{\tau}^\gamma_n(C)\geqslant\frac{4x_n|\delta_n|/n(1+o(1))}{2\widetilde{\lambda}^\gamma_{n,\lceil n/x_n\rceil}}\sim\frac{4x_n|\delta_n|/n}{2\widetilde{\lambda}^\gamma_{n,1}}\sim\frac{nx_n}{|\delta_n|}\quad\forall C>0.\tag{8.29}$$

Hence, $\widetilde{\tau}_n^\gamma(C)\widetilde{\lambda}^\gamma_{n,\widetilde{j}_n^\gamma(C)} \gtrsim \frac{x_n|\delta_n|}{n} \to \infty$ and

$$\lim_{n\to\infty} \sum_{j=1}^{\widetilde{j}_n^\gamma(C)-1} \left|\widetilde{\psi}^\gamma_{n,j}(x_n)\right|^2 e^{-2\widetilde{\tau}_n^\gamma(C)\widetilde{\lambda}^\gamma_{n,j}} \leqslant \lim_{n\to\infty} Ce^{-2\widetilde{\tau}_n^\gamma(C)\widetilde{\lambda}^\gamma_{n,1}} = 0.$$

By Theorems 5.1 and 5.3, both families have an L^2-cutoff.

To see $nx_n/|\delta_n|$ is a cutoff time, we need the following facts. For some universal constant $N > 0$,

$$\widetilde{\lambda}^\gamma_{n,j} \geqslant \widetilde{\lambda}^\gamma_{n,1}\left(1 + \frac{j^2-1}{N\delta_n^2}\right) \quad \forall 1 \leqslant j \leqslant n,\ n \geqslant 1,\ \gamma \in \{c,d\},$$

and

$$\left|\widetilde{\psi}^\gamma_{n,j}(x_n)\right|^2 \lesssim \left(\frac{p_n}{q_n}\right)^{x_n} \frac{1}{|\delta_n|} \quad \text{uniformly for } 1 \leqslant j \leqslant n.$$

The former comes immediate from the definition of $\lambda^\gamma_{n,2j-1}$ whereas the latter is a simple corollary of (8.23). Using these two inequalities, one can prove that

$$\begin{aligned}
\frac{\log\left(1 + \sum_{i=1}^j |\widetilde{\psi}^\gamma_{n,i}(x_n)|^2\right)}{2\widetilde{\lambda}^\gamma_{n,i}} &\lesssim \frac{x_n|\delta_n|/n + [\log(j+1)/|\delta_n|]}{\widetilde{\lambda}^\gamma_{n,1}(j^2-1)/\delta_n^2} \\
&= o\left(\frac{nx_n}{|\delta_n|}\right) \quad \text{uniformly for } \delta_n^2 x_n/n \leqslant j \leqslant n.
\end{aligned}$$

and

$$\begin{aligned}
\frac{\log\left(1 + \sum_{i=1}^j |\widetilde{\psi}^\gamma_{n,i}(x_n)|^2\right)}{2\widetilde{\lambda}^\gamma_{n,i}} &\leqslant \frac{x_n\log(p_n/q_n) + \log(x_n|\delta_n|/n) + O(1)}{2\widetilde{\lambda}^\gamma_{n,1}} \\
&\sim \frac{nx_n}{|\delta_n|} \quad \text{uniformly for } 1 \leqslant j \leqslant x_n\delta_n^2/n.
\end{aligned}$$

As a consequence of the above computations and (8.29), the L^2-cutoff time for both families is $nx_n/|\delta_n|$.

Case 3: $\delta_n \to \infty$ and $\delta_n = o(n)$. In this case, one can use (8.13) to get

$$\frac{a_n^{-2n-1} - a_n^{2n+1}}{n(a_n^{-1} - a_n)} \sim \frac{\exp\{4\delta_n(1+o(1))\}}{4\delta_n} \to \infty.$$

This implies $C_{n,1} \sim 1$ and

$$\begin{aligned}
\left|\psi_{n,1}(x_n)\right|^2 &\sim \left(\frac{p_n}{q_n}\right)^{x_n}\left(a_n^{-x_n} - a_n^{x_n}\right)^2 \\
&\sim \left(1 - a_n^{2x}\right)^2 \begin{cases} \asymp 1 & \text{if } x_n\delta_n/n \gtrsim 1, \\ = o(1) & \text{if } x_n\delta_n/n = o(1). \end{cases}
\end{aligned} \tag{8.30}$$

We discuss the L^2-cutoff by considering these two subcase. In the assumption $x_n\delta_n/n \gtrsim 1$, one may choose C_2 such that

$$j_n^\gamma(C_2) = 1,$$

which implies $\tau_n^\gamma(C_2) \gtrsim 1/\lambda_{n,1}^\gamma$. To find $\tau_n^\gamma(C_2)$, note that (8.20) implies

$$C_{2j-1}^{-2} \asymp nc_n \asymp \delta_n \left(\frac{p_n}{q_n}\right)^n.$$

Thus, we have

$$\left|\psi_{n,2j-1}(x_n)\right|^2 \lesssim \frac{1}{\delta_n}\left(\frac{q_n}{p_n}\right)^n \quad \text{uniformly for } 1 \leqslant j \leqslant n. \tag{8.31}$$

Observe that $C_{n,2j}^{-2} \sim 2\delta_n(1-\beta_{n,2j})(p_n/q_n)^n$ uniformly for $1 \leqslant j \leqslant n$. Using the notations introduced at (8.21), one can derive

$$\left|\psi_{n,2j}(x_n)\right|^2 \lesssim \frac{1}{\delta_n}\left(\frac{q_n}{p_n}\right)^n \quad \text{uniformly for } 1 \leqslant j \leqslant n. \tag{8.32}$$

By the fact

$$\lambda_{n,j}^\gamma \gtrsim \frac{\delta_n^2 + j^2}{n^2} \quad \text{uniformly for } 1 < j \leqslant 2n,$$

(8.31) and (8.32) yield

$$\begin{aligned}\tau_n^\gamma(C_2) &\lesssim \max\left\{\frac{1}{\lambda_{n,1}^\gamma}, \frac{n^2}{\delta_n^2} \sup_{2\leqslant j\leqslant 2n} \frac{n\log(q_n/p_n) + \log(j/\delta_n)}{1 + j^2/\delta_n^2}\right\}\\ &\asymp \max\left\{\frac{1}{\lambda_{n,1}^\gamma}, \frac{n^2}{\delta_n}\right\} \asymp \frac{1}{\lambda_{n,1}^\gamma},\end{aligned}$$

where the last asymptotic is a result of (8.15). Consequently, $\tau_n^\gamma(C_2)\lambda_{n,1}^\gamma \asymp 1$ for $\gamma \in \{c,d\}$ and, by Theorems 5.1 and 5.3, there is no L^2-cutoff in either case.

In the case $x_n\delta_n/n = o(1)$, recall (8.28). By Theorem 7.2, the family $\{L_{n,2}^\gamma\colon n = 1,2,\ldots\}$ has an L^2-cutoff with cutoff time

$$\frac{(n-x_n)\log(q_n/p_n)}{2\lambda_{n,2}^\gamma} \sim \frac{n^2}{\delta_n} \quad \forall\gamma \in \{c,d\}.$$

For $L_{n,1}^\gamma(t)$, let $\widetilde{\lambda}_{n,j}^\gamma$ and $\widetilde{\psi}_{n,j}^\gamma$ be those defined in Case 3. Note that one may choose a universal constant $\widetilde{N} > 0$ such that

$$\widetilde{\lambda}_{n,j}^\gamma \geqslant \frac{2\delta_n^2}{n^2}\left(1 + \frac{\widetilde{N}j^2}{\delta_n^2}\right) \quad \forall 2 \leqslant j \leqslant n,\ \gamma \in \{c,d\}.$$

As before, (8.20) implies

$$\left|\widetilde{\psi}_{n,j}(x_n)\right|^2 \lesssim \frac{1}{\delta_n}\left(\frac{q_n}{p_n}\right)^n = \frac{e^{4\delta_n(1+o(1))}}{\delta_n} \quad \text{uniformly for } 1 < j \leqslant n.$$

By setting $t_n = n^2/\delta_n$, we may compute using (8.30) that, for any $\epsilon > 0$ and $\gamma \in \{c, d\}$,

$$\begin{aligned}
L^{\gamma}_{n,1}\big((1+\epsilon)t_n\big) &= \sum_{j=2}^{n}\left|\widetilde{\psi}^{\gamma}_{n,j}(x_n)\right|^2 \exp\left\{-2(1+\epsilon)t_n\widetilde{\lambda}^{\gamma}_{n,j}\right\} + o(1) \\
&\lesssim \frac{1}{\delta_n}\sum_{j=2}^{\infty}\exp\left\{\frac{-4\widetilde{N}j^2}{\delta_n}\right\} + o(1) \\
&\leqslant \frac{1}{\sqrt{\delta_n}}\left(1 + \frac{1}{\sqrt{\delta_n}}\sum_{j\geqslant\sqrt{\delta_n}}^{\infty}\exp\left\{\frac{-4\widetilde{N}j}{\sqrt{\delta_n}}\right\}\right) + o(1) \\
&\leqslant \frac{1}{\sqrt{\delta_n}}\left(1 + \int_0^{\infty} e^{-4\widetilde{N}z}\,dz\right) + o(1) = o(1).
\end{aligned}$$

Consequently, for $\epsilon \in (0,1)$ and $\gamma \in \{c, d\}$,

$$\begin{aligned}
&\lim_{n\to\infty} D^{\gamma}_{n,2}\big(x_n, (1+\epsilon)n^2/\delta_n\big) \\
&\quad = \lim_{n\to\infty} L^{\gamma}_{n,1}\big((1+\epsilon)n^2/\delta_n\big) + \lim_{n\to\infty} L^{\gamma}_{n,2}\big((1+\epsilon)n^2/\delta_n\big) = 0
\end{aligned}$$

and

$$\lim_{n\to\infty} D^{\gamma}_{n,2}\big(x_n, (1-\epsilon)n^2/\delta_n\big) \geqslant \lim_{n\to\infty} L^{\gamma}_{n,2}\big((1-\epsilon)n^2/\delta_n\big) = \infty.$$

This means that both families have an L^2-cutoff with cutoff time n^2/δ_n as desired.

Case 4: $|\delta_n| \asymp n$. We first deal with the case $\delta_n > 0$. Recall that

$$\left(\frac{q_n/p_n}{a_n^2}\right)^n \sim 1.$$

Using this fact, it is easy to check

$$\left|\psi_{n,1}(x_n)\right|^2 \asymp \left(1 - a_n^{2x_n}\right)^2 \asymp 1,$$

where the last asymptote uses the assumption $x_n \geqslant 1$. Thus, by (8.15), we have $\lambda^{\gamma}_{n,1} \asymp (p_n/q_n)^{n+1/2}$ and

$$\sum_{j=2}^{2n} \left|\psi_{n,j}^{\gamma}(x_n)\right|^2 \exp\left\{-2\left(\lambda_{n,1}^{\gamma}\right)^{-1}\lambda_{n,j}^{\gamma}\right\} \leqslant \frac{1}{\pi_n(x_n)} \exp\left\{-2\lambda_{n,2}^{\gamma}/\lambda_{n,1}^{\gamma}\right\}$$
$$\leqslant \exp\left\{n\log(q_n/p_n) - C(q_n/p_n)^{n+1/2} + O(1)\right\} = o(1)$$

where C is a universal positive constant. This yields

$$D_{n,2}^{\gamma}\left(x_n, \epsilon/\lambda_{n,1}^{\gamma}\right) \asymp e^{-\epsilon} \quad \forall \epsilon > 0$$

which means that both families have no L^2-cutoff.

In the case $\delta_n < 0$, (8.7) becomes $x_n/q_n \to \infty$. First, assume that $x_n/q_n = O(1)$ or equivalently $x_n = O(1)$ and $1/q_n = O(1)$. For continuous time cases, since $1/\pi_n(x_n)$ is bounded, Corollary 5.2 implies that no L^2-cutoff exists. For discrete time cases, using the notation in (8.28), one can show without difficulty that

$$\forall t > 0, \quad \liminf_{n\to\infty} D_{n,2}^{d}(x_n, t) \geqslant \liminf_{n\to\infty} L_{n,2}^{d}(t) > 0.$$

Hence, by Corollary 3.3, these is no L^2-cutoff.

To see the sufficiency of $x_n/q_n \to \infty$, set

$$s_n^c = \frac{x_n(\log p_n - \log q_n)}{2(1 - 2\sqrt{p_n q_n})}, \qquad s_n^d = \left\lfloor \frac{x_n(\log p_n - \log q_n)}{-\log(4p_n q_n)} \right\rfloor.$$

By Theorem 7.2, the family $\{L_{n,2}^{\gamma}\colon n = 1, 2, \ldots\}$ has an L^2-cutoff with cutoff time s_n^{γ}. This implies for $\epsilon \in (0, 1)$ and $\gamma \in \{c, d\}$,

$$\liminf_{n\to\infty} D_{n,2}^{\gamma}\left(x_n, (1-\epsilon)s_n^{\gamma}\right) \geqslant L_{n,2}^{\gamma}\left((1-\epsilon)s_n^{\gamma}\right) = \infty.$$

To get an upper bound on the L^2-distance, note that

$$\lambda_{n,1}^{c} \sim 1 - 2\sqrt{p_n q_n}, \qquad \lambda_{n,1}^{d} \sim -\log(4p_n q_n).$$

This implies, for $\epsilon > 0$,

$$D_{n,2}^{\gamma}\left(x_n, (1+\epsilon)s_n^{\gamma}\right)^2 \leqslant \frac{1}{\pi_n(x_n)} \exp\left\{-2(1+\epsilon)s_n^{\gamma}\lambda_{n,1}^{\gamma}\right\}$$
$$= \exp\left\{x_n(\log p_n - \log q_n) - 2(1+\epsilon)s_n^{\gamma}\lambda_{n,1}^{\gamma} + O(1)\right\}.$$

Hence, in the assumption $x_n/q_n \to \infty$, we have

$$s_n^d \sim \frac{x_n(\log p_n - \log q_n)}{-\log(4p_n q_n)}, \quad x_n(\log p_n - \log q_n) \to \infty,$$

and, for $\gamma \in \{c, d\}$ and $\epsilon > 0$,

$$D_{n,2}^{\gamma}\left(x_n, (1+\epsilon)s_n^{\gamma}\right)^2 \leqslant \exp\left\{-2\epsilon\left(1 + o(1)\right)x_n(\log p_n - \log q_n)\right\} = o(1). \qquad \square$$

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Appendix A. Techniques and proofs

Proof of Lemma 3.2. There is no loss of generality in assuming that $t_n = 1$ for all n since one may always consider the following sequence of functions.

$$g_n(t) = f_n(tt_n) = \int_{(0,\infty)} e^{-t\lambda}\, dV_n'(\lambda)$$

where $V_n'(\lambda) = V_n(\lambda/t_n)$. By letting $V_n(0) = \lim_{\lambda\downarrow 0} V_n(\lambda)$ and

$$\forall s \in (0,1), \quad h_n(s) = \sup\bigl\{e^{-\lambda}\colon\ V_n(\lambda) - V_n(0) > sf_n(0)\bigr\},$$

we may express f_n as follows.

$$f_n(t) = f_n(0) \int_{(0,1)} h_n^t(s)\, ds, \quad \forall t > 0. \tag{A.1}$$

It is clear that h_n is a non-increasing non-negative function bounded from above by 1. Using the sequential compactness of monotonic functions, we may choose a subsequence n_k such that h_{n_k} converges almost surely to a non-increasing function h and $f_{n_k}(0)$ converges to $C \geqslant 0$. Consequently, one can show without difficulty that

$$\lim_{k\to\infty} f_{n_k}(a) = C \int_{(0,1)} h^a(s)\, ds, \quad \forall a > 0.$$

Using a similar argument as before, one may show that the right-hand side above is in fact a Laplace transform and then, by Lemma 3.1, is analytic on $(0,\infty)$.

It remains to prove that such a convergence is uniform on any compact subset of $(0,\infty)$. Note that

$$\bigl|x^b - y^b\bigr| \leqslant \frac{b}{a}\bigl|x^a - y^a\bigr|, \quad \forall x, y \in (0,1),\ b > a > 0.$$

Using this fact, one can show that

$$\begin{aligned}\sup_{b\in[2a,3a]} \bigl|f_{n_k}(b) - f_{n_l}(b)\bigr| &\leqslant \bigl|f_{n_k}(0) - f_{n_l}(0)\bigr| + f_{n_k}(0) \sup_{b\in[2a,3a]} \int_{(0,1)} \bigl|h_{n_k}^b(s) - h_{n_l}^b(s)\bigr|\, ds \\ &\leqslant \bigl|f_{n_k}(0) - f_{n_l}(0)\bigr| + 3f_{n_k}(0) \int_{(0,1)} \bigl|h_{n_k}^a(s) - h_{n_l}^a(s)\bigr|\, ds\end{aligned}$$

which converges to 0 as k, l tends to infinity. This proves that f_{n_k} converges uniformly on $[2a, 3a]$ for all $a > 0$ as desired. The last part of this lemma is easy to show using the locally uniform convergence of f_{n_k} and the continuity of the limiting function. □

Proof of Corollary 3.3. By scaling the time t up to a constant, one only needs to prove the continuity of F_1, F_2 at $t = 1$. We give the proof for F_1 but omit the similar proof for F_2. By Lemma 3.2, we may choose a subsequence f_{n_k} such that

$$\lim_{k\to\infty} f_{n_k}(at_{n_k}) = f(a) \quad \forall a > 0$$

and $f(1) = F_1(1)$, where f is continuous on $(0, \infty)$. Clearly, F_1 and f are non-increasing and satisfy $f \leqslant F_1$. This implies

$$F_1(1) = f(1) = \lim_{a\downarrow 1} f(a) \leqslant \liminf_{a\downarrow 1} F_1(a) \leqslant \limsup_{a\downarrow 1} F_1(a) \leqslant F_1(1)$$

which proves the right-continuity of F_1 at 1.

Concerning the left-continuity, set

$$L = \lim_{a\uparrow 1} F_1(a).$$

Let $m_0 = 1$. For $k \geqslant 1$, we may choose $x_k \in (1 - 2^{-k}, 1)$ and $m_k \geqslant m_{k-1}$ such that $f_{m_k}(x_k t_{m_k}) \in (L - 1/k, L + 1/k)$. Referring to the subsequence sequence m_k, we may choose by Lemma 3.2 a further subsequence m'_k such that the function $a \mapsto f_{m'_k}(at_{m'_k})$ converges uniformly to a continuous function g on any compact subset of $(0, \infty)$. This implies

$$L = \lim_{k\to\infty} f_{m'_k}(x_k t_{m'_k}) = \lim_{k\to\infty} g(x_k) = g(1).$$

Again, since F_1 is non-increasing and $g \leqslant F_1$, we get

$$F_1(1) \leqslant L = g(1) \leqslant F_1(1),$$

that is, F_1 is left-continuous.

For the second part of this corollary, assume that $F_1(c) > 0$ for some $c > 0$. As before, we may choose, by Lemma 3.2, a subsequence n_k such that f_{n_k} converges to an analytic function f and $f(c) = F_1(c) > 0$. Clearly, $F_1 \geqslant f$ and then, by the analyticity of f on $(0, \infty)$, $F_1 > 0$. □

Proof of Corollary 3.4. Set $g_n(s) = f_n(t_n + s)$. It is clear that

$$g_n(s) = f_n(t_n) \int_{(0,\infty)} e^{-s\lambda}\, d\widetilde{V}_n(\lambda) \quad \text{for } s > 0,$$

where $\widetilde{V}$ is a probability distribution defined by

$$\widetilde{V}_n(\gamma) = \frac{\int_{(0,\gamma]} e^{-t_n\lambda}\, dV_n(\lambda)}{\int_{(0,\infty)} e^{-t_n\lambda}\, dV_n(\lambda)} \quad \forall \gamma > 0.$$

Part (i) is then obtained by applying Corollary 3.3 to g_n and b_n. In the first case of (ii), assume the inverse that $\overline{F}(c_0) < \infty$ for some $c_0 < 0$. For $n \geqslant 1$, let $\widetilde{g}_n(s) = f_n(t_n + c_0 b_n + s)$. Since $\mathcal{F}$ has a (t_n, b_n)-cutoff, h_n is well-defined on $[0, \infty)$ for n large enough. For $c > 0$, let

$$G(c) = \liminf_{n\to\infty} \widetilde{g}_n(cb_n).$$

Obviously,

$$G(c) = \underline{F}(c + c_0), \qquad G(-c_0) = \underline{F}(0) > 0, \qquad H(0) \leqslant \overline{F}(c_0) < \infty.$$

As a consequence of Corollary 3.3, the analyticity of H implies that $G > 0$ on $(0, \infty)$ or equivalently $\underline{F} > 0$ on (c_0, ∞). This contradicts the assumption that $\underline{F}(c) = 0$ for some $c > 0$. Thus, $\overline{F} = \infty$ on $(-\infty, 0)$. In the second case of (ii), we prove as before by contradiction. Assume the inverse that $\underline{F}(c_1) < \infty$ for some $c_1 < 0$. This is equivalent to the existence of a subsequence n_k such that $f_{n_k}(c_1)$ is bounded. By considering the subsequence f_{n_k}, a similar proof as that of the first case will derive a confliction. Hence, $\underline{F} = \infty$ on $(-\infty, 0)$.

For (iii), let $t_n = T(f_n, \delta)$ with $\delta > 0$ and set

$$\overline{F}_\epsilon(c) = \overline{F}(c + \epsilon), \qquad \underline{F}_\epsilon(c) = \underline{F}(c + \epsilon), \quad \forall \epsilon \in \mathbb{R}.$$

According to the definition of the δ-mixing time, it can be easily shown that

$$\overline{F}_\epsilon(0) = \overline{F}(\epsilon) \leqslant \delta < \infty, \quad \forall \epsilon > 0,$$

and

$$\underline{F}_\epsilon(0) = \underline{F}(\epsilon) \geqslant \delta > 0, \quad \forall \epsilon < 0.$$

By [5, Corollary 2.4], the family $\mathcal{F}$ also presents a $(t_n + \epsilon b_n, b_n)$-cutoff for all $\epsilon \in \mathbb{R}$. Using the former inequality in the above, we may conclude from (i) that, for $\epsilon > 0$, either $\overline{F}_\epsilon > 0$ or $\overline{F}_\epsilon \equiv 0$ on $(0, \infty)$. This is equivalent to say that either $\overline{F} > 0$ or $\overline{F} \equiv 0$ on $(0, \infty)$. The proof for (ii) in this case is similar to that of (i) using the latter inequality. □

Proof of Theorem 3.5. Part (i) is an immediate result of Corollary 3.3. For (ii), we assume that there is a cutoff for $\mathcal{F} = \{f_n\colon n = 1, 2, \ldots\}$. By [5, Corollary 2.5(i)], the cutoff time sequence can be chosen to be $t_n = T(f_n, \delta)$ for any $\delta > 0$. Let C be any positive number and $\lambda_n = \lambda_n(C)$ be the constant defined in (3.2). Note that, for $n \geqslant 1$,

$$f_n(2t_n) \geqslant \int\limits_{(0,2\lambda_n]} e^{-2\lambda t_n}\, dV_n(\lambda) \geqslant \max\left\{Ce^{-4\lambda_n t_n}, \int\limits_{(0,\lambda_n)} e^{-2\lambda t_n}\, dV_n(\lambda)\right\}.$$

Then, the existence of the cutoff for $\mathcal{F}$ implies that $f_n(2t_n) \to 0$ as $n \to \infty$. This proves (a) and (b) with arbitrary $C > 0$, $\delta > 0$ and $\epsilon = 2$. In fact, (c) is true for all $\epsilon > 0$. To see this, let V_n' be a function defined by

$$V_n'(\lambda) = \begin{cases} V_n(\lambda) & \text{if } \lambda \in (0, \lambda_n), \\ \lim_{t\uparrow\lambda_n} V_n(t) & \text{if } \lambda \in [\lambda_n, \infty) \end{cases}$$

and set

$$g_n(t) = \int_{(0,\infty)} e^{-\lambda t}\, dV_n'(\lambda) = \int_{(0,\lambda_n)} e^{-\lambda t}\, dV_n(\lambda).$$

Clearly, $g_n(0) = V_n((0,\lambda_n)) \leqslant C$ for all $n \geqslant 1$ and

$$\limsup_{n\to\infty} g_n(2t_n) = 0.$$

By Corollary 3.3, we obtain

$$\limsup_{n\to\infty} g_n(\epsilon t_n) = 0 \quad \forall \epsilon > 0.$$

For the other direction, assume that C, δ, ϵ are positive constants such that (a) and (b) hold and let

$$t_n = T(f_n, \delta), \qquad b_n = 1/\lambda_n = 1/\lambda_n(C).$$

In this setting, one can show that for $c > 0$ and $n \geqslant N = N(c)$,

$$f_n(t_n + cb_n) \leqslant \int_{(0,\lambda_n)} e^{-\lambda t_n/2}\, dV_n(\lambda) + \delta e^{-c/2}$$

and

$$f_n(t_n - cb_n) \geqslant e^{c/2}\left(\delta - \int_{(0,\lambda_n)} e^{-\lambda t_n/2}\, dV_n(\lambda)\right).$$

By Corollary 3.3, (b) implies $\int_{(0,\lambda_n)} e^{-\lambda t_n/2}\, dV_n(\lambda) \to 0$ as $n \to \infty$. Consequently, $\mathcal{F}$ has a (t_n, b_n)-cutoff as desired. □

Proof of Theorem 3.6. Part (i) is immediate from Remark 3.2. For (ii), let $T^d(f_n, \delta)$ and $T^c(f_n, \delta)$ be respectively the mixing time for f_n with domain $\mathbb{N}$ and $[0,\infty)$. By Definition 2.3, $|T^d(f_n,\delta) - T^c(f_n,\delta)| \leqslant 1$ and using the assumption $T^d(f_n,\delta) \to \infty$, we know that $T^d(f_n,\delta) \sim T^c(f_n,\delta)$ for all $\delta > 0$. This implies that Theorem 3.5 (a)–(b) hold for $T^c(f_n,\delta), \lambda_n(C)$ if and only if they are true for $T^d(f_n,\delta), \lambda_n(C)$. Also [5, Propositions 2.3–2.4], $\{f_n : [0,\infty) \to [0,\infty] \mid n = 1,2,\ldots\}$ has a cutoff if and only if $\{f_n : \mathbb{N} \to [0,\infty] \mid n = 1,2,\ldots\}$ has a cutoff. Consequently, Theorem 3.6 is then a corollary of Theorem 3.5.

To see a cutoff window, note that, by Theorem 3.5, $\{f_n : [0,\infty) \to [0,\infty] \mid n = 1,2,\ldots\}$ has a $(T^c(f_n,\delta), \lambda_n^{-1})$-cutoff. Recall the fact $|T^d(f_n,\delta) - T^c(f_n,\delta)| \leqslant 1$. Then, by [5, Propositions 2.3–2.4], $\{f_n : \mathbb{N} \to [0,\infty] \mid n = 1,2,\ldots\}$ has a $(T^d(f_n,\delta), \gamma_n^{-1})$-cutoff. □

Proof of Proposition 3.7. We only consider the case where the domain of f_n is $[0,\infty)$. To see why the assumption $b_n \to \infty$ arises in the case of discrete domain, confer [5, Remark 2.9].

Since $\mathcal{F}$ has a cutoff, Theorem 3.5 implies that $M = \limsup_n f_n(0) = \infty$. Let $\overline{F}, \underline{F}$ be functions in (2.1). Part (ii) is an immediate result of Corollary 3.4. For (i), we first assume that

$\underline{F}(c) > 0$ for some $c > 0$. From the definition of mixing time and the monotonicity of f_n, it is clear that $\overline{F}(c/2) \leqslant \delta$. Then, by [5, Proposition 2.2], the (t_n, b_n)-cutoff is optimal. Since an optimal cutoff must be a weakly optimal cutoff, it remains to show that if there is a weakly optimal cutoff, then $\underline{F}(c) > 0$ for some $c > 0$, which is equivalent to $\underline{F}(c) > 0$ for all $c > 0$ using Corollary 3.4. Assume the inverse that $\underline{F}(c_0) = 0$ for some $c_0 > 0$ and let n_k be a subsequence such that $f_{n_k}(t_{n_k} + c_0 b_{n_k}) \to 0$ as $k \to \infty$. Consider the subfamily $\mathcal{G} = \{f_{n_k}\colon k \geqslant 1\}$ and let

$$\underline{G}(c) = \liminf_{k\to\infty} f_{n_k}(t_{n_k} + cb_{n_k}), \qquad \overline{G}(c) = \limsup_{k\to\infty} f_{n_k}(t_{n_k} + cb_{n_k}).$$

Obviously, $\underline{G}(c_0) = \overline{G}(c_0) = 0$ and, by Corollary 3.4, this implies

$$\underline{G}(c) = \overline{G}(c) = 0 \quad \forall c > 0, \qquad \underline{G}(c) = \overline{G}(c) = \infty \quad \forall c < 0.$$

Then, by [5, Proposition 2.2], the (t_{n_k}, b_{n_k})-cutoff for $\mathcal{G}$ can not be weakly optimal and this contradicts Proposition 2.1. □

Proof of Theorem 3.8. We first assume that (a) and (b) hold for some positive constants C, ϵ. Note that (a) restricts us to case (ii) of Theorem 3.5 because one may choose a sequence $\lambda_n' > \lambda_n$ such that

$$\frac{\log(1 + V_n((0, \lambda_n']))}{\lambda_n'} \geqslant \tau_n/2.$$

This implies

$$\tau_n\lambda_n \leqslant \tau_n\lambda_n' \leqslant 2\log\bigl(1 + V_n\bigl((0, \lambda_n']\bigr)\bigr) \leqslant 2\log\bigl(1 + V_n(0, \infty)\bigr) \to \infty,$$

as $n \to \infty$. By Corollary 3.3, (b) is true for all $\epsilon > 0$. Note that, for $n \geqslant 1$, we may choose a non-decreasing sequence $(\lambda_{n,k})_{k=1}^{\infty}$ such that

$$\lambda_{n,k} \geqslant \lambda_n \quad \forall k \geqslant 1, \qquad r_{n,k} = \frac{\log(1 + V_n((0, \lambda_{n,k}]))}{\lambda_{n,k}} \to \tau_n \quad \text{as } k \to \infty.$$

In this setting, it is easy to see that, for $k \geqslant 1$,

$$\begin{aligned} f_n(r_{n,k}) &\geqslant \int\limits_{(0,\lambda_{n,k}]} e^{-\lambda r_{n,k}}\, dV_n(\lambda) \geqslant e^{-\lambda_{n,k} r_{n,k}} V_n\bigl((0, \lambda_{n,k}]\bigr) \\ &= \frac{V_n((0, \lambda_{n,k}])}{1 + V_n((0, \lambda_{n,k}])} \geqslant \frac{V_n((0, \lambda_n])}{1 + V_n((0, \lambda_n])} \geqslant \frac{C}{1 + C}. \end{aligned}$$

By letting $\overline{C} = C/(1 + C)$, we obtain from the above computations that $\tau_n \leqslant T(f_n, \overline{C})$. Consequently,

$$\lim_{n\to\infty} T(f_n, \overline{C})\lambda_n \geqslant \lim_{n\to\infty} \tau_n\lambda_n = \infty$$

and

$$\int_{(0,\lambda_n)} e^{-\lambda T(f_n,\overline{C})}\,dV_n(\lambda) \leqslant \int_{(0,\lambda_n)} e^{-\lambda\tau_n}\,dV_n(\lambda) = 0.$$

By Theorem 3.5, $\mathcal{F}$ presents a cutoff.

For the inverse direction, assume the existence of the cutoff for $\mathcal{F}$. By Theorem 3.5 and Remark 3.4, the following are true for any positive constants C, δ, ϵ.

$$\lambda_n T(f_n,\delta) \to \infty, \qquad \int_{(0,\lambda_n)} e^{-\epsilon\lambda T(f_n,\delta)}\,dV_n(\lambda) \to 0,$$

where $\lambda_n = \lambda_n(C)$ is the constant defined in (3.2). Using these facts, it remains to show that, for some $\delta > 0$, $T(f_n,\delta) = O(\tau_n)$. Let $C > 0$ and $\tau_n = \tau_n(C)$ be the quantity defined in (3.3). For $\eta > 0$ and $n \geqslant 1$, we let $A_{n,j} = [\lambda_n(1+\eta)^j, \lambda_n(1+\eta)^{j+1})$ for $j \geqslant 0$. Consider the following computations.

$$\begin{aligned} f_n(t) &= \int_{(0,\infty)} e^{-\lambda t}\,dV_n(\lambda) \leqslant C + \sum_{j\geqslant 0}\int_{A_{n,j}} e^{-\lambda t}\,dV_n(\lambda) \\ &\leqslant C + \sum_{j\geqslant 0} e^{-\lambda_n(1+\eta)^j t} V_n\big((0,\lambda_n(1+\eta)^{j+1})\big) \\ &\leqslant C + \sum_{j\geqslant 0} \exp\big\{-\lambda_n(1+\eta)^{j+1}\big(t/(1+\eta) - \tau_n\big)\big\}. \end{aligned}$$

By letting $t = (1+\eta)^2\tau_n$, we have

$$f_n\big((1+\eta)^2\tau_n\big) \leqslant C + \frac{\exp\{-\eta\tau_n\lambda_n\}}{1-\exp\{-\eta^2\tau_n\lambda_n\}}. \tag{A.2}$$

Let $v : (0,\infty) \to (0,\infty)$ be any function satisfying

$$\sup\big\{v(t)/t\colon\ t \geqslant \log(1+C)\big\} < \infty, \qquad \inf_{t>0} e^{v(t)}\big(1 - e^{-v^2(t)/t}\big) = L > 0.$$

If one puts $\eta = v(\tau_n\lambda_n)/(\tau_n\lambda_n)$ in (A.2), then there exists some positive constant N such that

$$\forall n \geqslant N, \quad f_n(\tau_n + d_n) \leqslant \widetilde{C},$$

where

$$\widetilde{C} = C + 2/L, \qquad d_n = 2b_n(1 + b_n/\tau_n), \qquad b_n = \lambda_n^{-1} v(\tau_n\lambda_n).$$

Thus, $T(f_n,\widetilde{C}) \leqslant \tau_n + d_n$ for $n \geqslant N$. To derive the desired identity $T(f_n,\widetilde{C}) = O(\tau_n)$, it suffices to show that $b_n = O(\tau_n)$, which can be easily computed out using the fact $\tau_n\lambda_n \geqslant \log(1+C) > 0$. For a realization of v, one may choose $v(t) = t^{1/2}$.

To see a cutoff sequence, assume that $\mathcal{F}$ has a cutoff or, equivalently, Theorem 3.8 (a)–(b) hold. In this case, one may go through all arguments in the above to choose positive constants $\overline{C}, \widetilde{C}$ such that

$$T(f_n, \widetilde{C}) - d_n \leqslant \tau_n \leqslant T(f_n, \overline{C}) \quad \text{for } n \text{ large enough}, \tag{A.3}$$

where $d_n = 2b_n(1 + b_n/\tau_n)$, $b_n = \lambda_n^{-1} w(\tau_n \lambda_n)$ and $w : (0,\infty) \to (0,\infty)$ is a function satisfying

$$\limsup_{t\to\infty} \frac{w(t)}{t} < \infty, \qquad \liminf_{t\to\infty} e^{w(t)}\big(1 - e^{-w^2(t)/t}\big) > 0. \tag{A.4}$$

Thus, τ_n is a cutoff sequence for $\mathcal{F}$ if and only if there exists a function w satisfying (A.4) such that $d_n = o(\tau_n)$. This is equivalent to $b_n = o(\tau_n)$ or $w(t) = o(t)$ as $t \to \infty$. As one can see that $w(t) = t^{1/2}$ is qualified for (3.4), τ_n is a cutoff sequence.

To get a window sequence corresponding to τ_n, assume that w is a function satisfying (3.4). By Theorem 3.5, $\mathcal{F}$ has a $(T(f_n,\delta), \lambda_n^{-1})$-cutoff for any $\delta > 0$ and, by [5, Proposition 2.3], there exists $C_1 > 0$ such that

$$T(f_n, \overline{C}) \leqslant T(f_n, \widetilde{C}) + C_1\lambda_n^{-1} \quad \text{for } n \text{ large enough}.$$

Putting this inequality and (A.3) together gives

$$\big|\tau_n - T(f_n, \widetilde{C})\big| = O\big(b_n + \lambda_n^{-1}\big) = O\big(\lambda_n^{-1}\big(w(\tau_n\lambda_n) + 1\big)\big).$$

Note that the second condition of (3.4) implies that $w(t) \to \infty$ as $t \to \infty$. This implies $|\tau_n - T(f_n, \widetilde{C})| = O(b_n)$ and then, by [5, Corollary 2.5(v)], $\mathcal{F}$ has a (τ_n, b_n)-cutoff. □

Proof of Theorem 3.9. Let $\mathcal{F}_c$ and $\mathcal{F}_d$ be families in Theorems 3.8 and 3.9 and, for $\delta > 0$, let $T^c(f_n,\delta)$ and $T^d(f_n,\delta)$ be respectively their mixing time sequences. In this setting, it is clear that

$$T^c(f_n,\delta) \leqslant T^d(f_n,\delta) \leqslant T^c(f_n,\delta) + 1. \tag{A.5}$$

Recall in the proof of Theorem 3.8 that

$$\tau_n(C) \leqslant T^c\big(f_n, C/(C+1)\big) \quad \forall C > 0,\ n \geqslant 1.$$

This implies

$$\tau_n(C) \to \infty \quad \Rightarrow \quad T^d\big(f_n, C/(C+1)\big) \to \infty.$$

Thus, in Theorem 3.9, we always have $T^d(f_n,\delta) \to \infty$ for some $\delta > 0$. Consequently, by [5, Propositions 2.3–2.4], the above fact and (A.5) imply

$$\mathcal{F}_c \text{ has a cutoff} \quad \Leftrightarrow \quad \mathcal{F}_d \text{ has a cutoff}$$

and, for b_n such that $\inf_n b_n > 0$,

$$\mathcal{F}_c \text{ has a } (t_n, b_n)\text{-cutoff} \quad \Leftrightarrow \quad \mathcal{F}_d \text{ has a } (t_n, b_n)\text{-cutoff}.$$

Hence, Theorem 3.9 is an immediate result of Theorem 3.8. □

References

[1] David Aldous, Random walks on finite groups and rapidly mixing Markov chains, in: Seminar on Probability, XVII, in: Lecture Notes in Math., vol. 986, Springer, Berlin, 1983, pp. 243–297.
[2] David Aldous, Persi Diaconis, Shuffling cards and stopping times, Amer. Math. Monthly 93 (5) (1986) 333–348.
[3] David Aldous, Persi Diaconis, Strong uniform times and finite random walks, Adv. Appl. Math. 8 (1) (1987) 69–97.
[4] Guan-Yu Chen, The cut-off phenomenon for finite Markov chains, PhD thesis, Cornell University, 2006.
[5] Guan-Yu Chen, Laurent Saloff-Coste, The cutoff phenomenon for ergodic Markov processes, Electron. J. Probab. 13 (2008) 26–78.
[6] Persi Diaconis, The cutoff phenomenon in finite Markov chains, Proc. Natl. Acad. Sci. USA 93 (4) (1996) 1659–1664.
[7] Persi Diaconis, Phil Hanlon, Eigen-analysis for some examples of the Metropolis algorithm, in: Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications, Tampa, FL, 1991, in: Contemp. Math., vol. 138, Amer. Math. Soc., Providence, RI, 1992, pp. 99–117.
[8] Persi Diaconis, Laurent Saloff-Coste, Separation cut-offs for birth and death chains, Ann. Appl. Probab. 16 (4) (2006) 2098–2122.
[9] William Feller, An Introduction to Probability Theory and Its Applications, vol. I, third ed., John Wiley & Sons Inc., New York, 1968.
[10] Mourad E.H. Ismail, David R. Masson, Jean Letessier, Galliano Valent, Birth and death processes and orthogonal polynomials, in: Orthogonal Polynomials, Columbus, OH, 1989, in: NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 294, Kluwer Acad. Publ., Dordrecht, 1990, pp. 229–255.
[11] Samuel Karlin, James McGregor, Random walks, Illinois J. Math. 3 (1959) 66–81.
[12] Samuel Karlin, James McGregor, Ehrenfest urn models, J. Appl. Probab. 2 (1965) 352–376.
[13] Samuel Karlin, James L. McGregor, The Hahn polynomials, formulas and an application, Scripta Math. 26 (1961) 33–46.
[14] R. Koekoek, R. Swarttouw, The askey-scheme of hypergeometric orthogonal polynomials and its q-analog, http://math.nist.gov/opsf/projects/koekoek.html, 1998.
[15] David A. Levin, Yuval Peres, Elizabeth L. Wilmer, Markov Chains and Mixing Times, Amer. Math. Soc., Providence, RI, 2009, with a chapter by James G. Propp and David B. Wilson.
[16] F.J. MacWilliams, N.J.A. Sloane, The Theory of Error-correcting Codes. I, North-Holland Math. Library, vol. 16, North-Holland Publishing Co., Amsterdam, 1977.
[17] P.R. Parthasarathy, R.B. Lenin, On the exact transient solution of finite birth and death processes with specific quadratic rates, Math. Sci. 22 (2) (1997) 92–105.
[18] Walter Rudin, Functional Analysis, second ed., Int. Ser. Pure Appl. Math., McGraw–Hill Inc., New York, 1991.
[19] Laurent Saloff-Coste, Lectures on finite Markov chains, in: Lectures on Probability Theory and Statistics, Saint-Flour, 1996, in: Lecture Notes in Math., vol. 1665, Springer, Berlin, 1997, pp. 301–413.
[20] W. Van Assche, P.R. Parthasarathy, R.B. Lenin, Spectral representation of four finite birth and death processes, Math. Sci. 24 (2) (1999) 105–112.
[21] David Vernon Widder, The Laplace Transform, Princeton Math. Ser., vol. 6, Princeton University Press, Princeton, NJ, 1941.